Functions and Function Notation
In this section, you will:

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

A **relation** is a set of ordered pairs. The set of the first components of each ordered pair is called the **domain** and the set of the second components of each ordered pair is called the **range**. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.
Equation:

$$\{(1,\ 2),\ (2,\ 4),\ (3,\ 6),\ (4,\ 8),\ (5,\ 10)\}$$

The domain is $\{1,\ 2,\ 3,\ 4,\ 5\}$. The range is $\{2,\ 4,\ 6,\ 8,\ 10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an **output** value, or **dependent variable**, and is often labeled lowercase letter y.

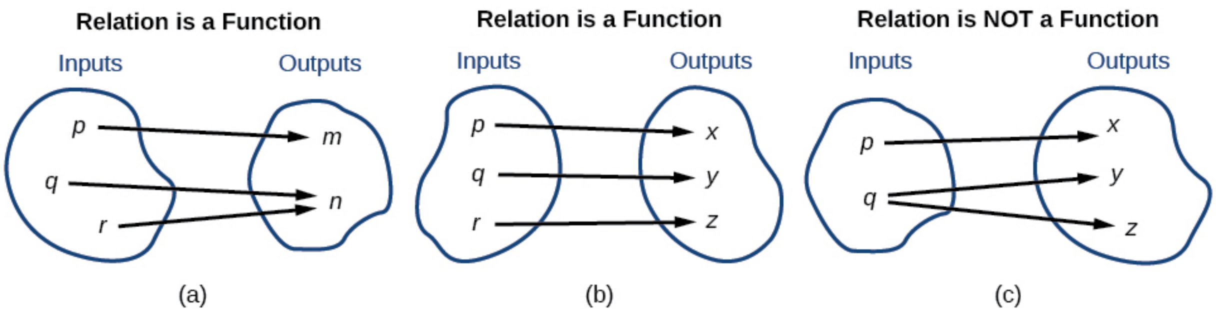
A function f is a relation that assigns a single value in the range to each value in the domain. In other words, no *x*-values are repeated. For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each element in the domain, $\{1,\ 2,\ 3,\ 4,\ 5\}$, is paired with exactly one element in the range, $\{2,\ 4,\ 6,\ 8,\ 10\}$.

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as
Equation:

$$\{(\text{odd},\ 1),\ (\text{even},\ 2),\ (\text{odd},\ 3),\ (\text{even},\ 4),\ (\text{odd},\ 5)\}$$

Notice that each element in the domain, $\{\text{even}, \text{odd}\}$ is *not* paired with exactly one element in the range, $\{1,\ 2,\ 3,\ 4,\ 5\}$. For example, the term "odd" corresponds to three values from the domain, $\{1,\ 3,\ 5\}$ and the term "even" corresponds to two values from the range, $\{2,\ 4\}$. This violates the definition of a function, so this relation is not a function.

[link] compares relations that are functions and not functions.



(a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Note:
Function
A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."
The **input** values make up the **domain**, and the **output** values make up the **range**.

Note:
Given a relationship between two quantities, determine whether the relationship is a function.

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example:
Exercise:

Problem:
Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in [link] consists of items and their prices.

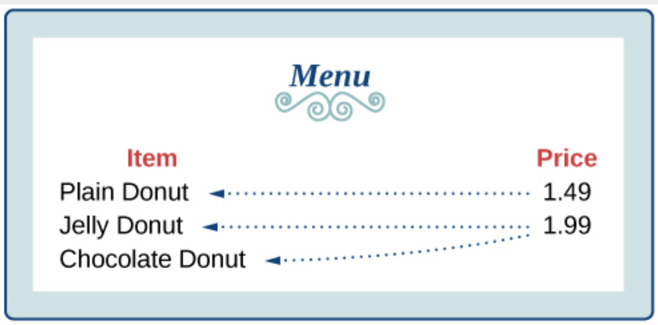
a. Is price a function of the item?
b. Is the item a function of the price?

Solution:

a. Let’s begin by considering the input as the items on the menu. The output values are then the prices. See [link].

Each item on the menu has only one price, so the price is a function of the item.

b. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See [link].



Therefore, the item is a not a function of price.

Example:
Exercise:

Problem:
Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? [link] shows a possible rule for assigning grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution:

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Note:
Exercise:

Problem:[link][footnote] lists the five greatest baseball players of all time in order of rank.
http://www.baseball-almanac.com/legendary/lisn100.shtml. Accessed 3/24/2014.

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Solution:

a. yes; b. yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters $f,\ g$,and h are often used to represent functions just as we use $x,\ y$,and$z$ to represent numbers and $A,\ B$, and C to represent sets.

Equation:

$$\begin{array}{ll} h \text{ is } f \text{ of } a & \text{We name the function } f\text{; height is a function of age.} \\ h = f(a) & \text{We use parentheses to indicate the function input.} \\ f(a) & \text{We name the function } f\text{; the expression is read as "}f \text{ of } a\text{."} \end{array}$$

Remember, we can use any letter to name the function; the notation $h\left(a\right)$ shows us that h depends on $a.$ The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f\left(a+b\right)$ means "first add a and b, and the result is the input for the function f." The operations must be performed in this order to obtain the correct result.

Note:
Function Notation
The notation $y=f\left(x\right)$ defines a function named $f.$ This is read as "y is a function of $x.$" The letter x represents the input value, or independent variable. The letter y, or $f\left(x\right),$ represents the output value, or dependent variable.

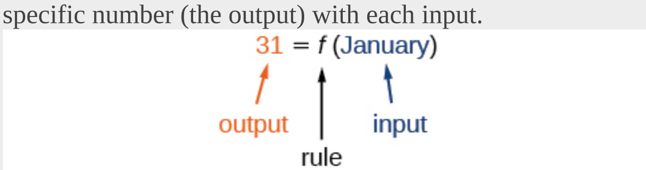
Example:
Exercise:

Problem:
Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month. Assume that the domain does not include leap years.

Solution:

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days} = f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.



For example, $f\left(\text{March}\right)=31,$ because March has 31 days. The notation $d=f\left(m\right)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example:
Exercise:

Problem:
Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, $N,$ in a town in year $y.$ What does $f(2005) = 300$ represent?

Solution:

When we read $f(2005) = 300,$ we see that the input year is 2005. The value for the output, the number of police officers $(N),$ is 300. Remember, $N = f(y).$ The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Note:
Exercise:

Problem: Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Solution:

$w = f(d)$

Note:
Instead of a notation such as $y = f(x\ ),$ could we use the same symbol for the output as for the function, such as $y = y(x\ ),$ meaning "*y* is a function of *x*?"
Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x), P = W(d),$ and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

[link] lists the input number of each month (January = 1, February = 2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

[link] defines a function $Q = g(n)$. Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q .

n	1	2	3	4	5
Q	8	6	7	6	8

[link] displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

Note:
Given a table of input and output values, determine whether the table represents a function.

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Example:
Exercise:

Problem:
Identifying Tables that Represent Functions

Which table, [link], [link], or [link], represents a function (if any)?

Input	Output
2	1
5	3
8	6

Input	Output
–3	5
0	1
4	5

Input	Output
1	0
5	2
5	4

Solution:

[link] and [link] define functions. In both, each input value corresponds to exactly one output value. [link] does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by [link] can be represented by writing
Equation:

$$f(2) = 1, f(5) = 3, \text{and } f(8) = 6$$

Similarly, the statements
Equation:

$$g\left(-3\right) = 5,\ g\left(0\right) = 1, \text{and } g\left(4\right) = 5$$

represent the function in [link].

[link] cannot be expressed in a similar way because it does not represent a function.

Note:
Exercise:

Problem: Does [link] represent a function?

Input	Output
1	10
2	100
3	1000

Solution:

yes

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we *evaluate* the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and *solve* for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f\left(x\right)=5-3{x}^{2}$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

Note:
Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example:
Exercise:

Problem:
Evaluating Functions at Specific Values

Evaluate $f\left(x\right)=x^2+3x-4$ at

a. 2
b. a
c. $a+h$
d. $\frac{f(a+h)-f(a)}{h}$

Solution:

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.
Equation:

$$\begin{aligned} f\left(2\right) &= 2^2+3\left(2\right)-4 \\ &= 4+6-4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.
Equation:

$$f\left(a\right)=a^2+3a-4$$

c. With an input value of $a+h,$ we must use the distributive property.
Equation:

$$\begin{aligned} f(a+h) &= (a+h)^2+3(a+h)-4 \\ &= a^2+2ah+h^2+3a+3h-4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that
Equation:

$$f\left(a+h\right)=a^2+2ah+h^2+3a+3h-4$$

and we know that
Equation:

$$f\left(a\right)=a^2+3a-4$$

Now we combine the results and simplify.
Equation:

$$
\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} \\
&= \frac{2ah+h^2+3h}{h} \\
&= \frac{h(2a+h+3)}{h} && \text{Factor out } h. \\
&= 2a+h+3 && \text{Simplify.}
\end{aligned}
$$

Example:
Exercise:

Problem:
Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

Solution:

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.
Equation:

$$
\begin{aligned}
h(p) &= p^2 + 2p \\
h(4) &= (4)^2 + 2(4) \\
&= 16 + 8 \\
&= 24
\end{aligned}
$$

Therefore, for an input of 4, we have an output of 24.

Note:
Exercise:

Problem: Given the function $g(m) = \sqrt{m-4}$, evaluate $g(5)$.

Solution:

$g(5) = 1$

Example:
Exercise:

Problem:
Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

Solution:

Equation:

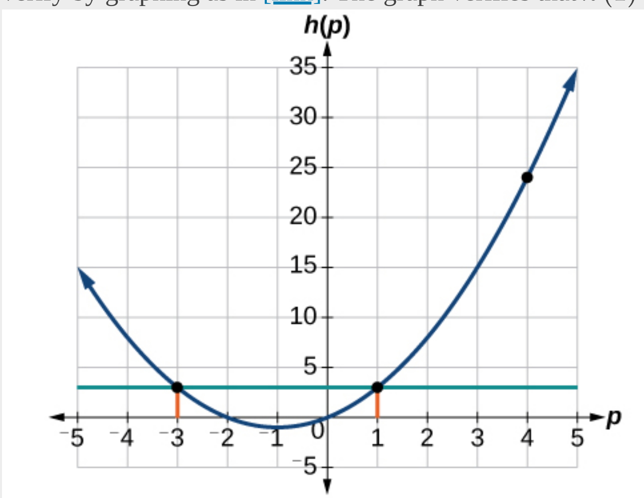
$$\begin{array}{rl} h(p) = 3 & \\ p^2 + 2p = 3 & \text{Substitute the original function } h(p) = p^2 + 2p. \\ p^2 + 2p - 3 = 0 & \text{Subtract 3 from each side.} \\ (p+3)(p-1) = 0 & \text{Factor.} \end{array}$$

If $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

Equation:

$$\begin{array}{l} (p+3) = 0, \quad p = -3 \\ (p-1) = 0, \quad p = 1 \end{array}$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing as in [link]. The graph verifies that $h(1) = h(-3) = 3$ and $h(4) = 24$.



p	−3	−2	0	1	4
h(*p*)	3	0	0	3	24

Note:
Exercise:

Problem: Given the function $g(m) = \sqrt{m-4}$, solve $g(m) = 2$.

Solution:

$m = 8$

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

Note:
Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves *only* the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example:
Exercise:

Problem:
Finding an Equation of a Function

Express the relationship $2n+6p=12$ as a function $p=f(n)$, if possible.

Solution:

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p=[\text{expression involving } n]$.
Equation:

$$\begin{array}{rll} 2n+6p=12 & & \\ 6p=12-2n & & \text{Subtract } 2n \text{ from both sides.} \\ p=\frac{12-2n}{6} & & \text{Divide both sides by 6 and simplify.} \\ p=\frac{12}{6}-\frac{2n}{6} & & \\ p=2-\frac{1}{3}n & & \end{array}$$

Therefore, p as a function of n is written as
Equation:

$$p=f(n)=2-\frac{1}{3}n$$

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example:
Exercise:

Problem:
Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution:

First we subtract x^2 from both sides.
Equation:

$$y^2 = 1 - x^2$$

We now try to solve for y in this equation.
Equation:

$$\begin{aligned} y &= \pm\sqrt{1 - x^2} \\ &= +\sqrt{1 - x^2} \text{ and } -\sqrt{1 - x^2} \end{aligned}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$.

Note:
Exercise:

Problem: If $x - 8y^3 = 0$, express y as a function of x.

Solution:

$y = f(x) = \frac{\sqrt[3]{x}}{2}$

Note:
Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?
Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy’s memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table. See [link].[footnote]
http://www.kgbanswers.com/how-long-is-a-dogs-memory-span/4221590. Accessed 3/24/2014.

Pet	Memory span in hours
Puppy	0.008
Adult dog	0.083
Cat	16
Goldfish	2160
Beta fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(\text{goldfish}) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

Note:
Given a function represented by a table, identify specific output and input values.

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example:
Exercise:

Problem:
Evaluating and Solving a Tabular Function

Using [link],

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

n	1	2	3	4	5
$g\left(n\right)$	8	6	7	6	8

Solution:

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. [link] shows two solutions: 2 and 4.

n	1	2	3	4	5
$g\left(n\right)$	8	6	7	6	8

When we input 2 into the function $g,$ our output is 6. When we input 4 into the function $g,$ our output is also 6.

Note:
Exercise:

Problem: Using [link], evaluate $g\left(1\right).$

Solution:

$g\left(1\right) = 8$

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

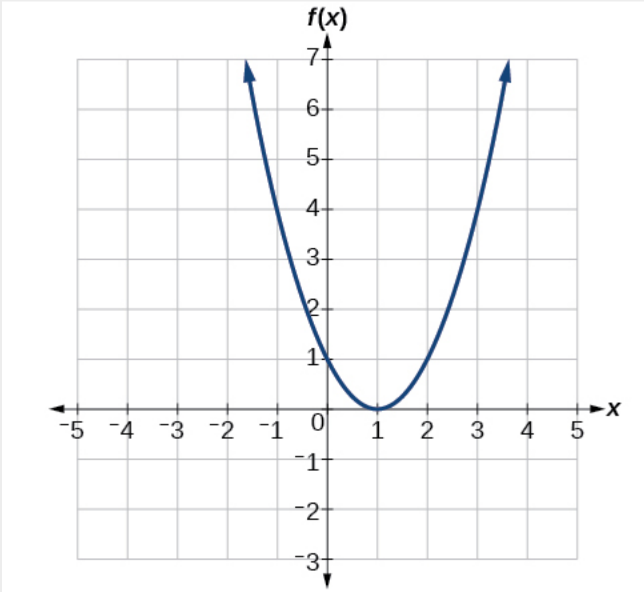
Example:
Exercise:

Problem:
Reading Function Values from a Graph

Given the graph in [link],

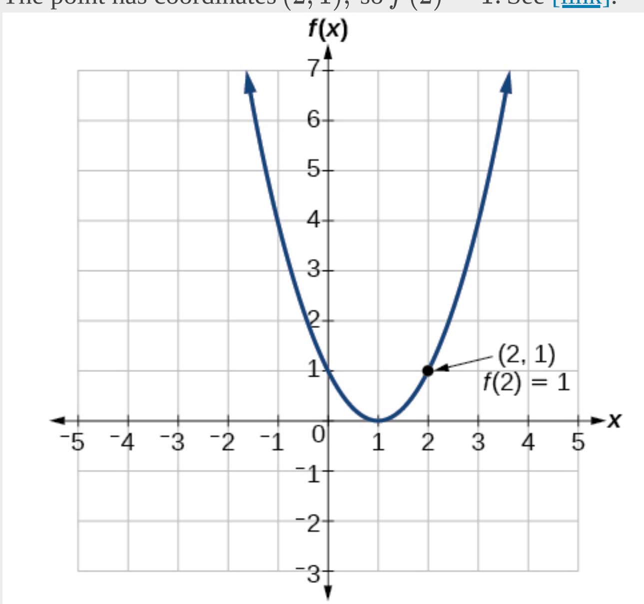
a. Evaluate $f\left(2\right).$

b. Solve $f(x) = 4.$

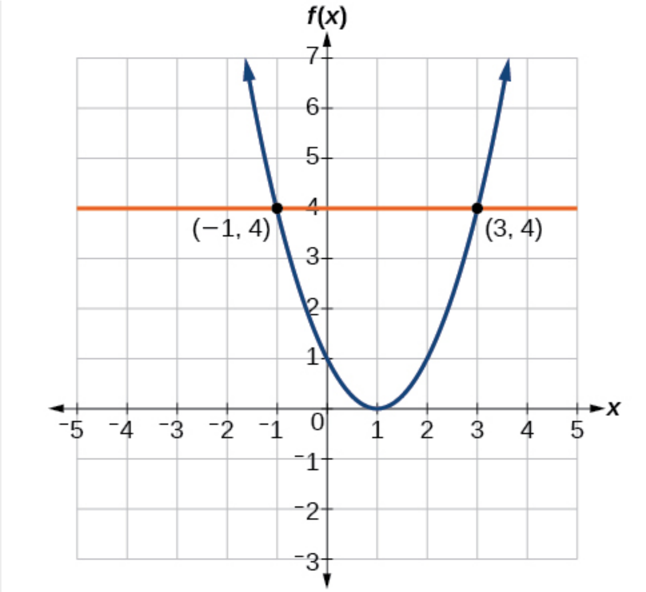


Solution:

a. To evaluate $f(2),$ locate the point on the curve where $x = 2,$ then read the *y*-coordinate of that point. The point has coordinates $(2, 1),$ so $f(2) = 1.$ See [link].



b. To solve $f(x) = 4,$ we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4,$ we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4).$ These points represent the two solutions to $f(x) = 4$: -1 or $3.$ This means $f(-1) = 4$ and $f(3) = 4,$ or when the input is -1 or$3,$ the output is $4.$ See [link].



Note:
Exercise:

Problem: Using [link], solve $f\left(x\right)=1$.

Solution:

$x=0$ or $x=2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in [link] at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in [link].

Letter grade	Grade point average
A	4.0
B	3.0

Letter grade	Grade point average
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let’s look again at the two simple functions sketched in [link]**(a)** and [link]**(b)**. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

Note:
One-to-One Function
A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

Example:
Exercise:

Problem:
Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution:

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius r.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle’s radius.

Note:
Exercise:

Problem:

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Solution:

a. yes, because each bank account has a single balance at any given time; b. no, because several bank account numbers may have the same balance; c. no, because the same output may correspond to more than one input.

Note:
Exercise:

Problem: Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

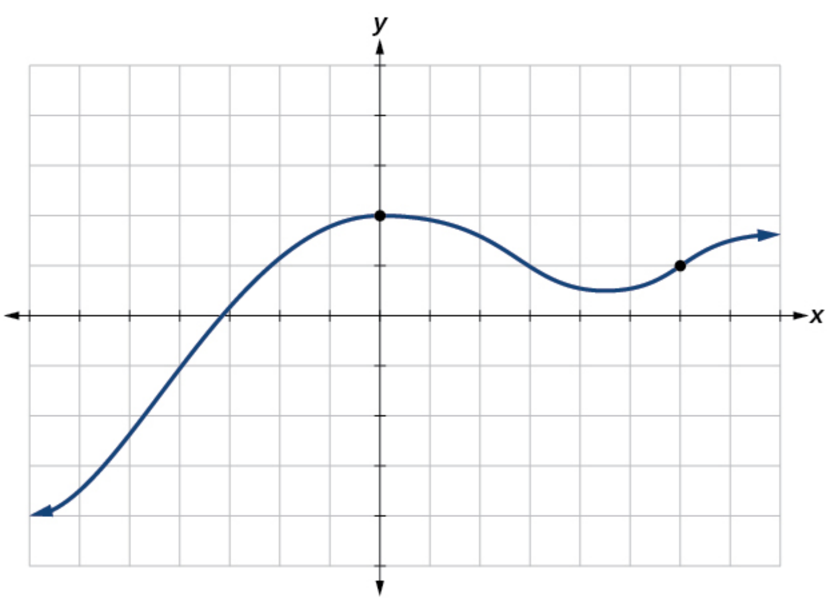
Solution:

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

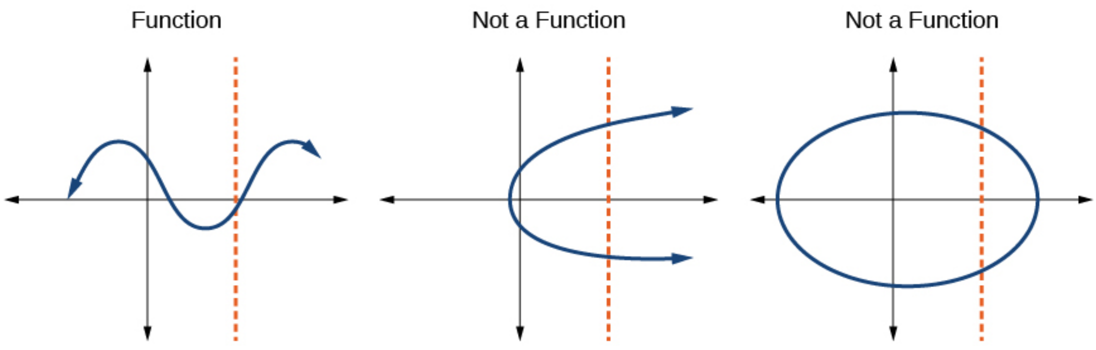
Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value $y,$ and we say y is a function of $x,$ or $y=f\left(x\right)$ when the function is named $f.$ The graph of the function is the set of all points $\left(x,y\right)$ in the plane that satisfies the equation$y=f\left(x\right).$ If the function is defined for only a few input values, then the graph of the function is only a few points, where the *x*-coordinate of each point is an input value and the *y*-coordinate of each point is the corresponding output value. For example, the black dots on the graph in [link] tell us that $f\left(0\right)=2$ and $f\left(6\right)=1.$ However, the set of all points $\left(x,y\right)$ satisfying $y=f\left(x\right)$ is a curve. The curve shown includes $\left(0,2\right)$ and $\left(6,1\right)$ because the curve passes through those points.



The **vertical line test** can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does *not* define a function because a function has only one output value for each input value. See [link].



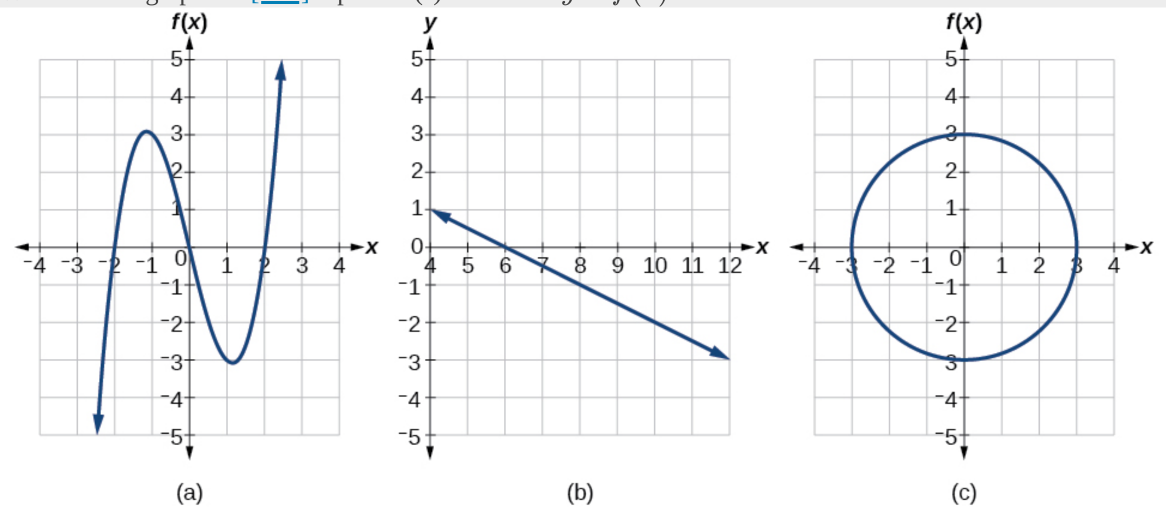
Note:
Given a graph, use the vertical line test to determine if the graph represents a function.

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example:
Exercise:

Problem:
Applying the Vertical Line Test

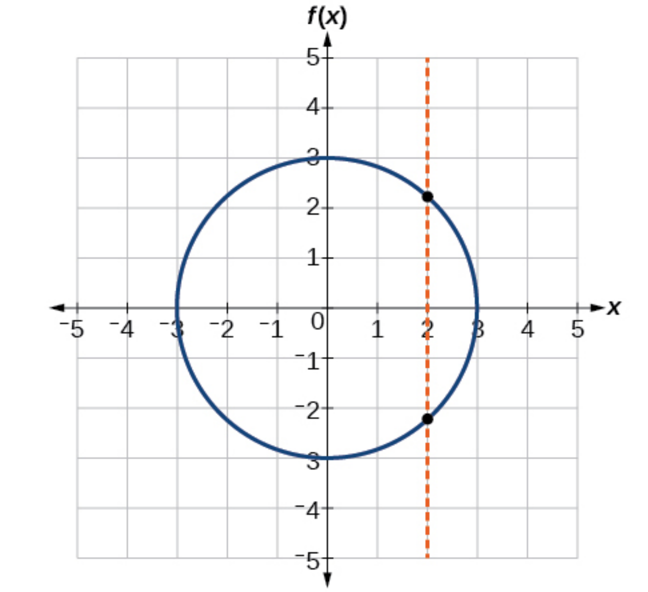
Which of the graphs in [link] represent(s) a function $y = f(x)$?



Solution:

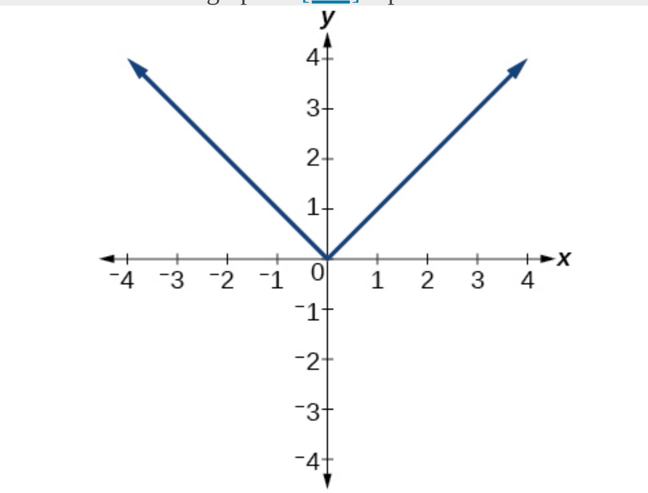
If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b)

of [link]. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in [link].



Note:
Exercise:

Problem: Does the graph in [link] represent a function?



Solution:

yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Note:
Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function.

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

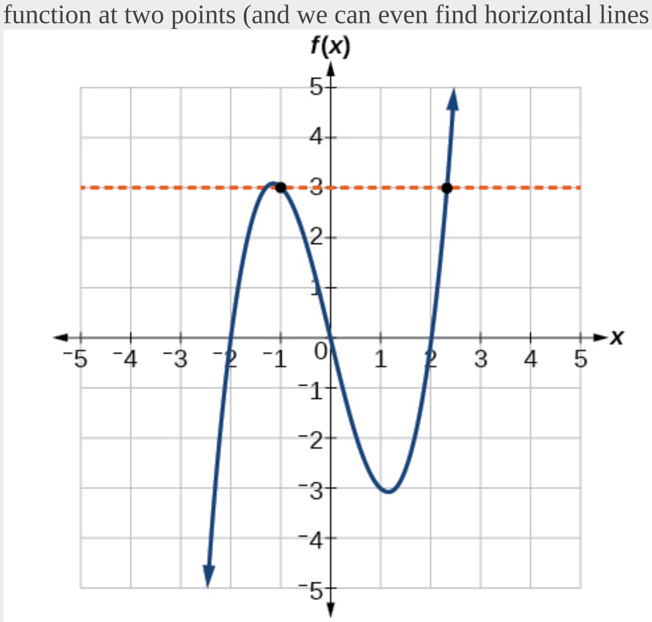
Example:
Exercise:

Problem:
Applying the Horizontal Line Test

Consider the functions shown in [link]**(a)** and [link]**(b)**. Are either of the functions one-to-one?

Solution:

The function in [link]**(a)** is not one-to-one. The horizontal line shown in [link] intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)



The function in [link]**(b)** is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Note:
Exercise:

Problem: Is the graph shown in [link] one-to-one?

Solution:

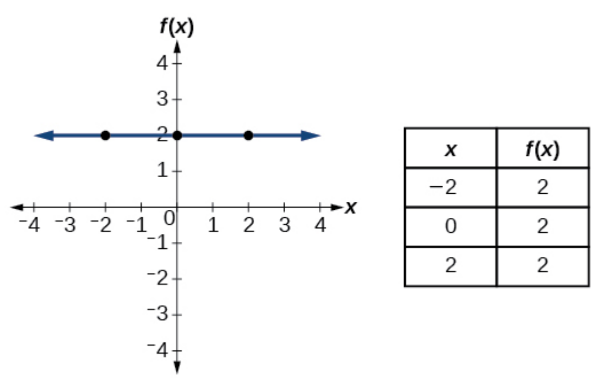
No, because it does not pass the horizontal line test.

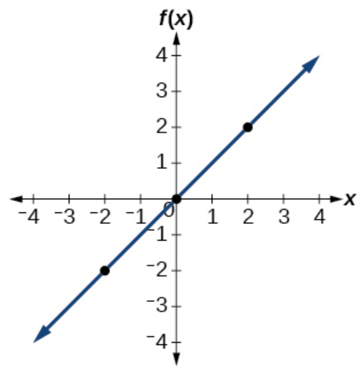
Identifying Basic Toolkit Functions

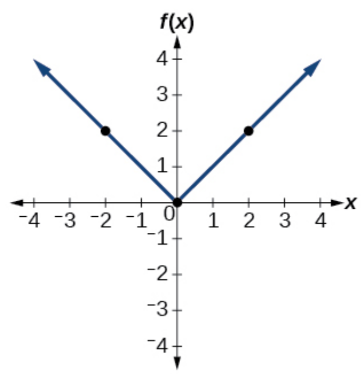
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

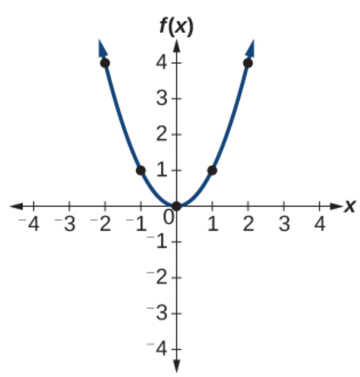
We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in [link].

Toolkit Functions

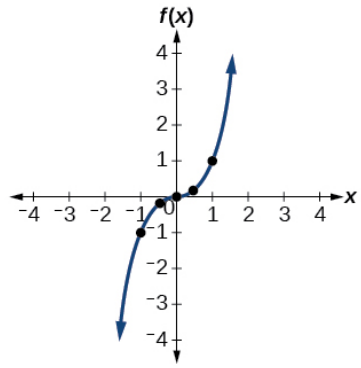
Name	Function	Graph
Constant	$f(x) = c$, where c is a constant	

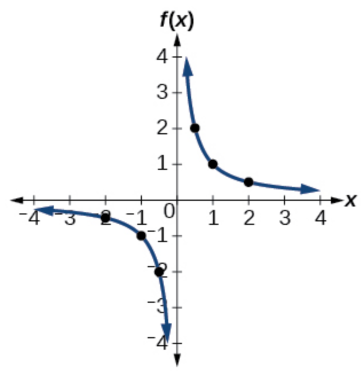
<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Identity</td><td>$f(x) = x$</td><td>

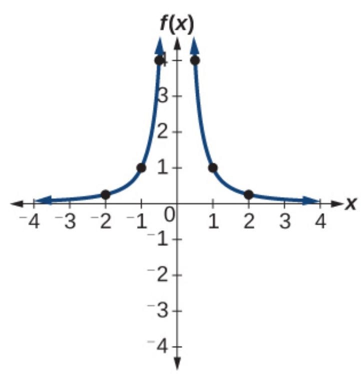
<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Absolute value</td><td>$f(x) = |x|$</td><td>

<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Quadratic</td><td>$f(x) = x^2$</td><td>

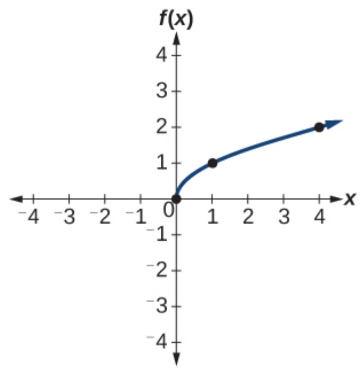
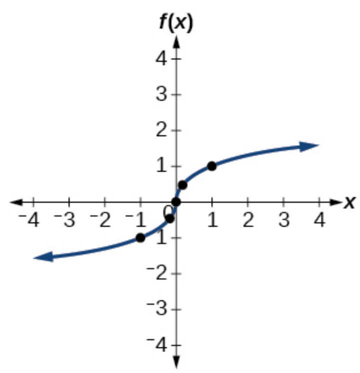
<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>4</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table></td></tr>
</table>

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Cubic</td><td>$f(x) = x^3$</td><td>


<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−0.125</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.5</td><td>0.125</td></tr>
<tr><td>1</td><td>1</td></tr>
</table>
</td></tr>
<tr><td>Reciprocal</td><td>$f(x) = \frac{1}{x}$</td><td>


<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−0.5</td></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−2</td></tr>
<tr><td>0.5</td><td>2</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.5</td></tr>
</table>
</td></tr>
<tr><td>Reciprocal squared</td><td>$f(x) = \frac{1}{x^2}$</td><td>


<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>0.25</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>−0.5</td><td>4</td></tr>
<tr><td>0.5</td><td>4</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.25</td></tr>
</table>
</td></tr>
</table>

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Square root</td><td>$f\left(x\right)=\sqrt{x}$</td><td><table><tr><th>x</th><th>f(x)</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>4</td><td>2</td></tr></table></td></tr>
<tr><td>Cube root</td><td>$f\left(x\right)=\sqrt[3]{x}$</td><td><table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.125</td><td>−0.5</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.125</td><td>0.5</td></tr><tr><td>1</td><td>1</td></tr></table></td></tr>
</table>

Note:
Access the following online resources for additional instruction and practice with functions.

- [Determine if a Relation is a Function]
- [Vertical Line Test]
- [Introduction to Functions]
- [Vertical Line Test on Graph]
- [One-to-one Functions]
- [Graphs as One-to-one Functions]

Key Equations

Constant function	$f(x) = c$, where c is a constant
Identity function	$f(x) = x$
Absolute value function	$f(x) = \lvert x \rvert$
Quadratic function	$f(x) = x^2$
Cubic function	$f(x) = x^3$
Reciprocal function	$f(x) = \frac{1}{x}$
Reciprocal squared function	$f(x) = \frac{1}{x^2}$
Square root function	$f(x) = \sqrt{x}$
Cube root function	$f(x) = \sqrt[3]{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output. See [link] and [link].
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$. See [link] and [link].
- In tabular form, a function can be represented by rows or columns that relate to input and output values. See [link].
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value. See [link] and [link].
- To solve for a specific function value, we determine the input values that yield the specific output value. See [link].
- An algebraic form of a function can be written from an equation. See [link] and [link].
- Input and output values of a function can be identified from a table. See [link].
- Relating input values to output values on a graph is another way to evaluate a function. See [link].
- A function is one-to-one if each output value corresponds to only one input value. See [link].
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point. See [link].
- The graph of a one-to-one function passes the horizontal line test. See [link].

Section Exercises

Verbal

Exercise:

Problem: What is the difference between a relation and a function?

Solution:

A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate.

Exercise:

Problem: What is the difference between the input and the output of a function?

Exercise:

Problem: Why does the vertical line test tell us whether the graph of a relation represents a function?

Solution:

When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function.

Exercise:

Problem: How can you determine if a relation is a one-to-one function?

Exercise:

Problem: Why does the horizontal line test tell us whether the graph of a function is one-to-one?

Solution:

When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input.

Algebraic

For the following exercises, determine whether the relation represents a function.

Exercise:

Problem: $\{(a,b),\ \ (c,d),\ \ (a,c)\}$

Exercise:

Problem: $\{(a,b),(b,c),(c,c)\}$

Solution:

function

For the following exercises, determine whether the relation represents y as a function of x.

Exercise:

Problem: $5x+2y=10$

Exercise:

Problem: $y=x^2$

Solution:

function

Exercise:

Problem: $x = y^2$

Exercise:

Problem: $3x^2 + y = 14$

Solution:

function

Exercise:

Problem: $2x + y^2 = 6$

Exercise:

Problem: $y = -2x^2 + 40x$

Solution:

function

Exercise:

Problem: $y = \frac{1}{x}$

Exercise:

Problem: $x = \frac{3y+5}{7y-1}$

Solution:

function

Exercise:

Problem: $x = \sqrt{1 - y^2}$

Exercise:

Problem: $y = \frac{3x+5}{7x-1}$

Solution:

function

Exercise:

Problem: $x^2 + y^2 = 9$

Exercise:

Problem: $2xy = 1$

Solution:

function

Exercise:

Problem: $x = y^3$

Exercise:

Problem: $y = x^3$

Solution:

function

Exercise:

Problem: $y = \sqrt{1 - x^2}$

Exercise:

Problem: $x = \pm\sqrt{1 - y}$

Solution:

function

Exercise:

Problem: $y = \pm\sqrt{1 - x}$

Exercise:

Problem: $y^2 = x^2$

Solution:

not a function

Exercise:

Problem: $y^3 = x^2$

For the following exercises, evaluate the function f at the indicated values $f(-3), f(2), f(-a), -f(a), f(a + h)$.

Exercise:

Problem: $f(x) = 2x - 5$

Solution:

$f(-3) = -11; \quad f(2) = -1; \quad f(-a) = -2a - 5; \quad -f(a) = -2a + 5; \quad f(a + h) = 2a + 2h - 5$

Exercise:

Problem: $f(x) = -5x^2 + 2x - 1$

Exercise:

Problem: $f(x) = \sqrt{2-x} + 5$

Solution:

$f(-3) = \sqrt{5} + 5; \quad f(2) = 5; \quad f(-a) = \sqrt{2+a} + 5; \quad -f(a) = -\sqrt{2-a} - 5; \quad f(a+h) = \sqrt{2-a-h} + 5$

Exercise:

Problem: $f(x) = \frac{6x-1}{5x+2}$

Exercise:

Problem: $f(x) = |x-1| - |x+1|$

Solution:

$f(-3) = 2; \quad f(2) = 1 - 3 = -2; \quad f(-a) = |-a-1| - |-a+1|; \quad -f(a) = -|a-1| + |a+1|; \quad f(a$ -

Exercise:

Problem: Given the function $g(x) = 5 - x^2$, evaluate $\frac{g(x+h)-g(x)}{h}$, $h \neq 0$.

Exercise:

Problem: Given the function $g(x) = x^2 + 2x$, evaluate $\frac{g(x)-g(a)}{x-a}$, $x \neq a$.

Solution:

$\frac{g(x)-g(a)}{x-a} = x + a + 2, \; x \neq a$

Exercise:

Problem: Given the function $k(t) = 2t - 1$:

a. Evaluate $k(2)$.
b. Solve $k(t) = 7$.

Exercise:

Problem: Given the function $f(x) = 8 - 3x$:

a. Evaluate $f(-2)$.
b. Solve $f(x) = -1$.

Solution:

a. $f(-2) = 14$; b. $x = 3$

Exercise:

Problem: Given the function $p(c) = c^2 + c$:

a. Evaluate $p(-3)$.
b. Solve $p(c) = 2$.

Exercise:

Problem: Given the function $f(x) = x^2 - 3x$:

a. Evaluate $f(5)$.
b. Solve $f(x) = 4$.

Solution:

a. $f(5) = 10$; b. $x = -1$ or $x = 4$

Exercise:

Problem: Given the function $f(x) = \sqrt{x+2}$:

a. Evaluate $f(7)$.
b. Solve $f(x) = 4$.

Exercise:

Problem: Consider the relationship $3r + 2t = 18$.

a. Write the relationship as a function $r = f(t)$.
b. Evaluate $f(-3)$.
c. Solve $f(t) = 2$.

Solution:

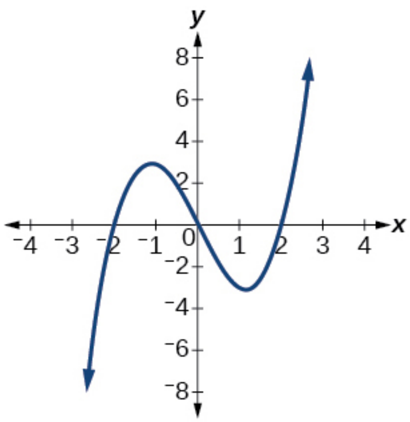
a. $f(t) = 6 - \frac{2}{3}t$; b. $f(-3) = 8$; c. $t = 6$

Graphical

For the following exercises, use the vertical line test to determine which graphs show relations that are functions.

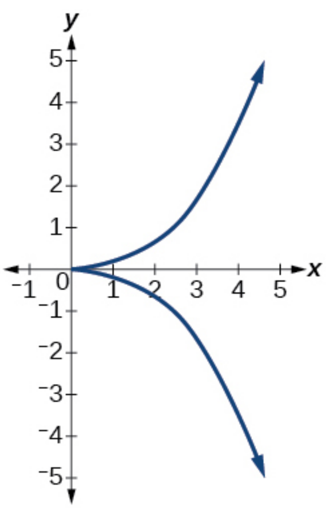
Exercise:

Problem:



Exercise:

Problem:

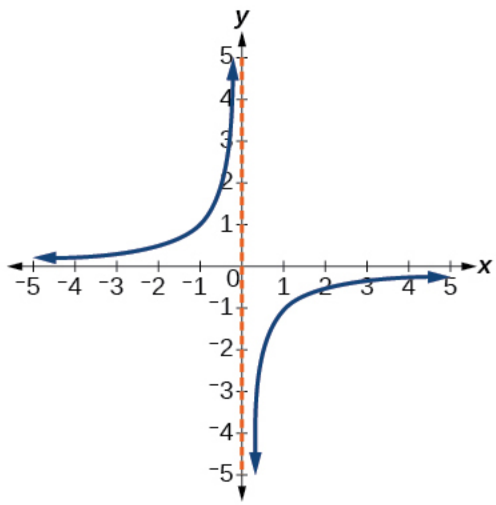


Solution:

not a function

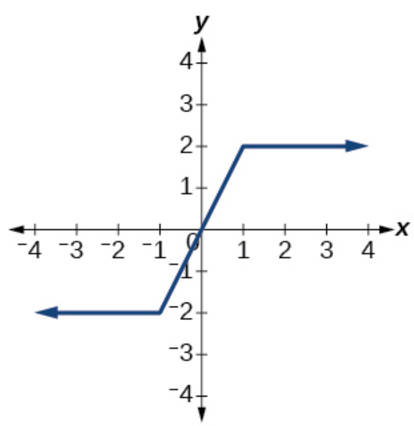
Exercise:

Problem:



Exercise:

Problem:

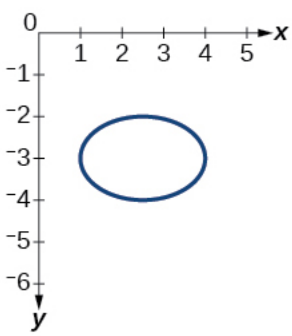


Solution:

function

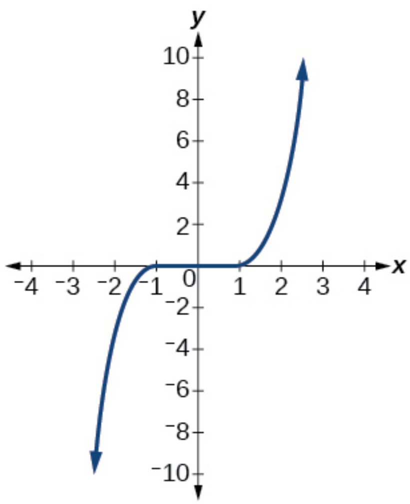
Exercise:

Problem:



Exercise:

Problem:

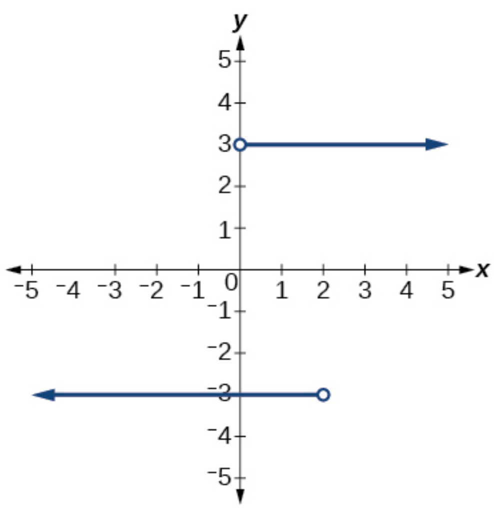


Solution:

function

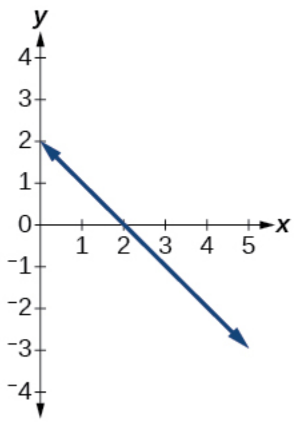
Exercise:

Problem:



Exercise:

Problem:

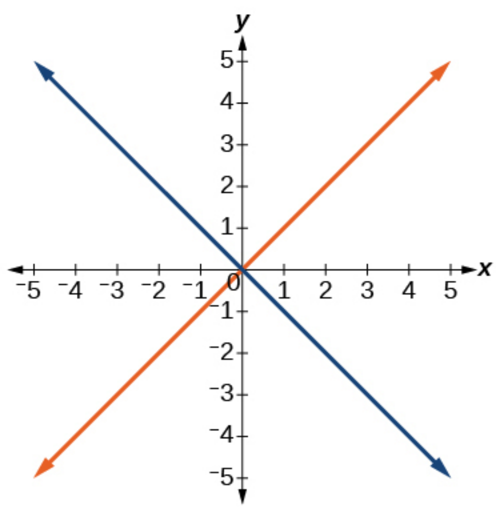


Solution:

function

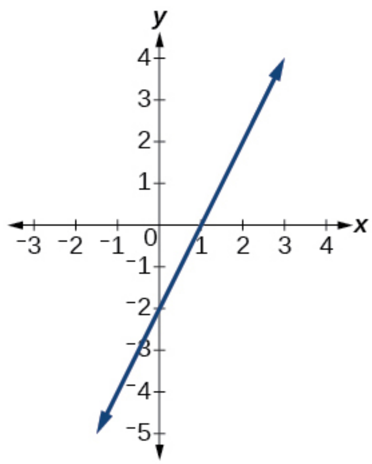
Exercise:

Problem:



Exercise:

Problem:

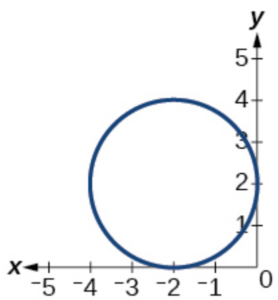


Solution:

function

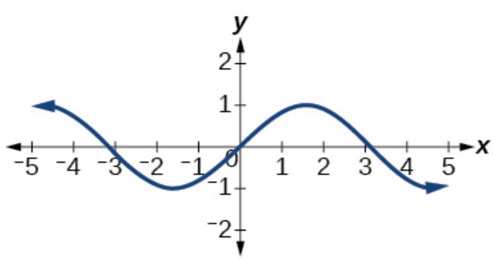
Exercise:

Problem:



Exercise:

Problem:



Solution:

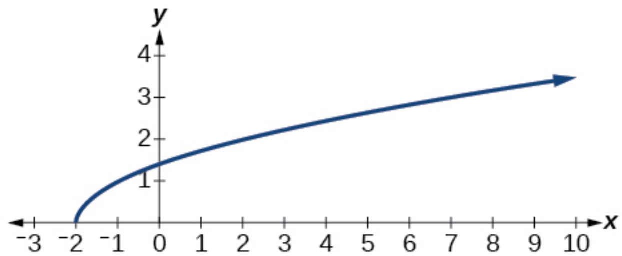
function

Exercise:

Problem: Given the following graph,

- Evaluate $f(-1)$.

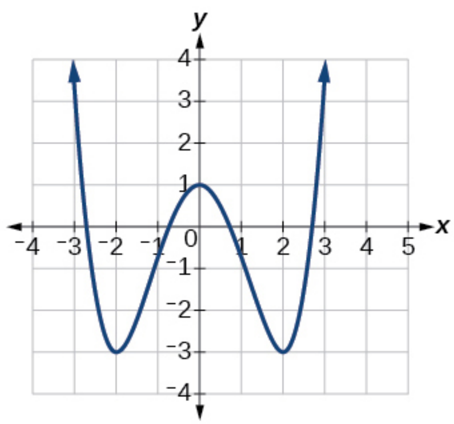
- Solve for $f(x) = 3$.



Exercise:

Problem: Given the following graph,

- Evaluate $f(0)$.
- Solve for $f(x) = -3$.



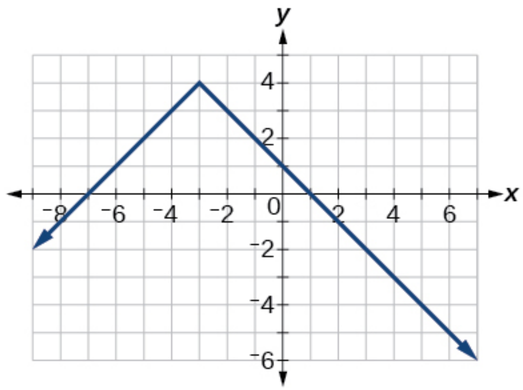
Solution:

a. $f(0) = 1$; b. $f(x) = -3$, $x = -2$ or $x = 2$

Exercise:

Problem: Given the following graph,

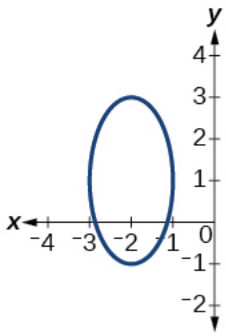
- Evaluate $f(4)$.
- Solve for $f(x) = 1$.



For the following exercises, determine if the given graph is a one-to-one function.

Exercise:

Problem:

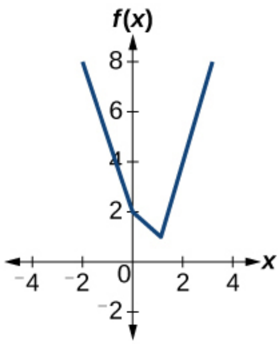


Solution:

not a function so it is also not a one-to-one function

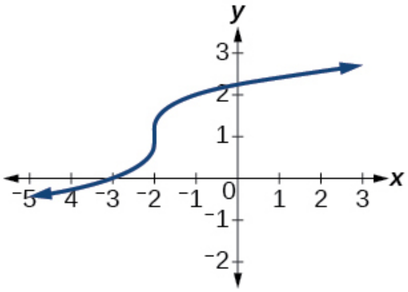
Exercise:

Problem:



Exercise:

Problem:

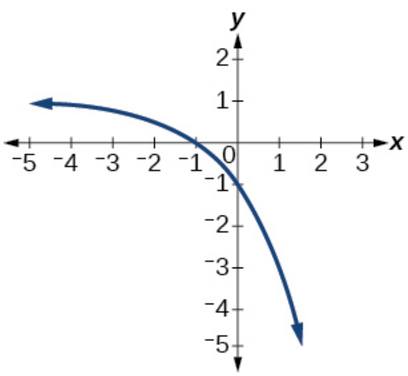


Solution:

one-to- one function

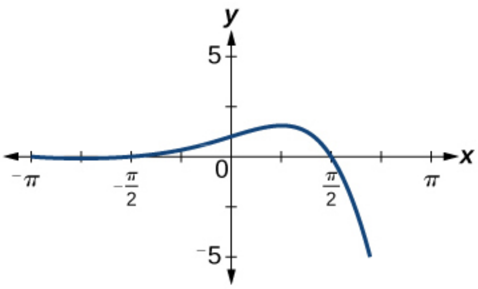
Exercise:

Problem:



Exercise:

Problem:



Solution:

function, but not one-to-one

Numeric

For the following exercises, determine whether the relation represents a function.

Exercise:

Problem: $\{(-1,-1),(-2,-2),(-3,-3)\}$

Exercise:

Problem: $\{(3,4),(4,5),(5,6)\}$

Solution:

function

Exercise:

Problem: $\{(2,5),(7,11),(15,8),(7,9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x.

Exercise:

Problem:

x	5	10	15
y	3	8	14

Solution:

function

Exercise:

Problem:

x	5	10	15
y	3	8	8

Exercise:

Problem:

x	5	10	10

y	3	8	14

Solution:

not a function

For the following exercises, use the function f represented in [link].

x	$f\left(x\right)$
0	74
1	28
2	1
3	53
4	56
5	3
6	36
7	45
8	14
9	47

Exercise:

Problem: Evaluate $f(3)$.

Exercise:

Problem: Solve $f(x)=1$.

Solution:

$f(x)=1,\ x=2$

For the following exercises, evaluate the function f at the values $f\left(-2\right),\ f(-1),\ f(0),\ f(1)$, and $f(2)$.

Exercise:

Problem: $f\left(x\right)=4-2x$

Exercise:

Problem: $f\left(x\right)=8-3x$

Solution:

$f(-2)=14;\quad f(-1)=11;\quad f(0)=8;\quad f(1)=5;\quad f(2)=2$

Exercise:

Problem: $f\left(x\right)=8{x}^{2}-7x+3$

Exercise:

Problem: $f\left(x\right)=3+\sqrt{x+3}$

Solution:

$f(-2)=4;\qquad f(-1)=4.414;\quad f(0)=4.732;\quad f(1)=4.5;\quad f(2)=5.236$

Exercise:

Problem: $f(x)=\frac{x-2}{x+3}$

Exercise:

Problem: $f\left(x\right)={3}^{x}$

Solution:

$f(-2)=\frac{1}{9};\quad f(-1)=\frac{1}{3};\quad f(0)=1;\quad f(1)=3;\quad f(2)=9$

For the following exercises, evaluate the expressions, given functions $f,\ g,$and h :

- $f(x)=3x-2$
- $g(x)=5-{x}^{2}$
- $h(x)=-2{x}^{2}+3x-1$

Exercise:

Problem: $3f\left(1\right)-4g\left(-2\right)$

Exercise:

Problem: $f\left(\frac{7}{3}\right)-h\left(-2\right)$

Solution:

20

Technology

For the following exercises, graph $y={x}^{2}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

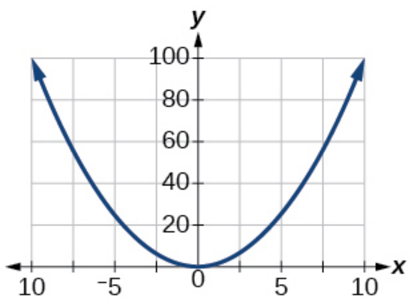
Problem: $[-0.1,\ 0.1]$

Exercise:

Problem:$[-10,\ 10]$

Solution:

$[0,\ 100]$



Exercise:

Problem: $[-100, 100]$

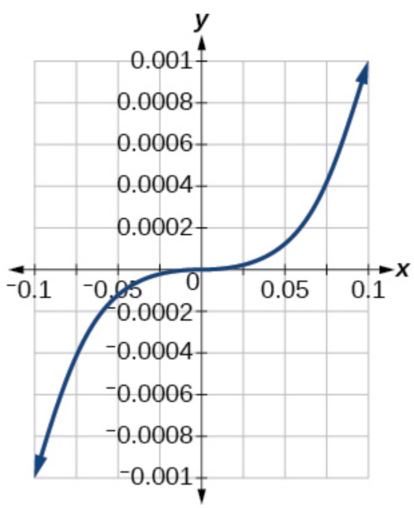
For the following exercises, graph $y = x^3$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

Problem: $[-0.1,\ 0.1]$

Solution:

$[-0.001,\ 0.001]$



Exercise:

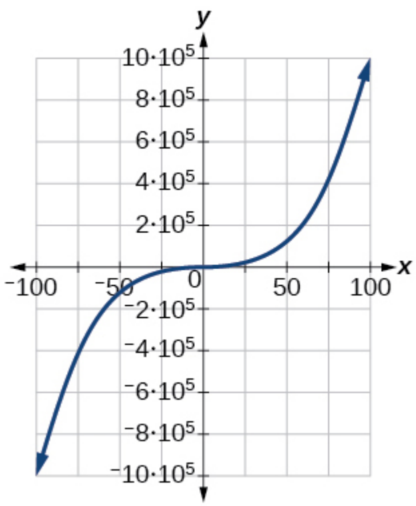
Problem: $[-10,\ 10]$

Exercise:

Problem:$[-100,\ 100]$

Solution:

$[-1,000,000,\ 1{,}000{,}000]$



For the following exercises, graph $y = \sqrt{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

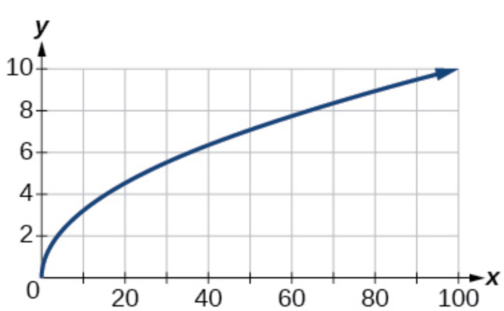
Problem: $[0,\ 0.01]$

Exercise:

Problem: $[0,\ 100]$

Solution:

$[0,\ 10]$



Exercise:

Problem: $[0,\ 10{,}000]$

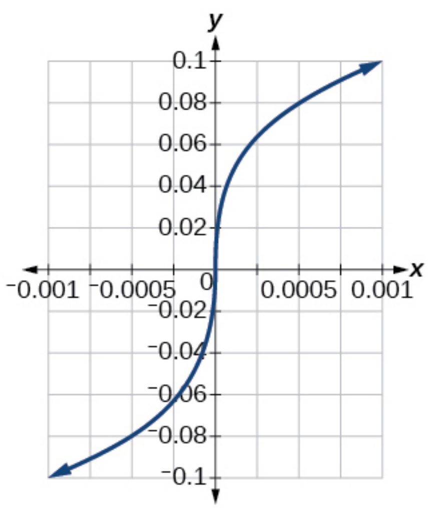
For the following exercises, graph $y = \sqrt[3]{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

Problem: $[-0.001, 0.001]$

Solution:

$[-0.1, 0.1]$



Exercise:

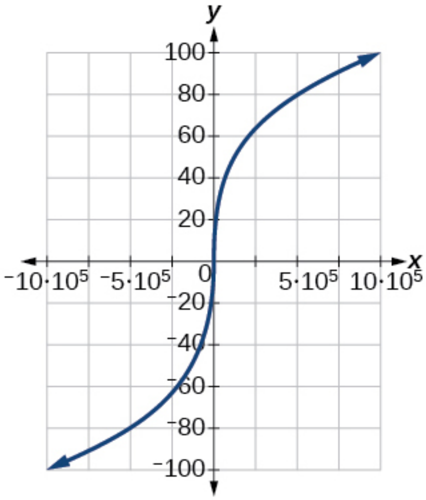
Problem: $[-1000, 1000]$

Exercise:

Problem: $[-1{,}000{,}000, 1{,}000{,}000]$

Solution:

$[-100,\ 100]$



Real-World Applications

Exercise:

Problem:

The amount of garbage, $G,$ produced by a city with population p is given by $G = f\left(p\right).$ G is measured in tons per week, and p is measured in thousands of people.

a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function $f.$
b. Explain the meaning of the statement $f\left(5\right) = 2.$

Exercise:

Problem:

The number of cubic yards of dirt, $D,$ needed to cover a garden with area a square feet is given by $D = g\left(a\right).$

a. A garden with area 5000 ft^2 requires 50 yd^3 of dirt. Express this information in terms of the function $g.$
b. Explain the meaning of the statement $g\left(100\right) = 1.$

Solution:

a. $g(5000) = 50$; b. The number of cubic yards of dirt required for a garden of 100 square feet is 1.

Exercise:

Problem:

Let $f\left(t\right)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:

a. $f\left(5\right) = 30$
b. $f\left(10\right) = 40$

Exercise:

Problem:

Let $h\left(t\right)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:

a. $h\left(1\right)=200$
b. $h\left(2\right)=350$

Solution:

a. The height of a rocket above ground after 1 second is 200 ft. b. the height of a rocket above ground after 2 seconds is 350 ft.

Exercise:

Problem: Show that the function $f\left(x\right)=3{\left(x-5\right)}^{2}+7$ is not one-to-one.

Glossary

dependent variable
: an output variable

domain
: the set of all possible input values for a relation

function
: a relation in which each input value yields a unique output value

horizontal line test
: a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
: an input variable

input
: each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
: a function for which each value of the output is associated with a unique input value

output
: each object or value in the range that is produced when an input value is entered into a function

range
: the set of output values that result from the input values in a relation

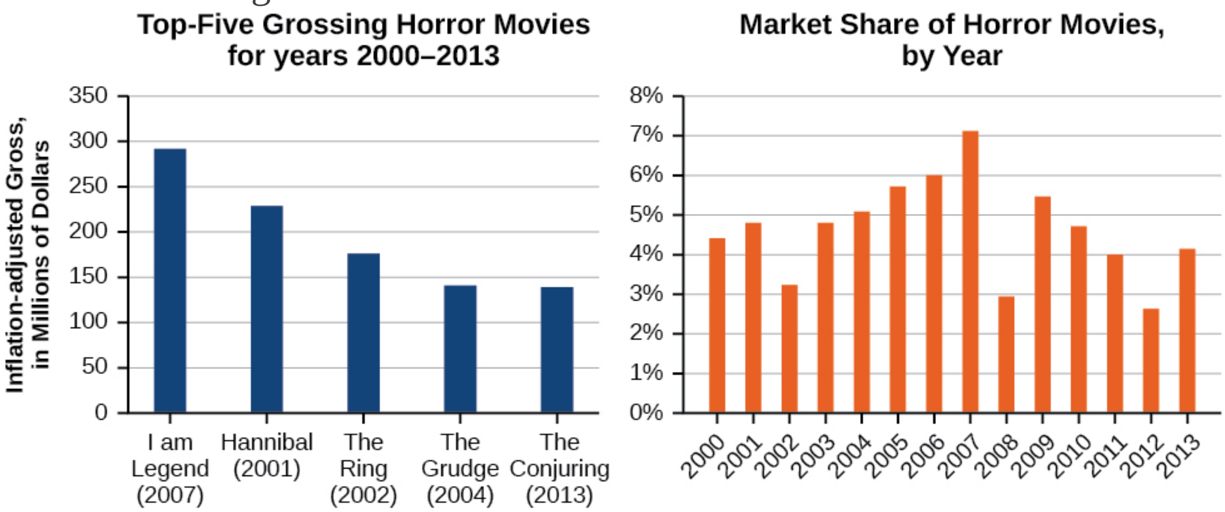
relation
: a set of ordered pairs

vertical line test
: a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

Domain and Range
In this section, you will:

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you’re in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—*I am Legend*, *Hannibal*, *The Ring*, *The Grudge*, and *The Conjuring*. [link] shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

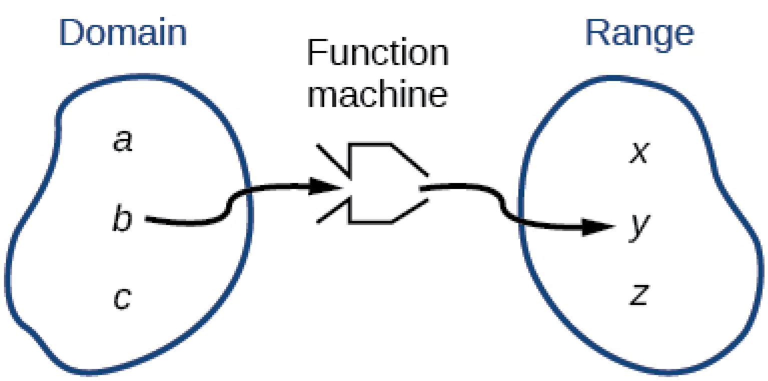


Based on data compiled by www.the-numbers.com.[footnote]
The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of domain and range. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a “holding area” that contains “raw materials” for a “function machine” and the range as another “holding area” for the machine’s products. See [link].



We can write the domain and range in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0,\ 100].$ We will discuss interval notation in greater detail later.

Let’s turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves

remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint is included, called inclusive.

See [link] for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Example:
Exercise:

Problem:
Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function:
$\{(2,\ 10), (3,\ 10), (4,\ 20), (5,\ 30), (6,\ 40)\}$.

Solution:

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

Equation:

$$\{2, 3, 4, 5, 6\}$$

Note:
Exercise:

Problem: Find the domain of the function:

$\{(-5, 4), (0, 0), (5, -4), (10, -8), (15, -12)\}$

Solution:

$\{-5,\ 0,\ 5,\ 10,\ 15\}$

Note:
Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example:
Exercise:

Problem:
Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution:

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

Note:
Exercise:

Problem: Find the domain of the function: $f(x) = 5 - x + x^3$.

Solution:

$(-\infty, \infty)$

Note:
Given a function written in an equation form that includes a fraction, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.

3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example:
Exercise:

Problem:
Finding the Domain of a Function Involving a Denominator

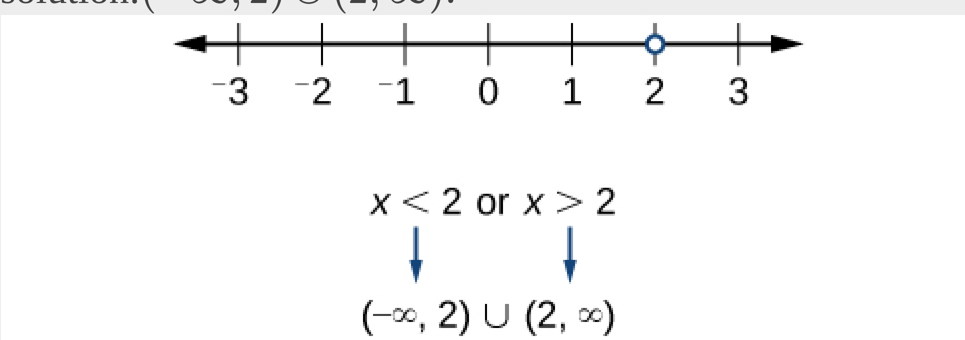
Find the domain of the function $f(x) = \frac{x+1}{2-x}$.

Solution:

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.
Equation:

$$\begin{aligned} 2 - x &= 0 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty, 2) \cup (2, \infty)$.



In interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Note:
Exercise:

Problem: Find the domain of the function: $f(x) = \frac{1+4x}{2x-1}$.

Solution:

$\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

Note:
Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example:
Exercise:

Problem:
Finding the Domain of a Function with an Even Root

Find the domain of the function $f(x) = \sqrt{7 - x}$.

Solution:

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.
Equation:

$$\begin{aligned} 7 - x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to $7,$ or $(-\infty, 7]$.

Note:
Exercise:

Problem: Find the domain of the function $f(x) = \sqrt{5 + 2x}$.

Solution:

$\left[-\frac{5}{2}, \infty\right)$

Note:
Can there be functions in which the domain and range do not intersect at all?
Yes. For example, the function $f(x) = -\frac{1}{\sqrt{x}}$ *has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be*

completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in **set-builder notation**. For example, $\{x|10 \le x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as “the set of,” and the vertical bar | is read as “such that,” so we would read $\{x|10 \le x < 30\}$ as “the set of *x*-values such that 10 is less than or equal to x, and x is less than 30.”

[link] compares inequality notation, set-builder notation, and interval notation.

	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \le 10$	$\{h \mid 5 < h \le 10\}$	$(5, 10]$
	$5 \le h < 10$	$\{h \mid 5 \le h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \ge 10$	$\{h \mid h \ge 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one *or* the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

Equation:

$$\{x| \quad |x| \ge 3\} = (-\infty, -3] \cup [3, \infty)$$

Note:

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|\text{ statement about }x\}$ which is read as, “the set of all x such that the statement about x is true.” For example,

Equation:

$$\{x|4<x\leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

Equation:

$$(4,12]$$

Note:

Given a line graph, describe the set of values using interval notation.

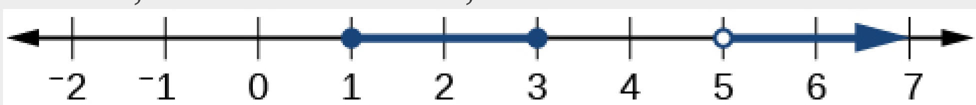
1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

Example:

Exercise:

Problem:
Describing Sets on the Real-Number Line

Describe the intervals of values shown in [link] using inequality notation, set-builder notation, and interval notation.



Solution:

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

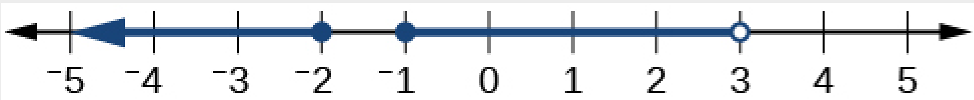
Inequality	$1 \leq x \leq 3 \text{ or } x > 5$
Set-builder notation	$\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$
Interval notation	$[1, 3] \cup (5, \infty)$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Note:
Exercise:

Problem: Given [link], specify the graphed set in

a. words
b. set-builder notation
c. interval notation

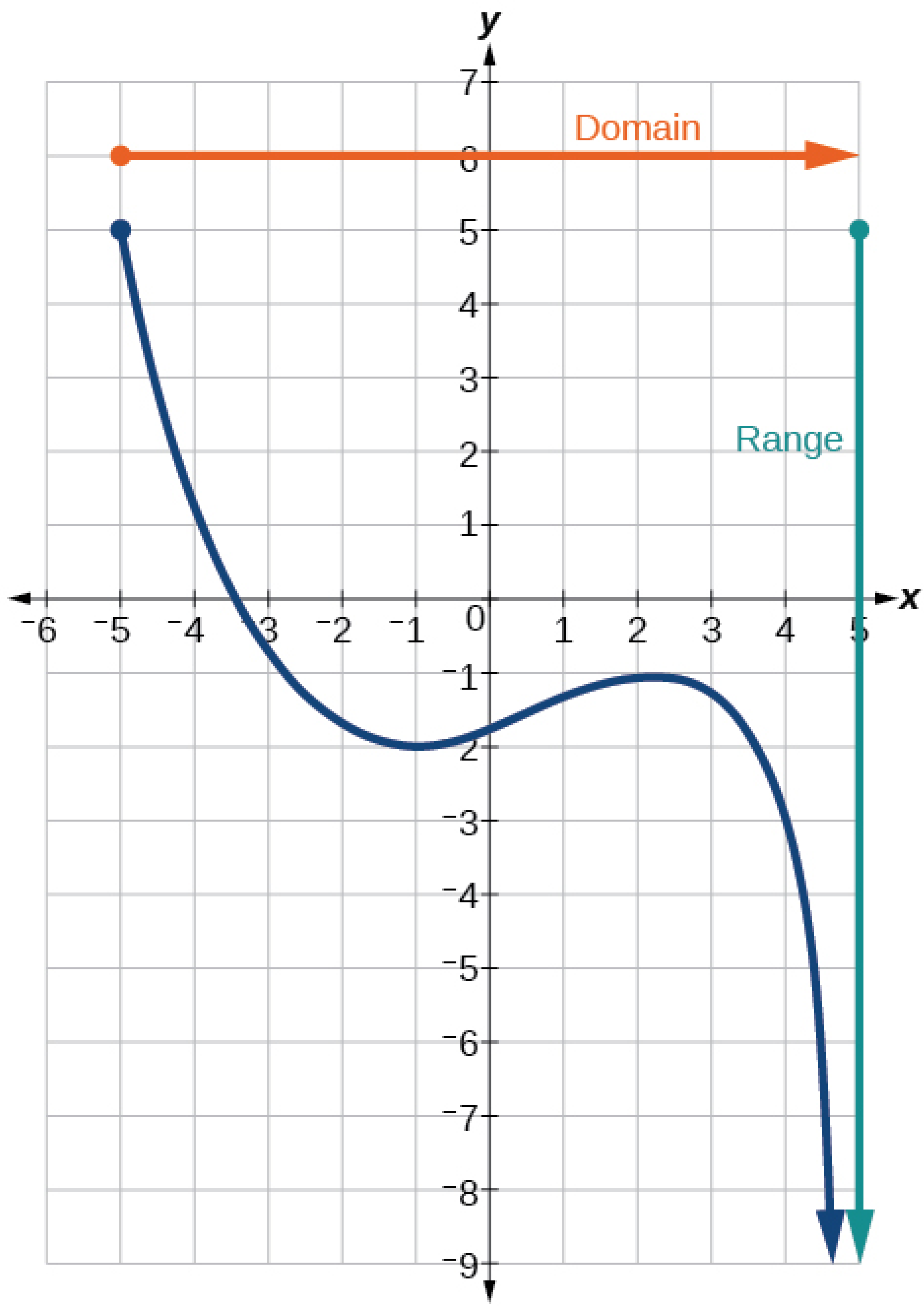


Solution:

a. values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;
b. $\{x|x \leq -2 \text{ or } -1 \leq x < 3\}$;
c. $(-\infty, -2] \cup [-1, 3)$

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the *x*-axis. The range is the set of possible output values, which are shown on the *y*-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See [link].

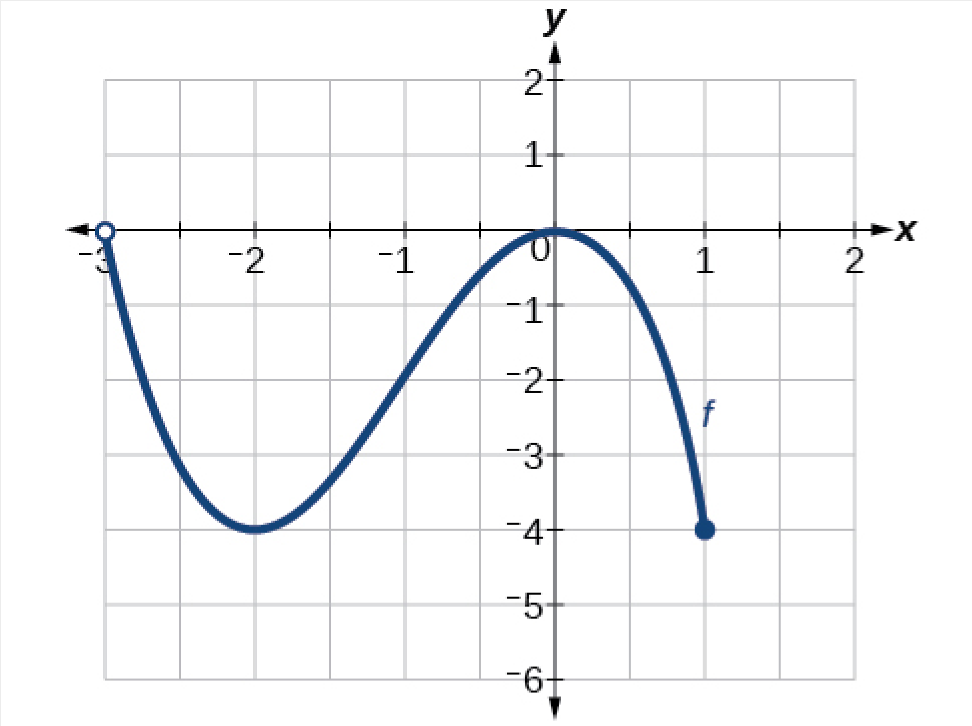


We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $\left[-5, \infty\right)$. The vertical extent of the graph is all range values 5 and below, so the range is $\left(-\infty, 5\right]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example:
Exercise:

Problem:
Finding Domain and Range from a Graph

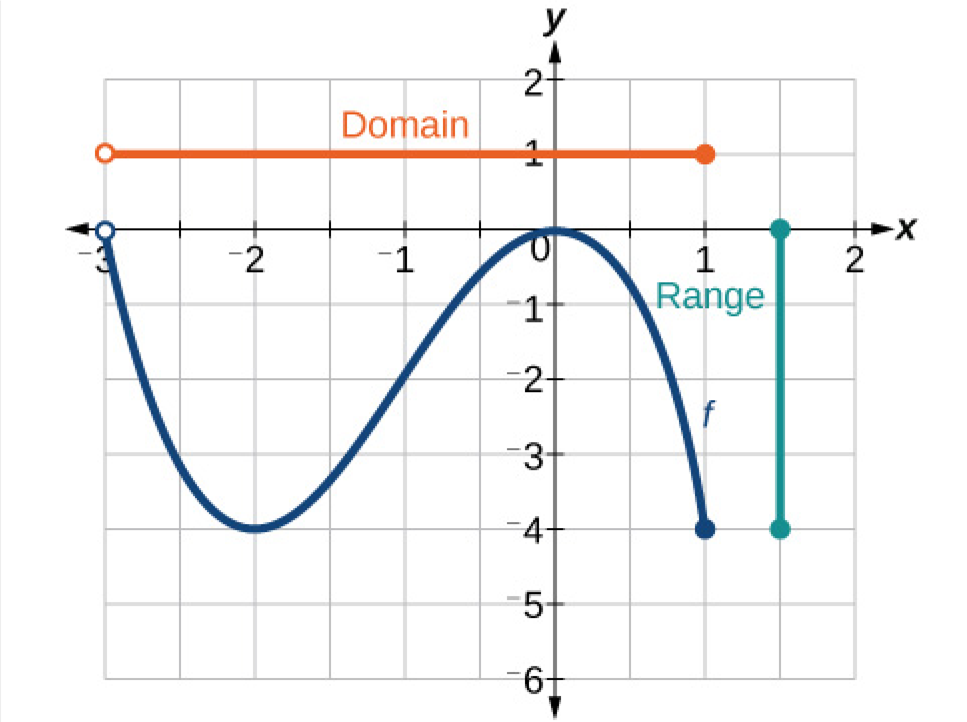
Find the domain and range of the function f whose graph is shown in [link].



Solution:

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

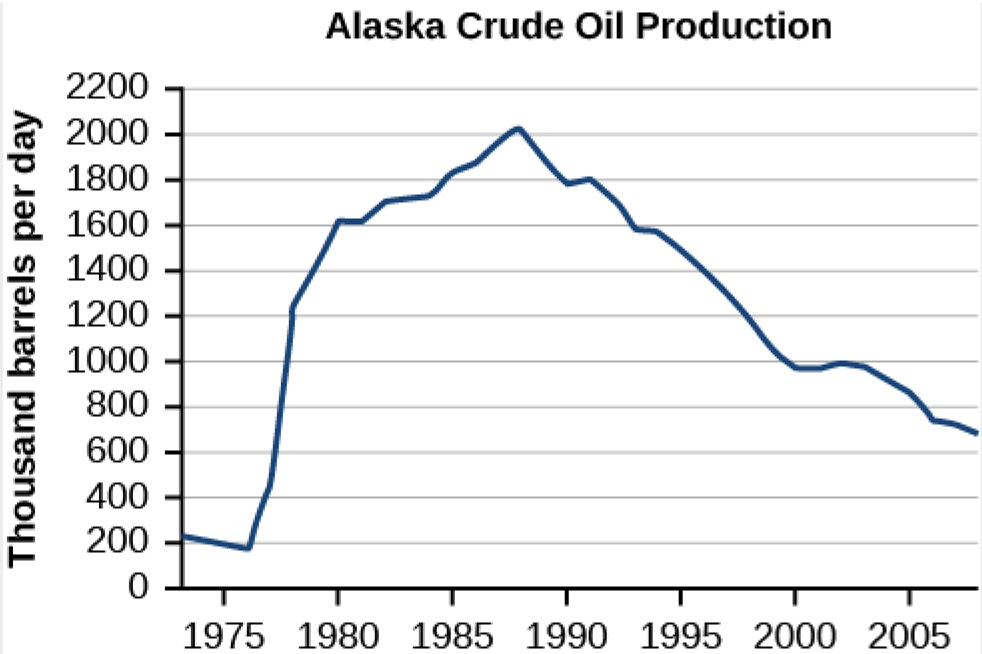
The vertical extent of the graph is 0 to –4, so the range is $[-4, 0]$. See [link].



Example:
Exercise:

Problem:
Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in [link].



(credit: modification of work by the U.S. Energy Information Administration)[footnote] http://www.eia.gov/dnav/pet/hist/LeafHandler.ashx?n=PET&s=MCRFPAK2&f=A.

Solution:

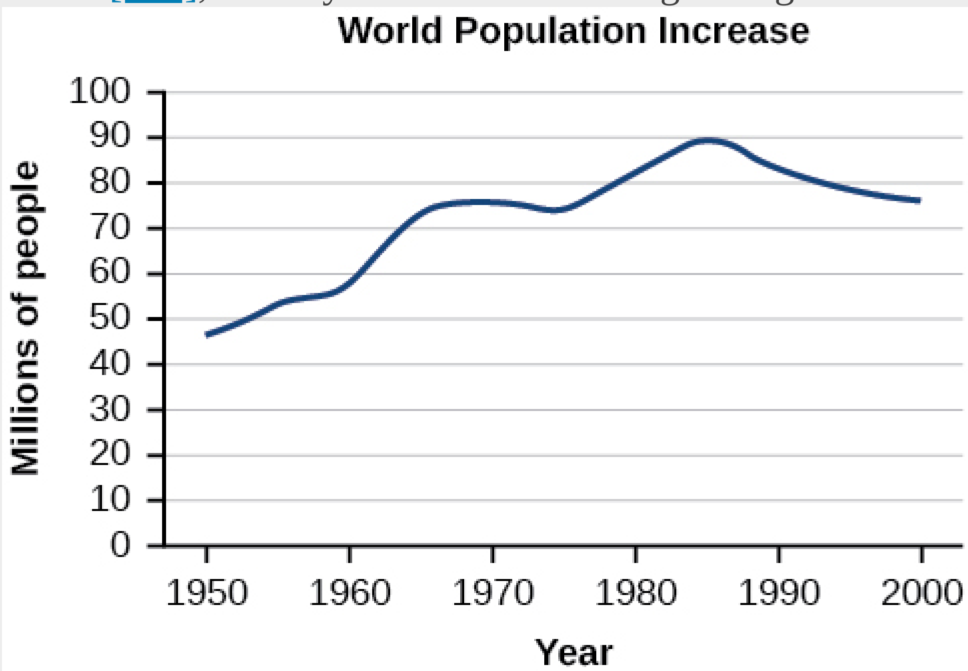
The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is [1973, 2008], and the range is about [180, 2010]. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Note:
Exercise:

Problem:

Given [link], identify the domain and range using interval notation.



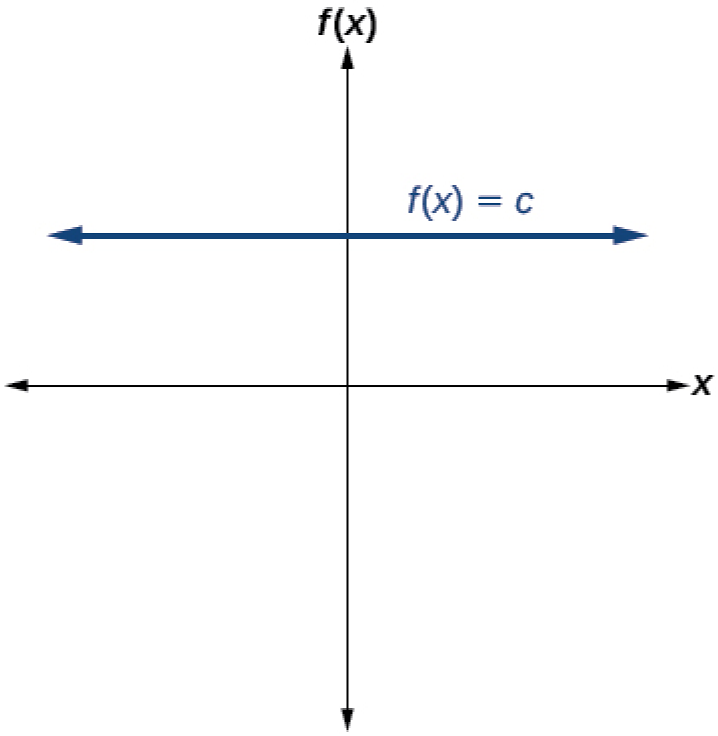
Solution:

domain =[1950,2002] range = [47,000,000,89,000,000]

Note:
Can a function's domain and range be the same?
Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

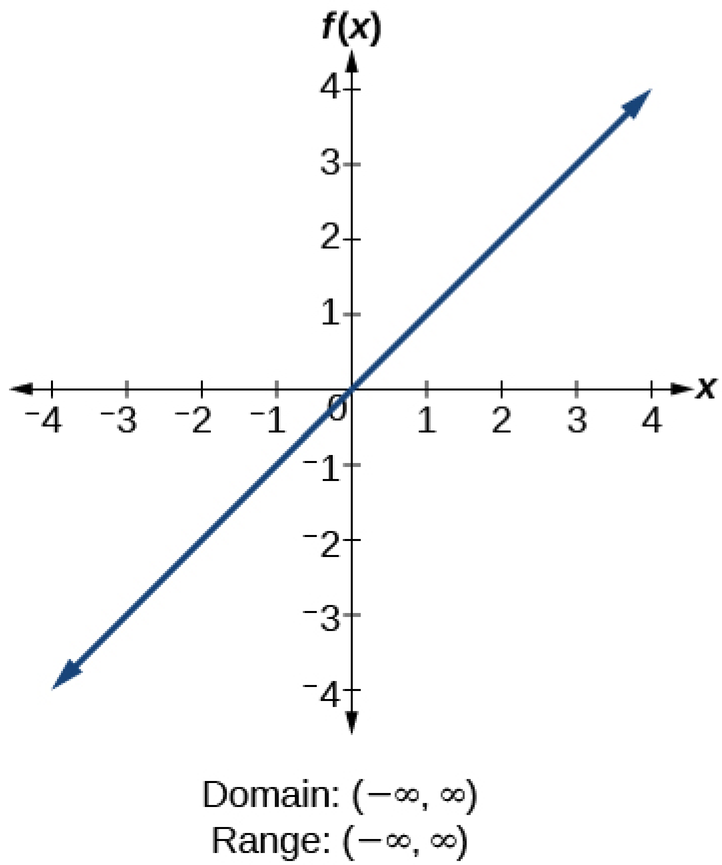
Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.

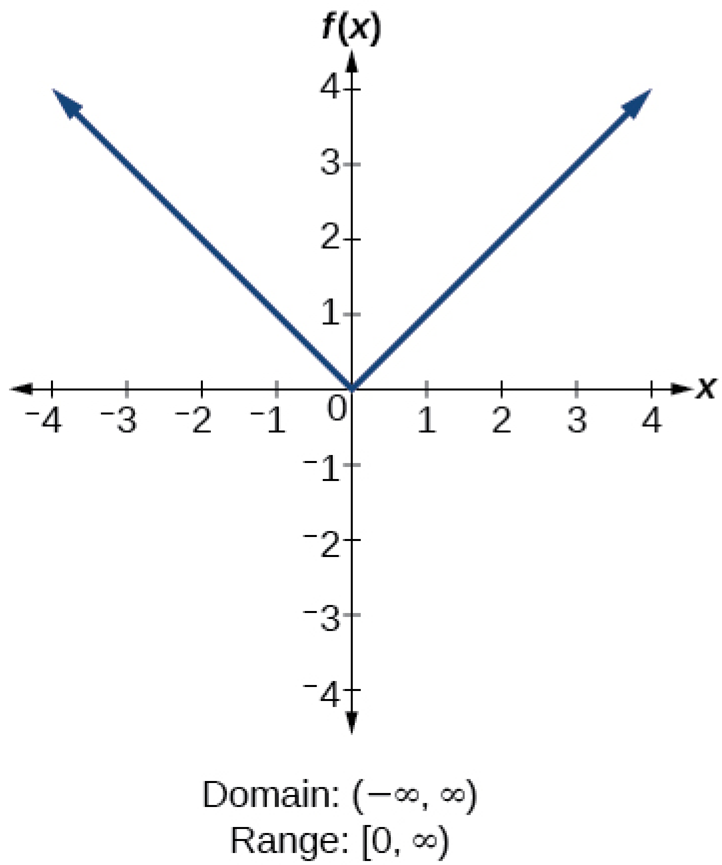


Domain: $(-\infty, \infty)$
Range: $[c,c]$

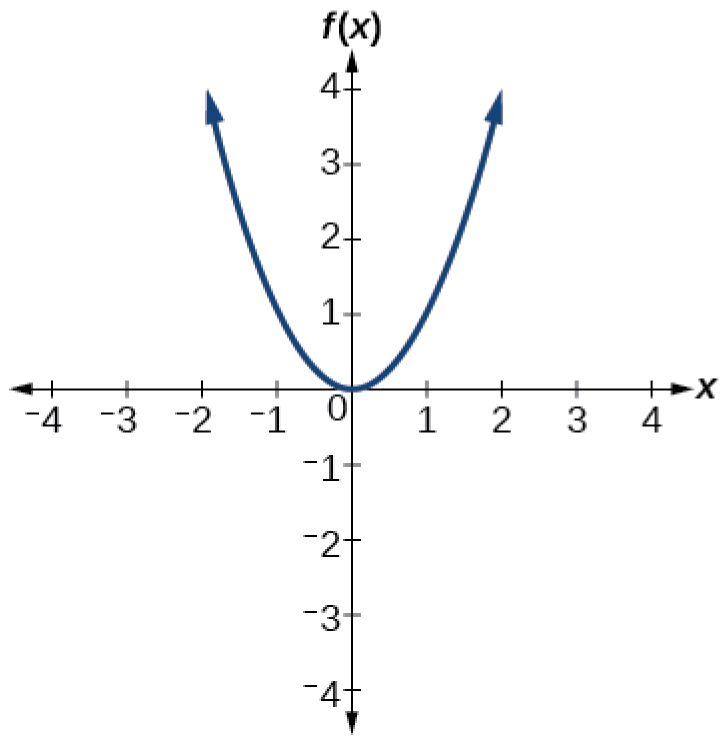
For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.



For the **identity function** $f(x) = x,$ there is no restriction on $x.$ Both the domain and range are the set of all real numbers.

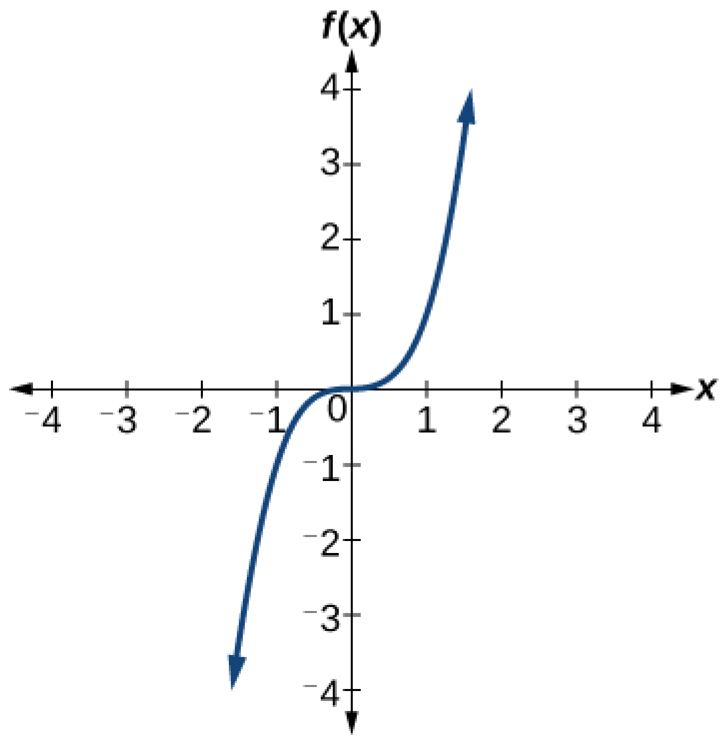


For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.



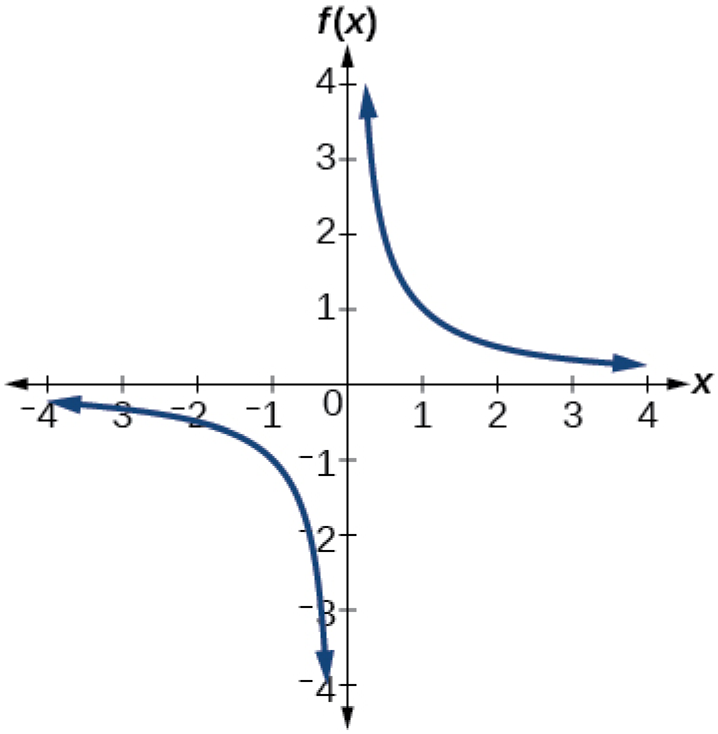
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.



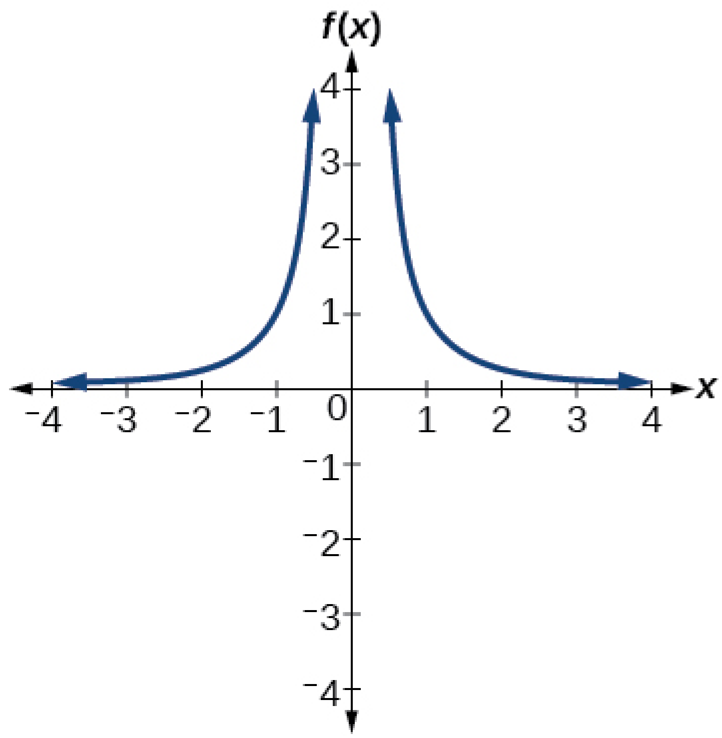
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.



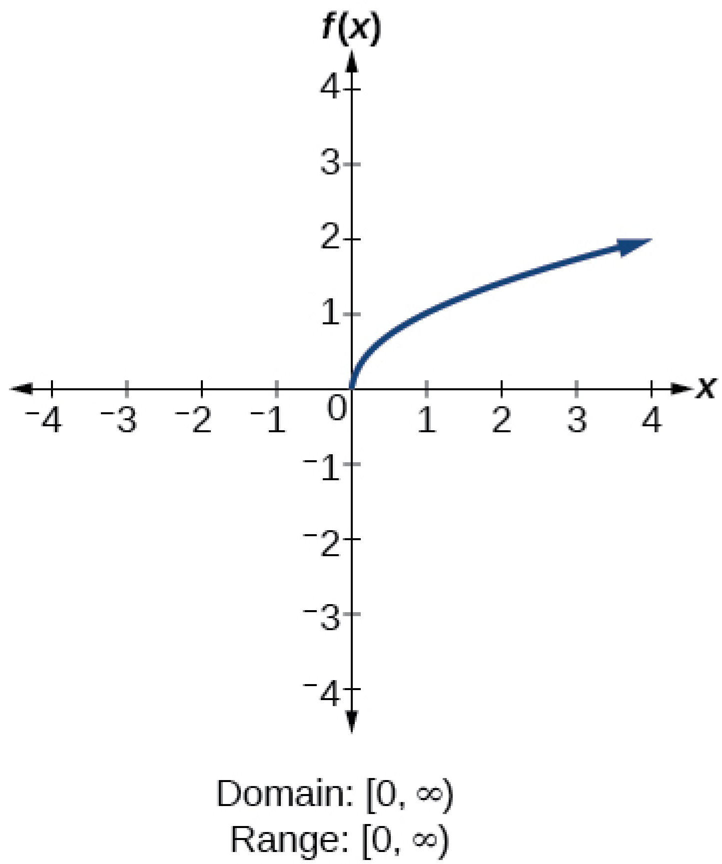
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x|\ x \neq 0\}$,the set of all real numbers that are not zero.

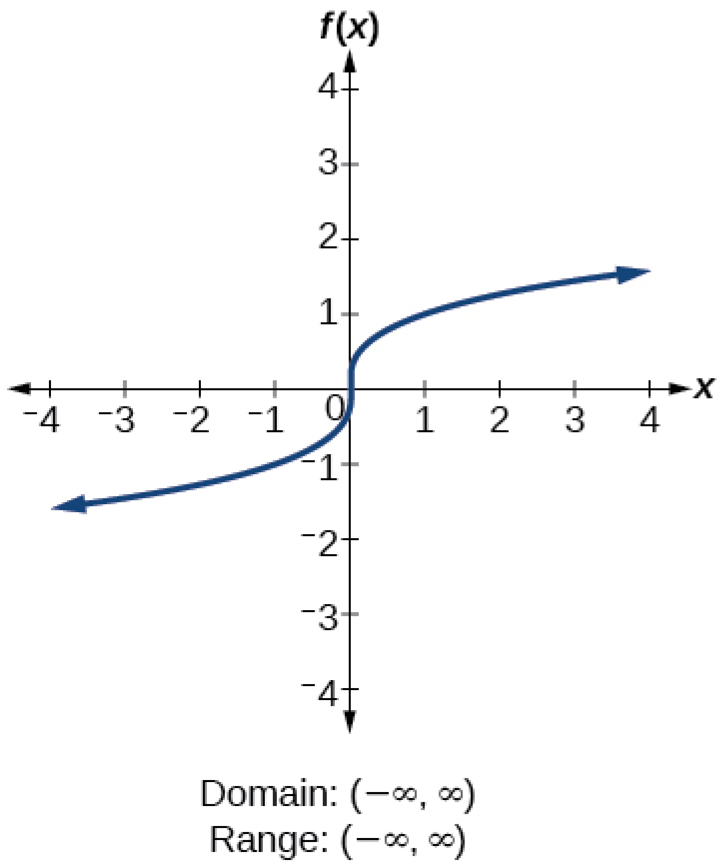


Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.



For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.



For the **cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

Note:
Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.

3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Example:
Exercise:

Problem:
Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution:

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example:
Exercise:

Problem:
Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution:

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example:
Exercise:

Problem:
Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution:

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.
Equation:

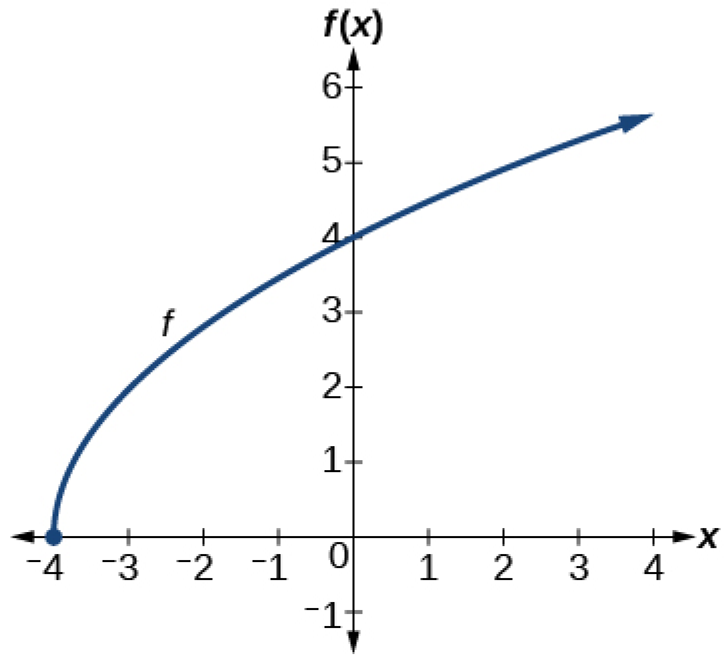
$$x + 4 \geq 0 \text{ when } x \geq -4$$

The domain of $f\left(x\right)$ is $\left[-4, \infty\right)$.

We then find the range. We know that $f\left(-4\right) = 0,$ and the function value increases as x increases without any upper limit. We conclude that the range of f is $\left[0, \infty\right)$.

Analysis

[link] represents the function f.



Note:
Exercise:

Problem: Find the domain and range of $f\left(x\right)=-\sqrt{2-x}.$

Solution:

domain: $\left(-\infty ,2\right];$ range: $\left(-\infty ,0\right]$

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x)=|x|.$ With a domain of all real numbers and a range of values greater than or equal to 0, absolute value can be defined as the magnitude, or modulus, of a real number value

regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.
Equation:

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.
Equation:

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a piecewise function. A **piecewise function** is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to $10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10{,}000$ and $\$1000 + 0.2(S - \$10{,}000)$ if $S > \$10{,}000$.

Note:
Piecewise Function
A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

Equation:

$$f(x) = \begin{array}{ll} \text{formula 1} & \text{if } x \text{ is in domain 1} \\ \text{formula 2} & \text{if } x \text{ is in domain 2} \\ \text{formula 3} & \text{if } x \text{ is in domain 3} \end{array}$$

In piecewise notation, the absolute value function is

Equation:

$$|x| = \begin{cases} x & \text{if } \ x \geq 0 \\ -x & \text{if } \ x < 0 \end{cases}$$

Note:

Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Example:

Exercise:

Problem:

Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a function relating the number of people, n, to the cost, C.

Solution:

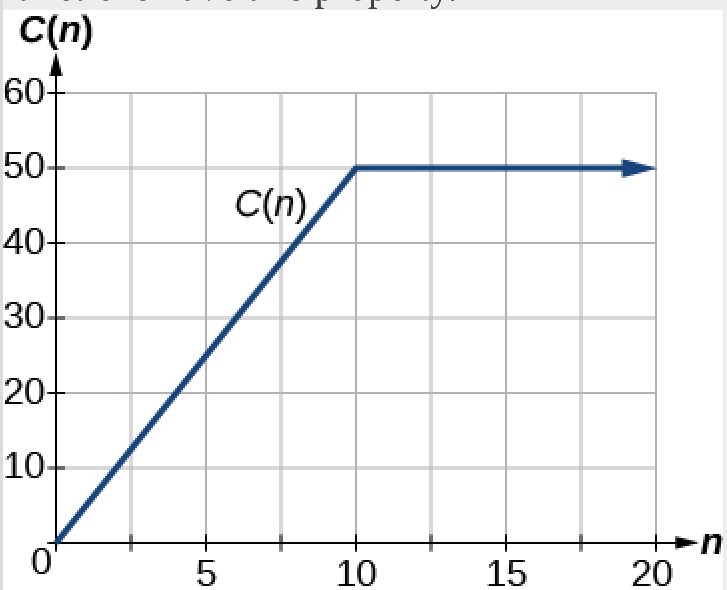
Two different formulas will be needed. For n-values under 10, $C = 5n.$ For values of n that are 10 or greater, $C = 50.$

Equation:

$$C(n) = \begin{cases} 5n & \text{if} \quad 0 < n < 10 \\ 50 & \text{if} \quad n \geq 10 \end{cases}$$

Analysis

The function is represented in [link]. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10,$ but not all piecewise functions have this property.



Example:
Exercise:

Problem:
Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, $C,$ in dollars for g gigabytes of data transfer.

Equation:

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g-2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Solution:

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

Equation:

$$C(1.5) = \$25$$

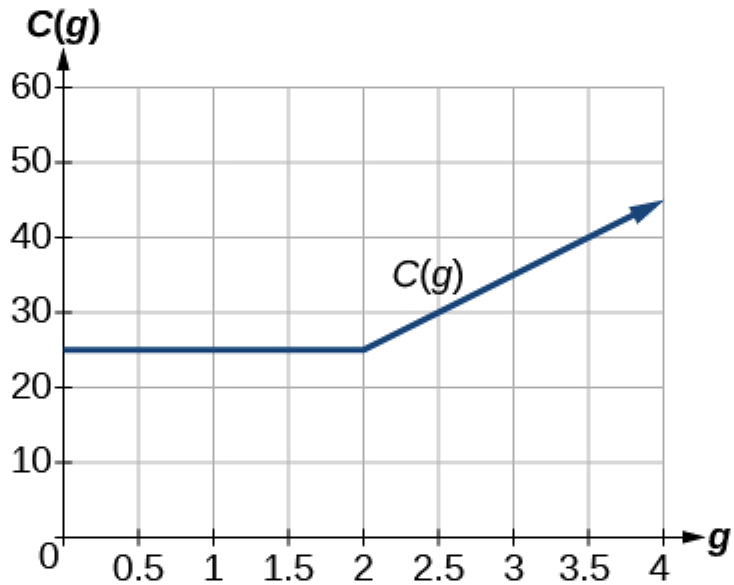
To find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we use the second formula.

Equation:

$$C(4) = 25 + 10(4-2) = \$45$$

Analysis

The function is represented in [link]. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.



Note:
Given a piecewise function, sketch a graph.

1. Indicate on the *x*-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example:
Exercise:

Problem:
Graphing a Piecewise Function

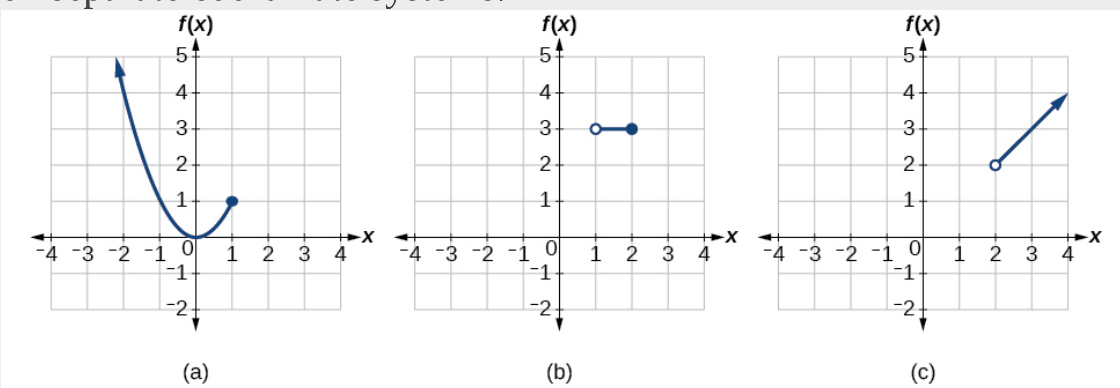
Sketch a graph of the function.
Equation:

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \le 1 \\ 3 & \text{if} \quad 1 < x \le 2 \\ x & \text{if} \quad x > 2 \end{cases}$$

Solution:

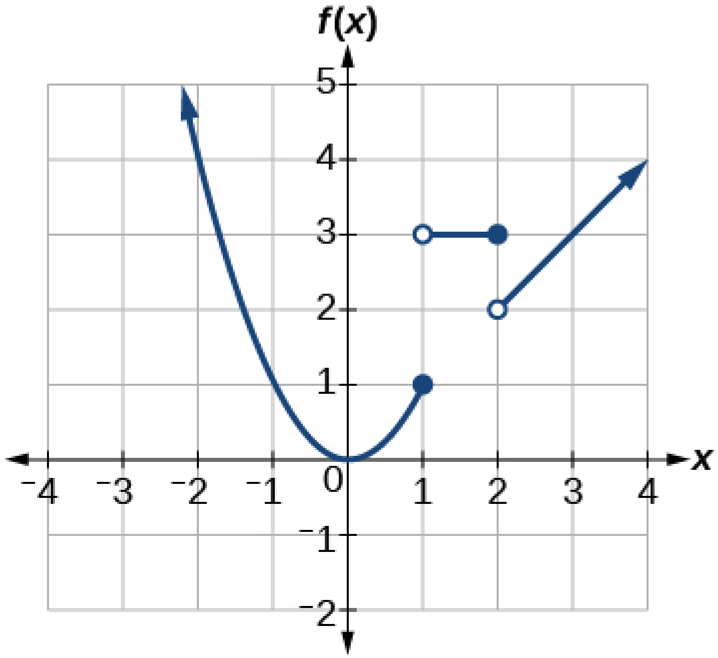
Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

[link] shows the three components of the piecewise function graphed on separate coordinate systems.



(a) $f(x) = x^2$ if $x \le 1$; (b) $f(x) = 3$ if $1 < x \le 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See [link].



Analysis

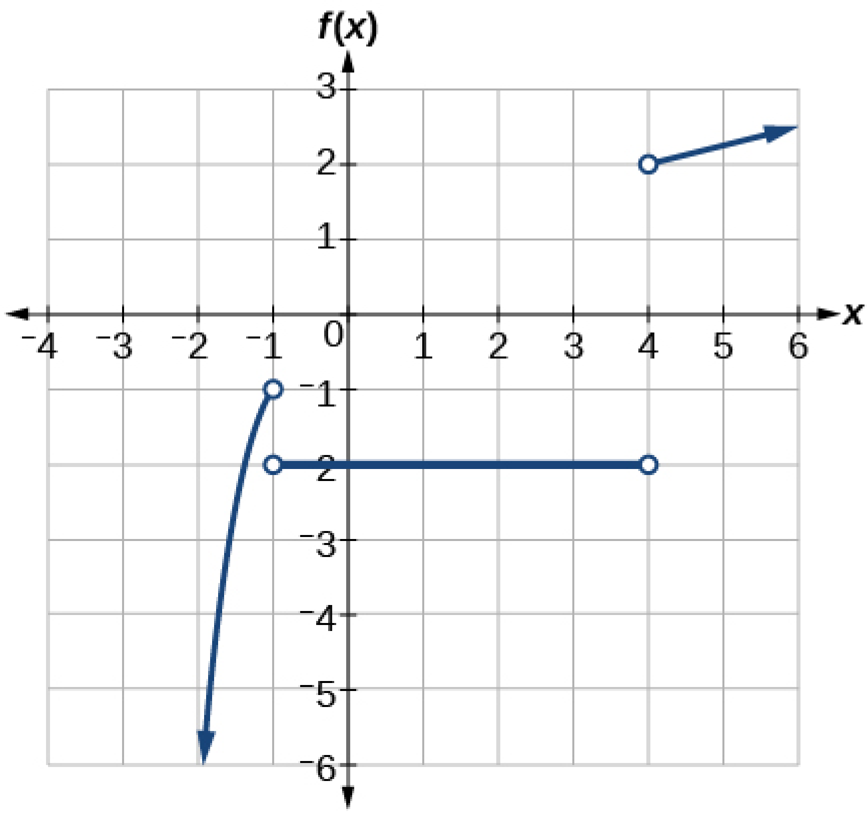
Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2,\ 3)$ are.

Note:
Exercise:

Problem: Graph the following piecewise function.
Equation:

$$f(x) = \begin{array}{ccc} x^3 & \text{if} & x < -1 \\ -2 & \text{if} & -1 < x < 4 \\ \sqrt{x} & \text{if} & x > 4 \end{array}$$

Solution:



Note:
Can more than one formula from a piecewise function be applied to a value in the domain?
No. Each value corresponds to one equation in a piecewise formula.

Note:
Access these online resources for additional instruction and practice with domain and range.

- Domain and Range of Square Root Functions
- Determining Domain and Range
- Find Domain and Range Given the Graph
- Find Domain and Range Given a Table

- [Find Domain and Range Given Points on a Coordinate Plane]

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs. See [link].
- The domain of a function can also be determined by identifying the input values of a function written as an equation. See [link], [link], and [link].
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation. See [link].
- For many functions, the domain and range can be determined from a graph. See [link] and [link].
- An understanding of toolkit functions can be used to find the domain and range of related functions. See [link], [link], and [link].
- A piecewise function is described by more than one formula. See [link] and [link].
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain. See [link].

Section Exercises

Verbal

Exercise:

Problem: Why does the domain differ for different functions?

Solution:

The domain of a function depends upon what values of the independent variable make the function undefined or imaginary.

Exercise:

Problem:

How do we determine the domain of a function defined by an equation?

Exercise:

Problem:

Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.

Solution:

There is no restriction on x for $f(x) = \sqrt[3]{x}$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty, \infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x-values are restricted for $f(x) = \sqrt{x}$ to nonnegative numbers and the domain is $[0, \infty)$.

Exercise:

Problem:

When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?

Exercise:

Problem: How do you graph a piecewise function?

Solution:

Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x-axis and y-axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints

with an open circle. Use an arrow to indicate $-\infty$ or ∞. Combine the graphs to find the graph of the piecewise function.

Algebraic

For the following exercises, find the domain of each function using interval notation.

Exercise:

Problem: $f(x) = -2x(x-1)(x-2)$

Exercise:

Problem: $f(x) = 5 - 2x^2$

Solution:

$(-\infty, \infty)$

Exercise:

Problem: $f\left(x\right) = 3\sqrt{x-2}$

Exercise:

Problem: $f\left(x\right) = 3 - \sqrt{6-2x}$

Solution:

$(-\infty, 3]$

Exercise:

Problem: $f(x) = \sqrt{4-3x}$

Exercise:

Problem: $f(x) = \sqrt{x^2 + 4}$

Solution:

$(-\infty, \infty)$

Exercise:

Problem: $f(x) = \sqrt[3]{1 - 2x}$

Exercise:

Problem: $f(x) = \sqrt[3]{x - 1}$

Solution:

$(-\infty, \infty)$

Exercise:

Problem: $f(x) = \frac{9}{x-6}$

Exercise:

Problem: $f\left(x\right) = \frac{3x+1}{4x+2}$

Solution:

$(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

Exercise:

Problem: $f\left(x\right) = \frac{\sqrt{x+4}}{x-4}$

Exercise:

Problem: $f(x) = \frac{x-3}{x^2+9x-22}$

Solution:

$(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$

Exercise:

Problem: $f(x) = \frac{1}{x^2-x-6}$

Exercise:

Problem: $f(x) = \frac{2x^3-250}{x^2-2x-15}$

Solution:

$(-\infty, -3) \cup (-3, 5) \cup (5, \infty)$

Exercise:

Problem: $\frac{5}{\sqrt{x-3}}$

Exercise:

Problem: $\frac{2x+1}{\sqrt{5-x}}$

Solution:

$(-\infty, 5)$

Exercise:

Problem: $f(x) = \frac{\sqrt{x-4}}{\sqrt{x-6}}$

Exercise:

Problem: $f(x) = \frac{\sqrt{x-6}}{\sqrt{x-4}}$

Solution:

$[6, \infty)$

Exercise:

Problem: $f(x) = \frac{x}{x}$

Exercise:

Problem: $f(x) = \frac{x^2-9x}{x^2-81}$

Solution:

$(-\infty, -9) \cup (-9, 9) \cup (9, \infty)$

Exercise:

Problem: Find the domain of the function $f(x) = \sqrt{2x^3 - 50x}$ by:

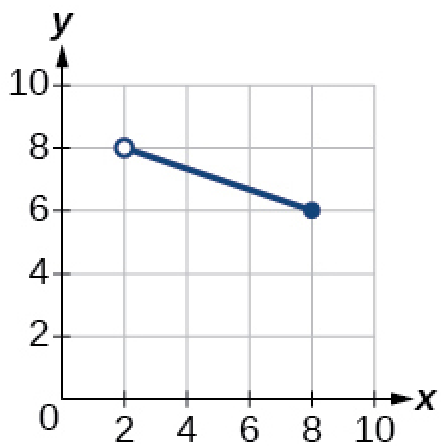
a. using algebra.
b. graphing the function in the radicand and determining intervals on the *x*-axis for which the radicand is nonnegative.

Graphical

For the following exercises, write the domain and range of each function using interval notation.

Exercise:

Problem:

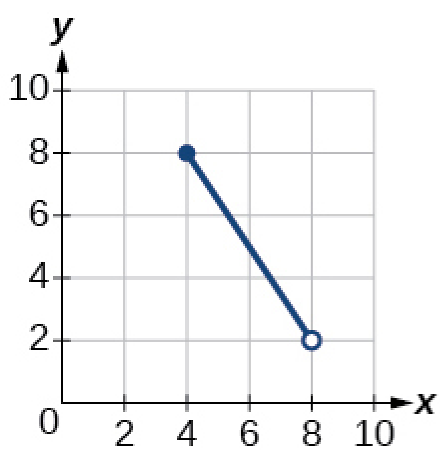


Solution:

domain: $(2, 8]$, range $[6, 8)$

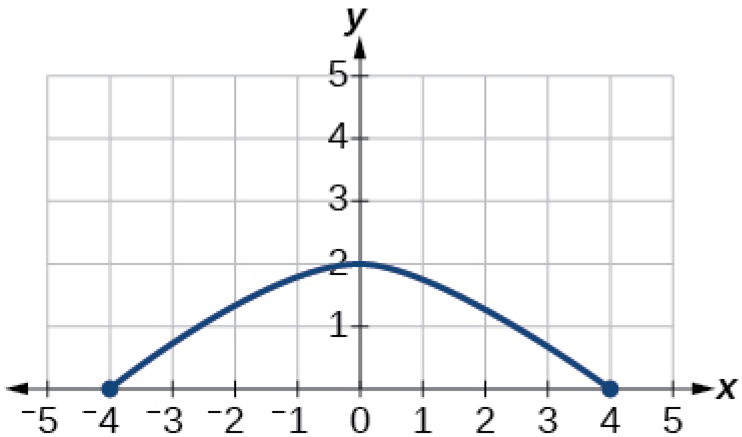
Exercise:

Problem:



Exercise:

Problem:

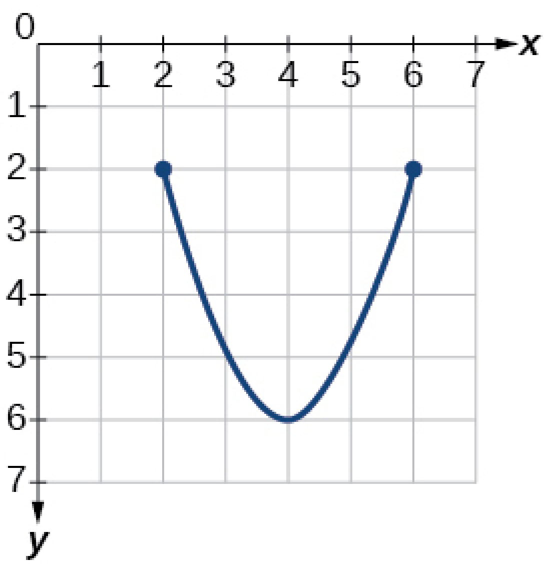


Solution:

domain: $[-4,\ 4]$, range: $[0,\ 2]$

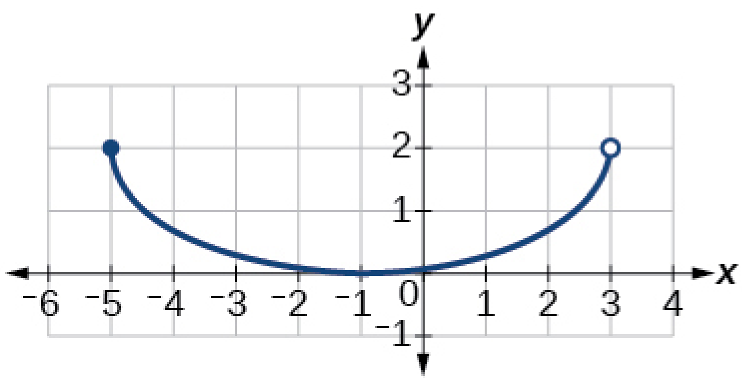
Exercise:

Problem:



Exercise:

Problem:

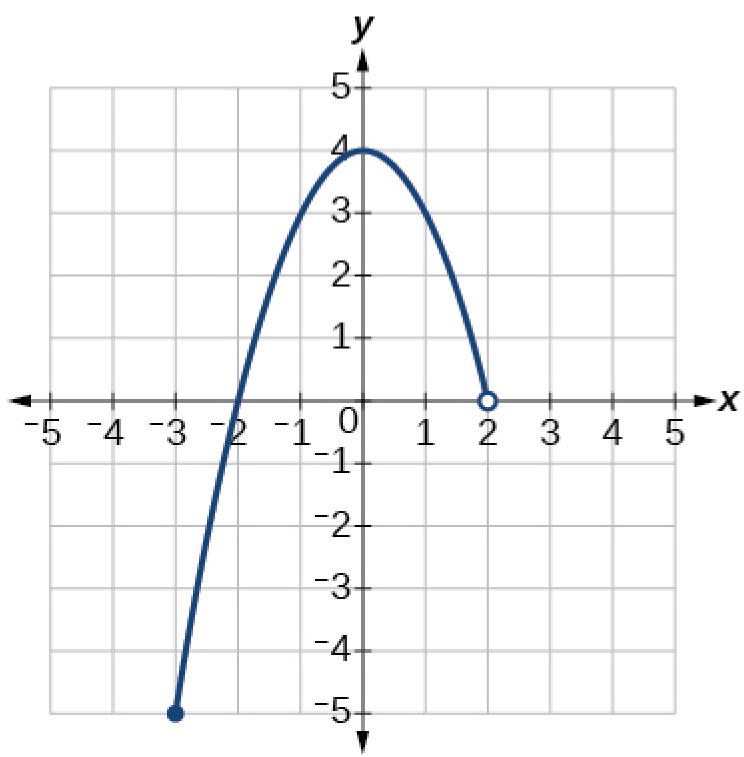


Solution:

domain: $[-5,\ 3)$, range: $[0, 2]$

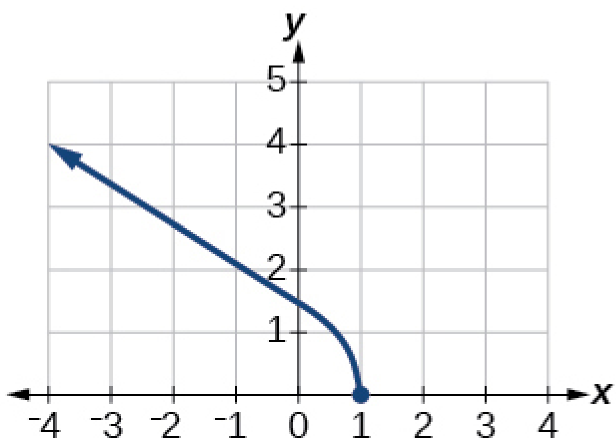
Exercise:

Problem:



Exercise:

Problem:

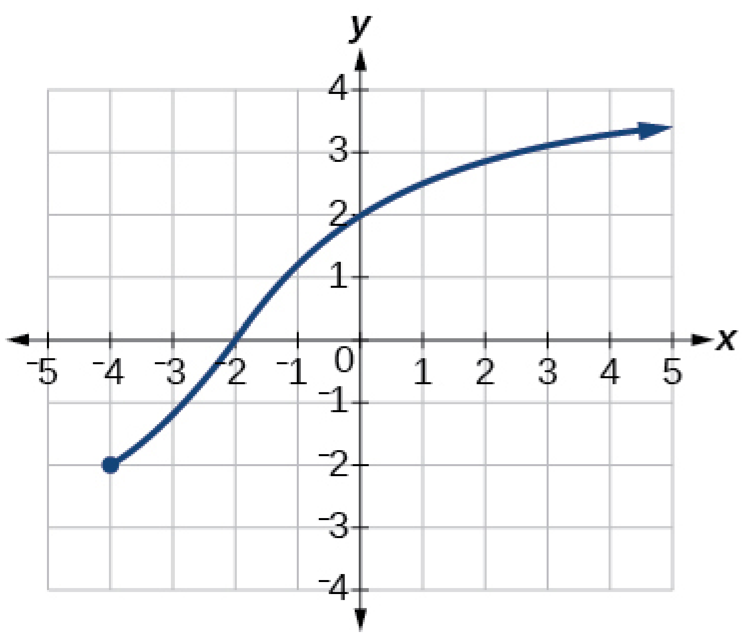


Solution:

domain: $(-\infty, 1]$, range: $[0, \infty)$

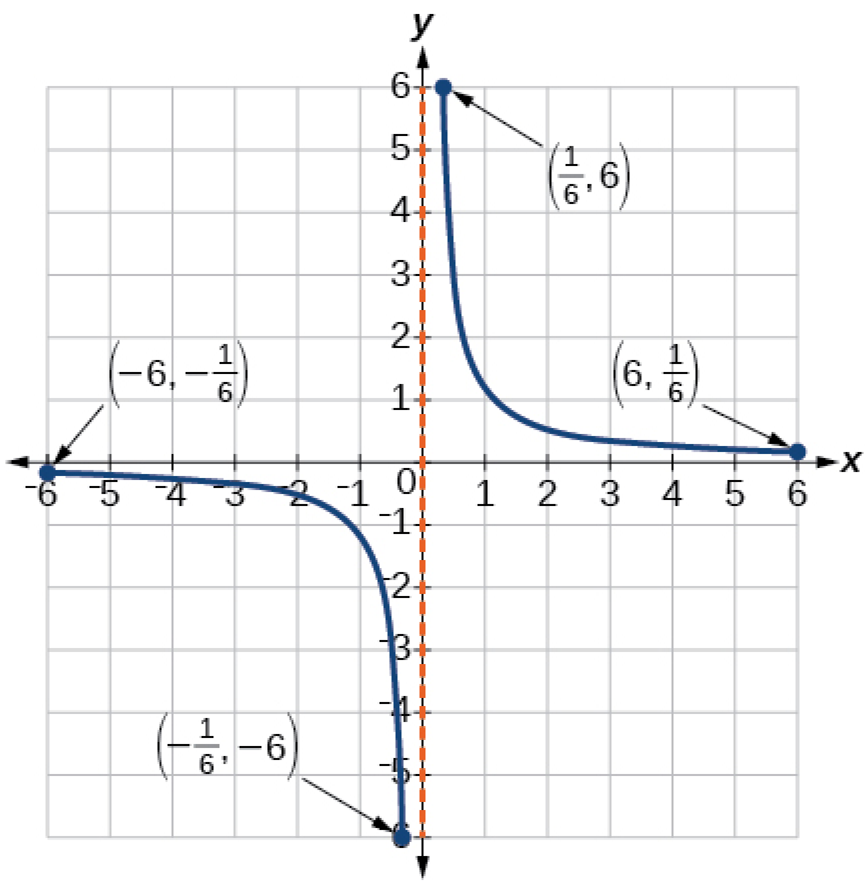
Exercise:

Problem:



Exercise:

Problem:

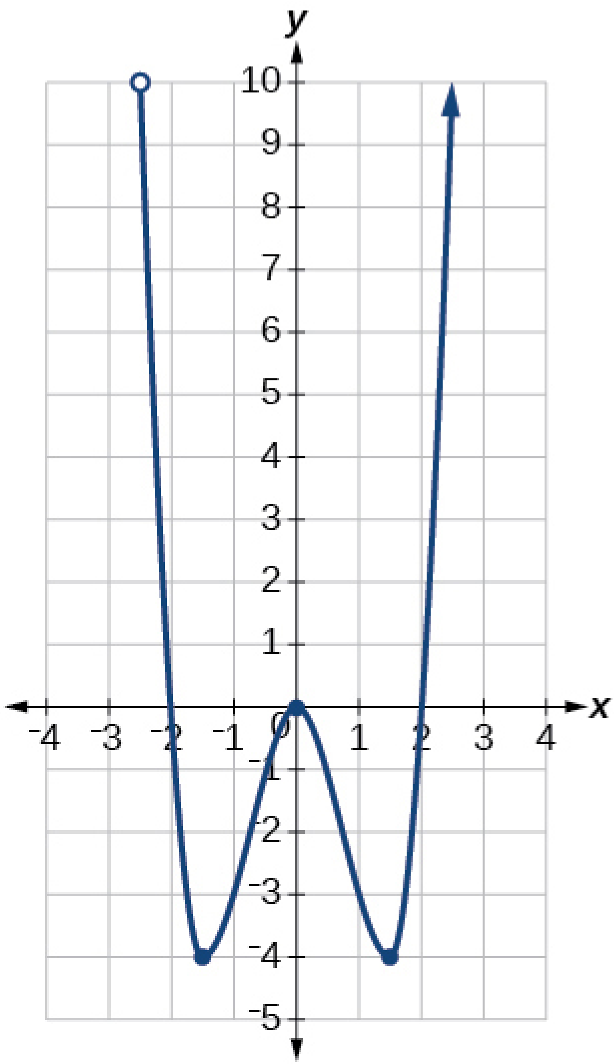


Solution:

domain: $\left[-6, -\frac{1}{6}\right] \cup \left[\frac{1}{6}, 6\right]$; range: $\left[-6, -\frac{1}{6}\right] \cup \left[\frac{1}{6}, 6\right]$

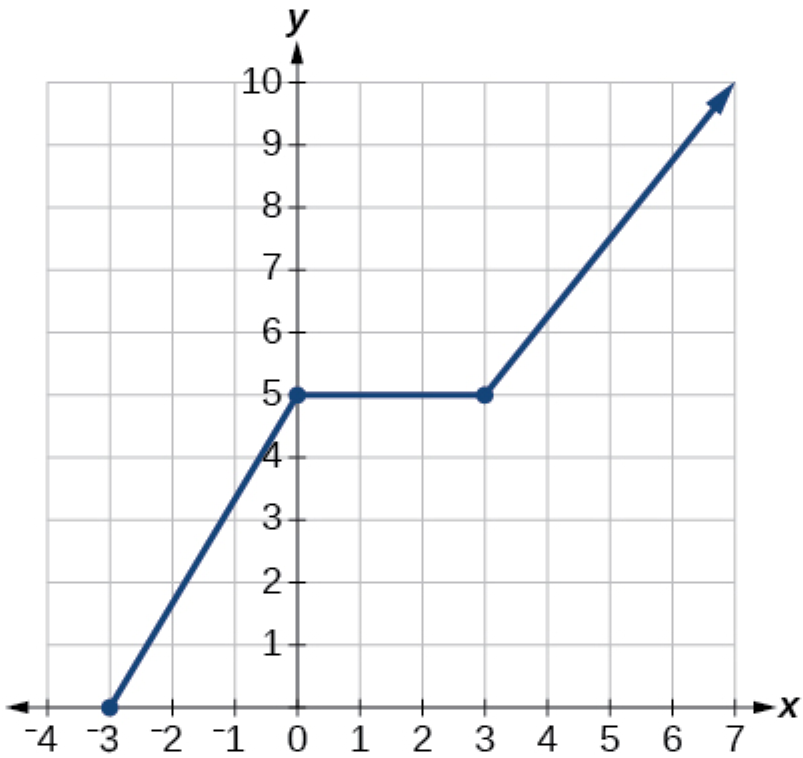
Exercise:

Problem:



Exercise:

Problem:



Solution:

domain: $[-3,\ \infty)$; range: $[0, \infty)$

For the following exercises, sketch a graph of the piecewise function. Write the domain in interval notation.

Exercise:

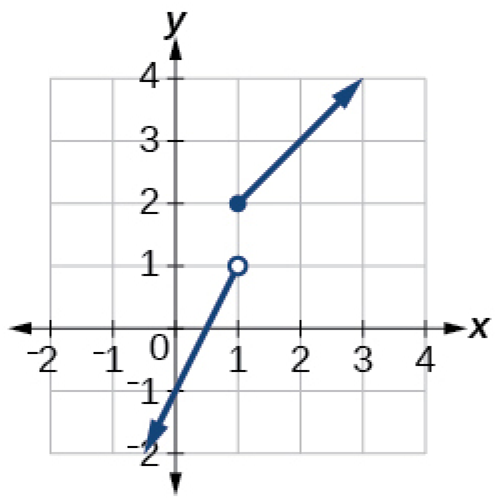
Problem: $f(x) = \begin{cases} x+1 & \text{if} \quad x < -2 \\ -2x-3 & \text{if} \quad x \geq -2 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} 2x-1 & \text{if} \quad x < 1 \\ 1+x & \text{if} \quad x \geq 1 \end{cases}$

Solution:

domain: $(-\infty, \infty)$



Exercise:

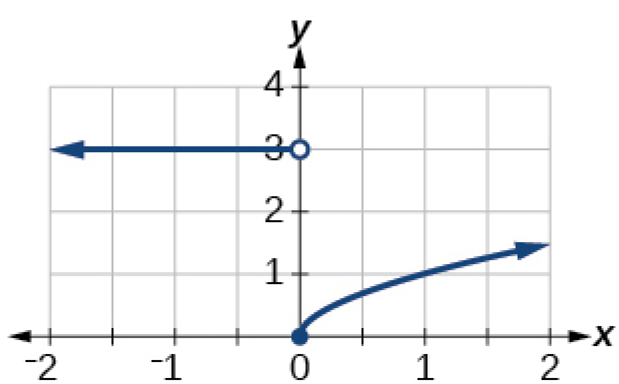
Problem: $f(x) = \begin{cases} x+1 \text{ if } x < 0 \\ x-1 \text{ if } \ x > 0 \end{cases}$

Exercise:

Problem: $f\left(x\right) = \begin{cases} 3 & \text{if} & x < 0 \\ \sqrt{x} & \text{if} & x \geq 0 \end{cases}$

Solution:

domain: $(-\infty, \infty)$



Exercise:

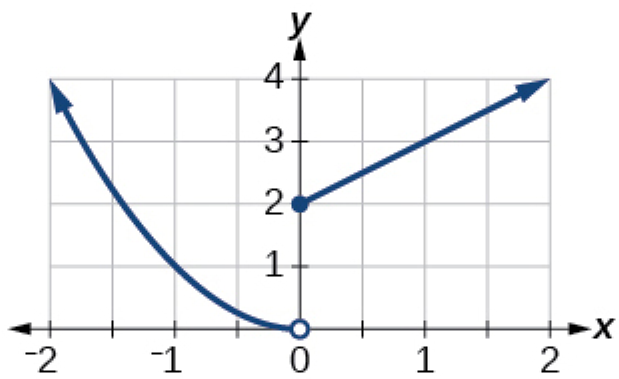
Problem: $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1 - x & \text{if } x > 0 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} x^2 \text{ if } & x < 0 \\ x + 2 \text{ if } & x \geq 0 \end{cases}$

Solution:

domain: $(-\infty, \infty)$



Exercise:

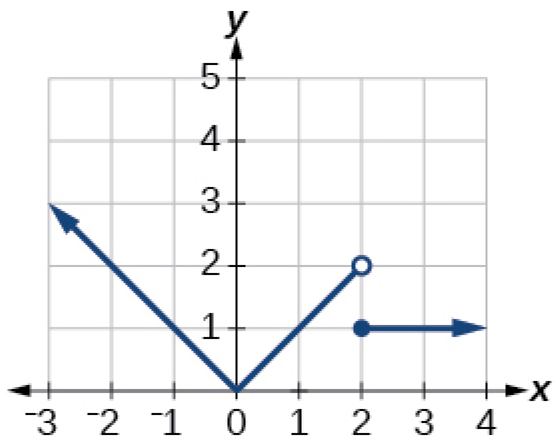
Problem: $f(x) = \begin{cases} x + 1 & \text{if } & x < 1 \\ x^3 & \text{if } & x \geq 1 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} |x| & \text{if } x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$

Solution:

domain: $(-\infty, \infty)$



Numeric

For the following exercises, given each function f,evaluate $f(-3),\ f(-2),\ f(-1)$, and $f(0)$.

Exercise:

Problem: $f(x) = \begin{cases} x+1 & \text{if} \quad x < -2 \\ -2x-3 & \text{if} \quad x \geq -2 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} 1 & \text{if } x \leq -3 \\ 0 & \text{if } x > -3 \end{cases}$

Solution:

$f(-3) = 1; \quad f(-2) = 0; \quad f(-1) = 0; \quad f(0) = 0$

Exercise:

Problem: $f(x) = \begin{cases} -2x^2 + 3 & \text{if } x \le -1 \\ 5x - 7 & \text{if } x > -1 \end{cases}$

For the following exercises, given each function f, evaluate $f(-1),\ f(0),\ f(2),$ and $f(4)$.

Exercise:

Problem: $f(x) = \begin{cases} 7x + 3 & \text{if} \quad x < 0 \\ 7x + 6 & \text{if} \quad x \ge 0 \end{cases}$

Solution:

$$f(-1) = -4; \quad f(0) = 6; \quad f(2) = 20; \quad f(4) = 34$$

Exercise:

Problem: $f\left(x\right) = \begin{cases} x^2 - 2 & \text{if} \quad x < 2 \\ 4 + \left|x - 5\right| & \text{if} \quad x \ge 2 \end{cases}$

Exercise:

Problem: $f\left(x\right) = \begin{array}{ccc} 5x & \text{if} & x < 0 \\ 3 & \text{if} & 0 \le x \le 3 \\ x^2 & \text{if} & x > 3 \end{array}$

Solution:

$$f(-1) = -5; \quad f(0) = 3; \quad f(2) = 3; \quad f(4) = 16$$

For the following exercises, write the domain for the piecewise function in interval notation.

Exercise:

Problem: $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} x^2-2 & \text{if } x < 1 \\ -x^2+2 & \text{if } x > 1 \end{cases}$

Solution:

domain: $(-\infty, 1) \cup (1, \infty)$

Exercise:

Problem: $f(x) = \begin{cases} 2x-3 & \text{if } \; x < 0 \\ -3x^2 & \text{if } \; x \geq 2 \end{cases}$

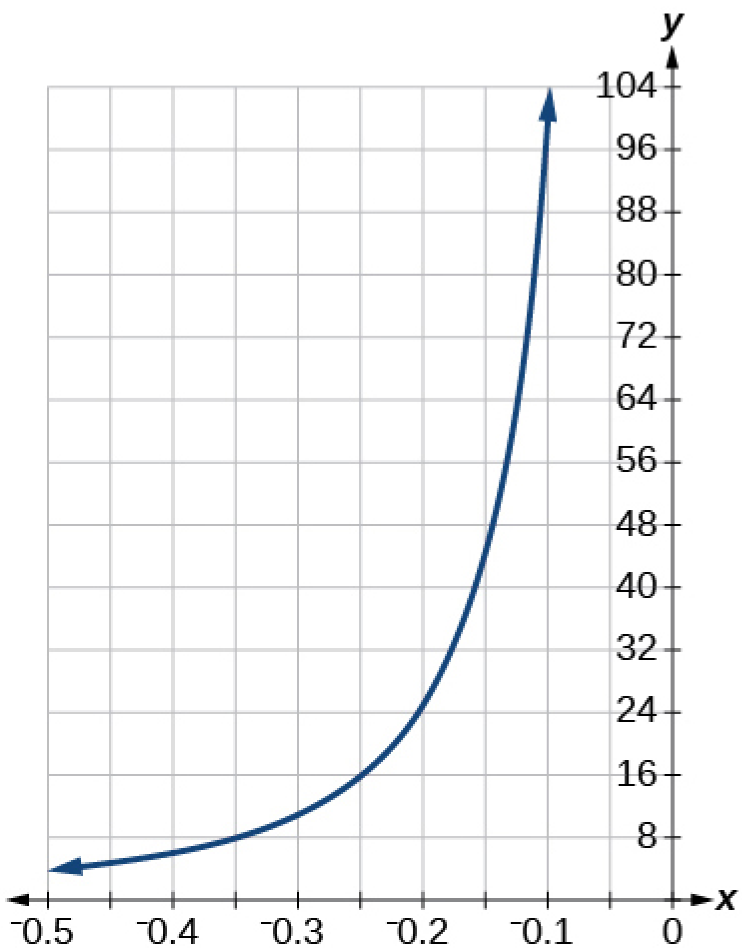
Technology

Exercise:

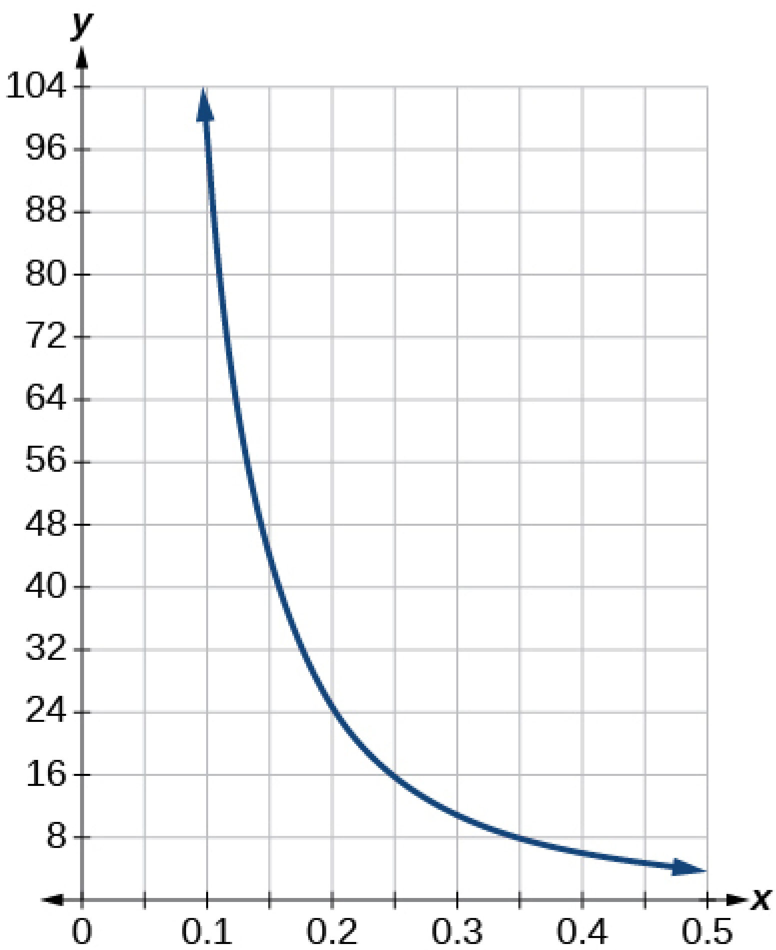
Problem:

Graph $y = \frac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Solution:



window: $[-0.5, -0.1]$; range: $[4,\ 100]$



window: $[0.1,\ 0.5]$; range: $[4,\ 100]$

Exercise:

Problem:

Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1,\ 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Extension

Exercise:

Problem:

Suppose the range of a function f is $[-5,\ 8]$. What is the range of $|f(x)|$?

Solution:

$[0,\ 8]$

Exercise:

Problem:

Create a function in which the range is all nonnegative real numbers.

Exercise:

Problem: Create a function in which the domain is $x > 2$.

Solution:

Many answers. One function is $f(x) = \frac{1}{\sqrt{x-2}}$.

Real-World Applications

Exercise:

Problem:

The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?

Solution:

The domain is $[0,\ 6];$ it takes 6 seconds for the projectile to leave the ground and return to the ground

Exercise:

Problem:

The cost in dollars of making x items is given by the function $C(x) = 10x + 500.$

a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.
b. What is the cost of making 25 items?
c. Suppose the maximum cost allowed is $1500. What are the domain and range of the cost function, $C(x)$?

Glossary

interval notation
: a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

piecewise function
: a function in which more than one formula is used to define the output

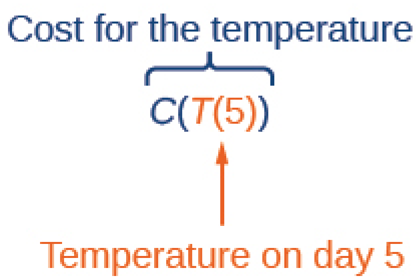
set-builder notation
: a method of describing a set by a rule that all of its members obey; it takes the form $\{x|\,\text{statement about } x\}$

Composition of Functions
In this section, you will:

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C\left(T\right)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T\left(d\right)$ gives the average daily temperature on day d of the year. For any given day, $\text{Cost}=C\left(T\left(d\right)\right)$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T\left(d\right).$ For example, we could evaluate $T\left(5\right)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the cost function at that temperature. We would write $C\left(T\left(5\right)\right).$



By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

Equation:

$$T\left(y\right)=h\left(y\right)+w\left(y\right)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

Equation:

$$T=h+w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f\left(x\right)$ and $g\left(x\right)$ with real number outputs, we define new functions $f+g,$ $f-g,$ $fg,$ and $\frac{f}{g}$ by the relations

Equation:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example:
Exercise:

Problem:
Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x) = x - 1$ and $g(x) = x^2 - 1$. Are they the same function?

Solution:

Begin by writing the general form, and then substitute the given functions.

Equation:

$$(g-f)(x) = g(x) - f(x)$$
$$\begin{aligned}(g-f)(x) &= x^2 - 1 - (x-1) \\ &= x^2 - x \\ &= x(x-1)\end{aligned}$$

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$$
$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1} \\ &= \frac{(x+1)(x-1)}{x-1} \qquad \text{where } x \neq 1 \\ &= x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x = 1$, the denominator is equal to 0, which makes the function undefined.

Note:
Exercise:

Problem: Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.
Equation:

$$f(x) = x - 1 \quad \text{and} \quad g(x) = x^2 - 1$$

Are they the same function?

Solution:

$$(fg)(x) = f(x)g(x) = (x-1)\left(x^2-1\right) = x^3 - x^2 - x + 1$$
$$(f-g)(x) = f(x) - g(x) = (x-1) - \left(x^2-1\right) = x - x^2$$

No, the functions are not the same.

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a composition of functions**.** The resulting function is known as a **composite function**. We represent this combination by the following notation:

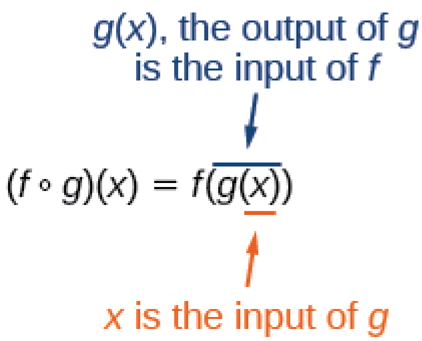
Equation:

$$(f \circ g)(x) = f(g(x))$$

We read the left-hand side as "f composed with g at x ," and the right-hand side as "f of g of x." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and

yields an output $g\left(x\right).$ Then the function f takes $g\left(x\right)$ as an input and yields an output $f\left(g\left(x\right)\right).$



In general, $f\circ g$ and $g\circ f$ are different functions. In other words, in many cases $f\left(g\left(x\right)\right)\ne g\left(f\left(x\right)\right)$ for all $x.$ We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f\left(x\right)={x}^{2}$ and $g\left(x\right)=x+2,$then

Equation:

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2+4x+4 \end{aligned}$$

but

Equation:

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2+2 \end{aligned}$$

These expressions are not equal for all values of $x,$ so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x=-\frac{1}{2}.$

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the

composition has to make sense in terms of inputs and outputs.

Note:
Composition of Functions
When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that
Equation:

$$(f \circ g)(x) = f(g(x))$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.
It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Example:
Exercise:

Problem:
Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is commutative.
Equation:

$$f(x) = 2x + 1 \qquad g(x) = 3 - x$$

Solution:

Let's begin by substituting $g(x)$ into $f(x)$.
Equation:

$$\begin{aligned} f(g(x)) &= 2(3-x)+1 \\ &= 6-2x+1 \\ &= 7-2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

Equation:

$$\begin{aligned} g(f(x)) &= 3-(2x+1) \\ &= 3-2x-1 \\ &= -2x+2 \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

Example:

Exercise:

Problem:

Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution:

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in

3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example:
Exercise:

Problem:
Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f\left(g(y)\right)$ or $g\left(f(x)\right)$?

Solution:

The function $y = f\left(x\right)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.
Equation:

$$\text{number of miles } = f\left(\text{number of hours}\right)$$

The function $g\left(y\right)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:
Equation:

$$\text{number of gallons } = g\left(\text{number of miles}\right)$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f\left(g(y)\right)$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as

the input. Using $f(x)$ (miles driven) as an input value for $g(y),$ where gallons of gas depends on miles driven, does make sense. The expression $g\left(f(x)\right)$ makes sense, and will yield the number of gallons of gas used, $g,$ driving a certain number of miles, $f(x),$ in x hours.

Note:
Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?
Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order.

Note:
Exercise:

Problem:

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Solution:

A gravitational force is still a force, so $a\left(G(r)\right)$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G\left(a(F)\right)$ does not make sense.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example:
Exercise:

Problem:
Using a Table to Evaluate a Composite Function

Using [link], evaluate $f(g(3))$ and $g(f(3))$.

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2

x	$f(x)$	$g(x)$
4	1	7

Solution:

To evaluate $f(g(3)),$ we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function $g:\ g(3)=2.$ We can then use that result as the input to the function $f,$ so $g(3)$ is replaced by 2 and we get $f(2).$ Then, using the table that defines the function $f,$ we find that $f(2)=8.$

Equation:

$$\begin{array}{c}g(3)=2\\ f(g(3))=f(2)=8\end{array}$$

To evaluate $g(f(3)),$ we first evaluate the inside expression $f(3)$ using the first table: $f(3)=3.$ Then, using the table for $g,$ we can evaluate

Equation:

$$g(f(3))=g(3)=2$$

[link] shows the composite functions $f\circ g$ and $g\circ f$ as tables.

x	$g\left(x\right)$	$f\left(g\left(x\right)\right)$	$f\left(x\right)$	$g\left(f\left(x\right)\right)$
3	2	8	3	2

Note:
Exercise:

Problem: Using [link], evaluate $f(g(1))$ and $g(f(4))$.

Solution:

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

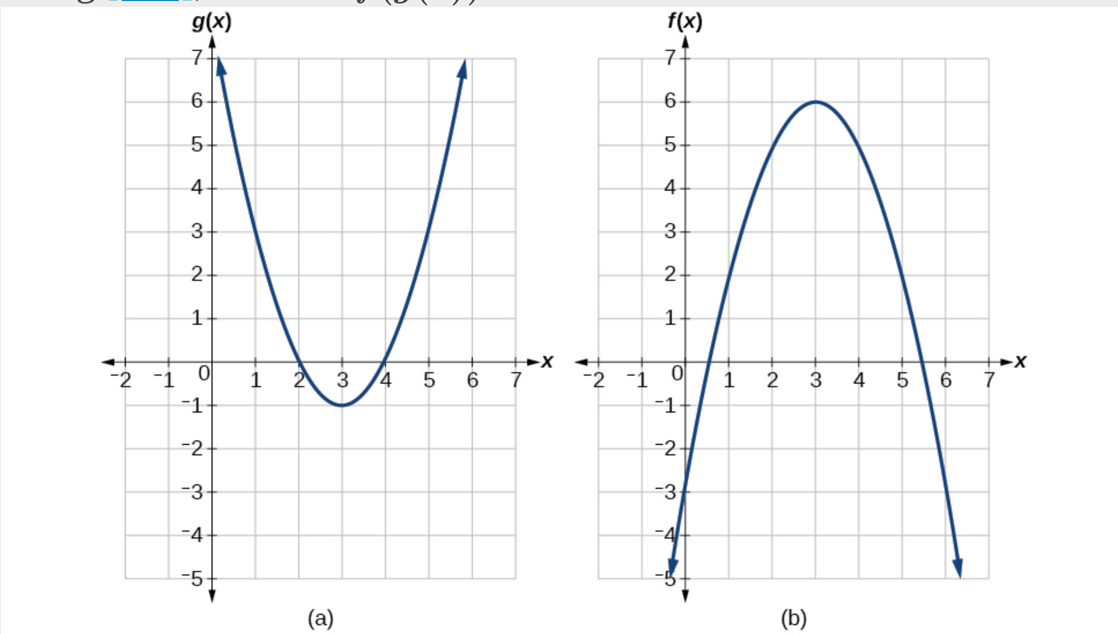
Note:
Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

Example:
Exercise:

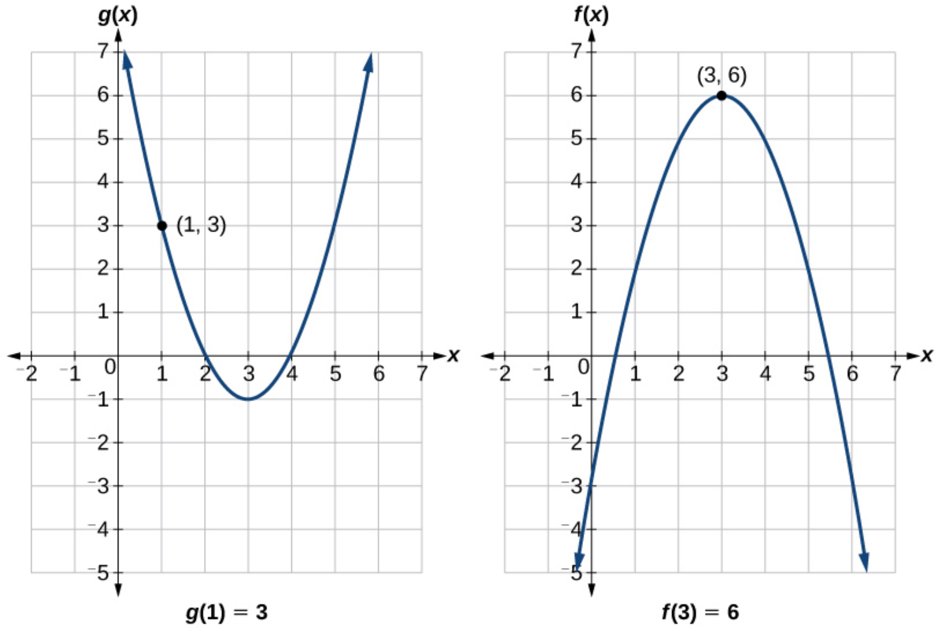
Problem:
Using a Graph to Evaluate a Composite Function

Using [link], evaluate $f(g(1))$.



Solution:

To evaluate $f(g(1))$, we start with the inside evaluation. See [link].



We evaluate $g(1)$ using the graph of $g(x),$ finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1)=3.$ We use this value as the input to the function $f.$

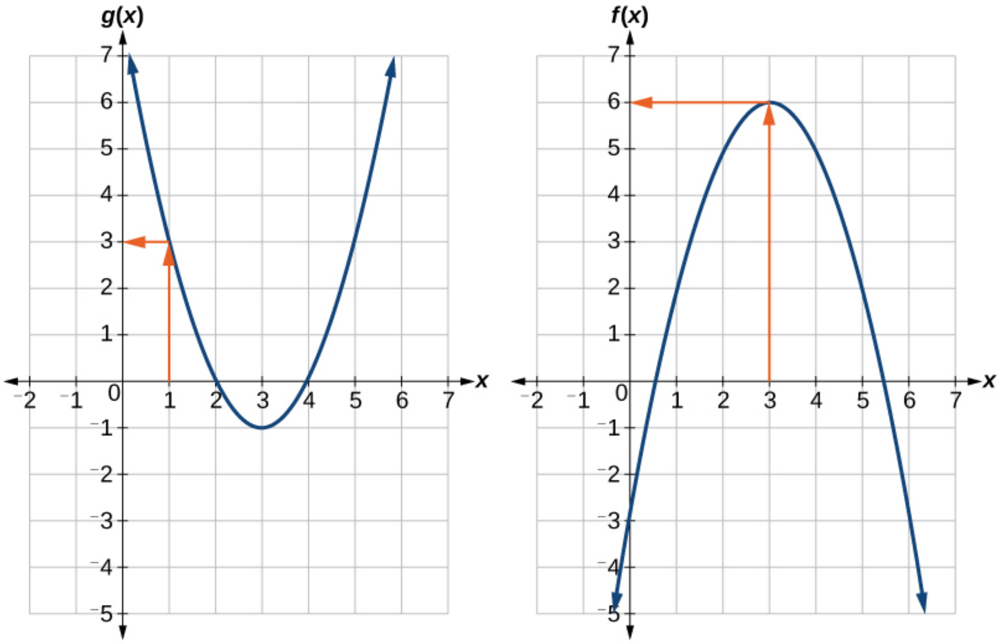
Equation:

$$f(g(1))=f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x),$ finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3)=6,$ so $f(g(1))=6.$

Analysis

[link] shows how we can mark the graphs with arrows to trace the path from the input value to the output value.



Note:
Exercise:

Problem: Using [link], evaluate $g(f(2))$.

Solution:

$g(f(2)) = g(5) = 3$

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f\left(g\left(x\right)\right).$ To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t)=t^2-t,$ we substitute the value inside the parentheses into the formula wherever we see the input variable.

Note:
Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Example:
Exercise:

Problem:
Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t)=t^2-t$ and $h(x)=3x+2,$ evaluate $f(h(1)).$

Solution:

Because the inside expression is $h(1),$ we start by evaluating $h(x)$ at 1.

Equation:

$$\begin{aligned}h(1)&=3(1)+2\\h(1)&=5\end{aligned}$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.
Equation:

$$f(h(1)) = f(5)$$
$$f(h(1)) = 5^2 - 5$$
$$f(h(1)) = 20$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Note:
Exercise:

Problem: Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Solution:

a. 8; b. 20

Finding the Domain of a Composite Function

As we discussed previously, the domain of a composite function such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions f and g separately. If we write the composite

function for an input x as $f\left(g\left(x\right)\right),$ we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g\left(x\right)$ must be a member of the domain of $f,$ otherwise the second function evaluation in $f\left(g\left(x\right)\right)$ cannot be completed, and the expression is still undefined. Thus the domain of $f\circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of $f.$ Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g\left(x\right)$ is in the domain of $f.$

Note:
Domain of a Composite Function
The domain of a composite function $f\left(g\left(x\right)\right)$ is the set of those inputs x in the domain of g for which $g\left(x\right)$ is in the domain of $f.$

Note:
Given a function composition $f(g(x))$,determine its domain.

1. Find the domain of g.
2. Find the domain of f.
3. Find those inputs x in the domain of g for which $g\left(x\right)$ is in the domain of $f.$ That is, exclude those inputs x from the domain of g for which $g\left(x\right)$ is not in the domain of $f.$ The resulting set is the domain of $f\circ g$.

Example:
Exercise:

Problem:
Finding the Domain of a Composite Function

Find the domain of
Equation:

$$(f \circ g)(x) \quad \text{where} \quad f(x) = \frac{5}{x-1} \text{ and } g(x) = \frac{4}{3x-2}$$

Solution:

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x) = 1$.
Equation:

$$\begin{aligned} \tfrac{4}{3x-2} &= 1 \\ 4 &= 3x - 2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that
Equation:

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as
Equation:

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

Example:
Exercise:

Problem:
Finding the Domain of a Composite Function Involving Radicals

Find the domain of
Equation:

$$(f \circ g)(x) \text{ where } \quad f(x) = \sqrt{x+2} \text{ and } \quad g(x) = \sqrt{3-x}$$

Solution:

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function
Equation:

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$$

For $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$, $\sqrt{3-x}+2 \geq 0$, since the radicand of a square root must be positive. Since square roots are positive, $\sqrt{3-x} \geq 0$, or, $3-x \geq 0$, which gives a domain of (-$\infty, 3]$.

Analysis

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

Note:

Exercise:

Problem: Find the domain of

Equation:

$$(f \circ g)(x) \text{ where } f(x) = \frac{1}{x-2} \text{ and } g(x) = \sqrt{x+4}$$

Solution:

$[-4, 0) \cup (0, \infty)$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to decompose a composite function, so we may choose the decomposition that appears to be most expedient.

Example:

Exercise:

Problem:

Decomposing a Function

Write $f(x) = \sqrt{5 - x^2}$ as the composition of two functions.

Solution:

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$.

As one possibility, we might notice that the expression $5 - x^2$ is the inside of the square root. We could then decompose the function as
Equation:

$$h(x) = 5 - x^2 \text{ and } g(x) = \sqrt{x}$$

We can check our answer by recomposing the functions.
Equation:

$$g(h(x)) = g\left(5 - x^2\right) = \sqrt{5 - x^2}$$

Note:
Exercise:

Problem: Write $f(x) = \frac{4}{3 - \sqrt{4 + x^2}}$ as the composition of two functions.

Solution:

Possible answer:

$g\left(x\right) = \sqrt{4 + x^2}$
$h\left(x\right) = \frac{4}{3 - x}$
$f = h \circ g$

Note:
Access these online resources for additional instruction and practice with composite functions.

- Composite Functions
- Composite Function Notation Application

- Composite Functions Using Graphs
- Decompose Functions
- Composite Function Values

Key Equation

Composite function	$(f \circ g)(x) = f(g(x))$

Key Concepts

- We can perform algebraic operations on functions. See [link].
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See [link] and [link].
- The order of function composition must be considered when interpreting the meaning of composite functions. See [link].
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See [link].
- A composite function can be evaluated from a graph. See [link].
- A composite function can be evaluated from a formula. See [link].
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See [link] and [link].
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.

- Functions can often be decomposed in more than one way. See [link].

Section Exercises

Verbal

Exercise:

Problem:

How does one find the domain of the quotient of two functions, $\frac{f}{g}$?

Solution:

Find the numbers that make the function in the denominator g equal to zero, and check for any other domain restrictions on f and g, such as an even-indexed root or zeros in the denominator.

Exercise:

Problem: What is the composition of two functions, $f \circ g$?

Exercise:

Problem:

If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

Solution:

Yes. Sample answer: Let $f(x) = x + 1$ and $g(x) = x - 1$. Then $f(g(x)) = f(x - 1) = (x - 1) + 1 = x$ and $g(f(x)) = g(x + 1) = (x + 1) - 1 = x$. So $f \circ g = g \circ f$.

Exercise:

Problem:

How do you find the domain for the composition of two functions, $f \circ g$?

Algebraic

Exercise:

Problem:

Given $f(x) = x^2 + 2x$ and $g(x) = 6 - x^2$,find $f + g,\ f - g,\ fg,$ and $\frac{f}{g}$.Determine the domain for each function in interval notation.

Solution:

$(f + g)(x) = 2x + 6,$ domain: $(-\infty, \infty)$

$(f - g)(x) = 2x^2 + 2x - 6,$ domain: $(-\infty, \infty)$

$(fg)(x) = -x^4 - 2x^3 + 6x^2 + 12x,$ domain: $(-\infty, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{x^2+2x}{6-x^2}$, domain: $(-\infty, -\sqrt{6}) \cup (-\sqrt{6}, \sqrt{6}) \cup (\sqrt{6}, \infty)$

Exercise:

Problem:

Given $f(x) = -3x^2 + x$ and $g(x) = 5,$ find $f + g,\ f - g,\ fg,$ and $\frac{f}{g}$. Determine the domain for each function in interval notation.

Exercise:

Problem:

Given $f(x) = 2x^2 + 4x$ and $g(x) = \frac{1}{2x}$,find $f + g,\ f - g,\ fg,$ and $\frac{f}{g}$. Determine the domain for each function in interval notation.

Solution:

$(f + g)\,(x) = \frac{4x^3 + 8x^2 + 1}{2x}$, domain: $(-\infty, 0) \cup (0, \infty)$

$(f - g)\,(x) = \frac{4x^3 + 8x^2 - 1}{2x}$, domain: $(-\infty, 0) \cup (0, \infty)$

$(fg)\,(x) = x + 2,$ domain: $(-\infty, 0) \cup (0, \infty)$

$\left(\frac{f}{g}\right)(x) = 4x^3 + 8x^2,$ domain: $(-\infty, 0) \cup (0, \infty)$

Exercise:

Problem:

Given $f(x) = \frac{1}{x - 4}$ and $g(x) = \frac{1}{6 - x}$, find $f + g,\ f - g,\ fg,$ and $\frac{f}{g}$. Determine the domain for each function in interval notation.

Exercise:

Problem:

Given $f(x) = 3x^2$ and $g(x) = \sqrt{x - 5}$, find $f + g,\ f - g,\ fg,$ and $\frac{f}{g}$. Determine the domain for each function in interval notation.

Solution:

$(f + g)(x) = 3x^2 + \sqrt{x - 5},$ domain: $[5, \infty)$

$(f - g)(x) = 3x^2 - \sqrt{x - 5},$ domain: $[5, \infty)$

$(fg)(x) = 3x^2\sqrt{x - 5},$ domain: $[5, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{3x^2}{\sqrt{x-5}}$, domain: $(5, \infty)$

Exercise:

Problem:

Given $f(x) = \sqrt{x}$ and $g(x) = |x - 3|$, find $\frac{g}{f}$. Determine the domain of the function in interval notation.

Exercise:

Problem:

Given $f(x) = 2x^2 + 1$ and $g(x) = 3x - 5$, find the following:

a. $f(g(2))$
b. $f(g(x))$
c. $g(f(x))$
d. $(g \circ g)(x)$
e. $(f \circ f)(-2)$

Solution:

a. 3; b. $f(g(x)) = 2(3x - 5)^2 + 1$; c. $f(g(x)) = 6x^2 - 2$; d. $(g \circ g)(x) = 3(3x - 5) - 5 = 9x - 20$; e. $(f \circ f)(-2) = 163$

For the following exercises, use each pair of functions to find $f(g(x))$ and $g(f(x))$. Simplify your answers.

Exercise:

Problem: $f(x) = x^2 + 1,\ g(x) = \sqrt{x + 2}$

Exercise:

Problem: $f(x) = \sqrt{x} + 2,\ g(x) = x^2 + 3$

Solution:

$$f(g(x)) = \sqrt{x^2+3}+2,\ g(f(x)) = x+4\sqrt{x}+7$$

Exercise:

Problem: $f(x) = |x|,\ g(x) = 5x+1$

Exercise:

Problem: $f(x) = \sqrt[3]{x},\ g(x) = \frac{x+1}{x^3}$

Solution:

$$f(g(x)) = \sqrt[3]{\frac{x+1}{x^3}} = \frac{\sqrt[3]{x+1}}{x},\ g(f(x)) = \frac{\sqrt[3]{x}+1}{x}$$

Exercise:

Problem: $f(x) = \frac{1}{x-6},\ g(x) = \frac{7}{x}+6$

Exercise:

Problem: $f(x) = \frac{1}{x-4},\ g(x) = \frac{2}{x}+4$

Solution:

$$(f \circ g)(x) = \frac{1}{\frac{2}{x}+4-4} = \frac{x}{2},\ \ (g \circ f)(x) = 2x-4$$

For the following exercises, use each set of functions to find $f\left(g\left(h(x)\right)\right)$. Simplify your answers.

Exercise:

Problem: $f(x) = x^4+6,\ g(x) = x-6,$ and $h(x) = \sqrt{x}$

Exercise:

Problem: $f(x) = x^2+1,\ g(x) = \frac{1}{x},$ and $h(x) = x+3$

Solution:

$f(g(h(x))) = \left(\frac{1}{x+3}\right)^2 + 1$

Exercise:

Problem: Given $f(x) = \frac{1}{x}$ and $g(x) = x - 3,$ find the following:

a. $(f \circ g)(x)$
b. the domain of $(f \circ g)(x)$ in interval notation
c. $(g \circ f)(x)$
d. the domain of $(g \circ f)(x)$
e. $\left(\frac{f}{g}\right)x$

Exercise:

Problem:

Given $f(x) = \sqrt{2 - 4x}$ and $g(x) = -\frac{3}{x}$, find the following:

a. $(g \circ f)(x)$
b. the domain of $(g \circ f)(x)$ in interval notation

Solution:

a. $(g \circ f)(x) = -\frac{3}{\sqrt{2-4x}}$; b. $\left(-\infty, \frac{1}{2}\right)$

Exercise:

Problem:

Given the functions $f(x) = \frac{1-x}{x}$ and $g(x) = \frac{1}{1+x^2}$, find the following:

a. $(g \circ f)(x)$
b. $(g \circ f)(2)$

Exercise:

Problem:

Given functions $p(x) = \frac{1}{\sqrt{x}}$ and $m(x) = x^2 - 4,$ state the domain of each of the following functions using interval notation:

a. $\frac{p(x)}{m(x)}$
b. $p(m(x))$
c. $m(p(x))$

Solution:

a. $(0, 2) \cup (2, \infty);$ b. $(-\infty, -2) \cup (2, \infty);$ c. $(0, \infty)$

Exercise:

Problem:

Given functions $q(x) = \frac{1}{\sqrt{x}}$ and $h(x) = x^2 - 9,$ state the domain of each of the following functions using interval notation.

a. $\frac{q(x)}{h(x)}$
b. $q\left(h(x)\right)$
c. $h\left(q(x)\right)$

Exercise:

Problem:

For $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x - 1},$ write the domain of $(f \circ g)(x)$ in interval notation.

Solution:

$(1, \infty)$

For the following exercises, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x) = f(g(x))$.

Exercise:

Problem: $h(x) = (x+2)^2$

Exercise:

Problem: $h(x) = (x-5)^3$

Solution:

sample: $f(x) = x^3$
$g(x) = x - 5$

Exercise:

Problem: $h(x) = \frac{3}{x-5}$

Exercise:

Problem: $h(x) = \frac{4}{(x+2)^2}$

Solution:

sample: $f(x) = \frac{4}{x}$
$g(x) = (x+2)^2$

Exercise:

Problem: $h(x) = 4 + \sqrt[3]{x}$

Exercise:

Problem: $h(x) = \sqrt[3]{\frac{1}{2x-3}}$

Solution:

sample: $f(x) = \sqrt[3]{x}$
$g(x) = \frac{1}{2x-3}$

Exercise:

Problem: $h(x) = \frac{1}{(3x^2-4)^{-3}}$

Exercise:

Problem: $h(x) = \sqrt[4]{\frac{3x-2}{x+5}}$

Solution:

sample: $f(x) = \sqrt[4]{x}$
$g(x) = \frac{3x-2}{x+5}$

Exercise:

Problem: $h(x) = \left(\frac{8+x^3}{8-x^3}\right)^4$

Exercise:

Problem: $h(x) = \sqrt{2x+6}$

Solution:

sample: $f(x) = \sqrt{x}$
$g(x) = 2x + 6$

Exercise:

Problem: $h(x) = (5x-1)^3$

Exercise:

Problem: $h(x) = \sqrt[3]{x-1}$

Solution:

sample: $f(x) = \sqrt[3]{x}$
$g(x) = (x-1)$

Exercise:

Problem: $h(x) = \left|x^2 + 7\right|$

Exercise:

Problem: $h(x) = \frac{1}{(x-2)^3}$

Solution:

sample: $f(x) = x^3$
$g(x) = \frac{1}{x-2}$

Exercise:

Problem: $h(x) = \left(\frac{1}{2x-3}\right)^2$

Exercise:

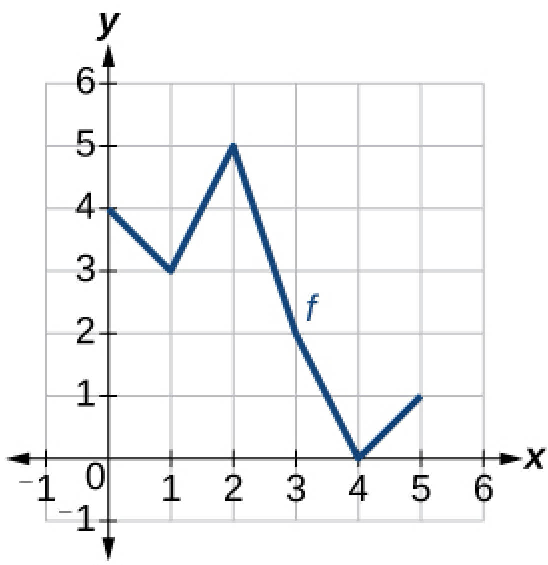
Problem: $h(x) = \sqrt{\frac{2x-1}{3x+4}}$

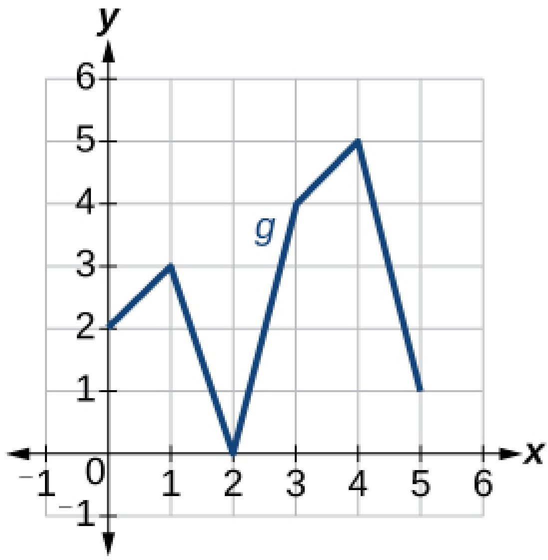
Solution:

sample: $f(x) = \sqrt{x}$
$g(x) = \frac{2x-1}{3x+4}$

Graphical

For the following exercises, use the graphs of f,shown in [link], and g, shown in [link], to evaluate the expressions.





Exercise:

Problem: $f\left(g(3)\right)$

Exercise:

Problem: $f\left(g(1)\right)$

Solution:

2

Exercise:

Problem: $g\left(f(1)\right)$

Exercise:

Problem: $g\left(f(0)\right)$

Solution:

5

Exercise:

Problem: $f\left(f(5)\right)$

Exercise:

Problem: $f\left(f(4)\right)$

Solution:

4

Exercise:

Problem: $g\left(g(2)\right)$

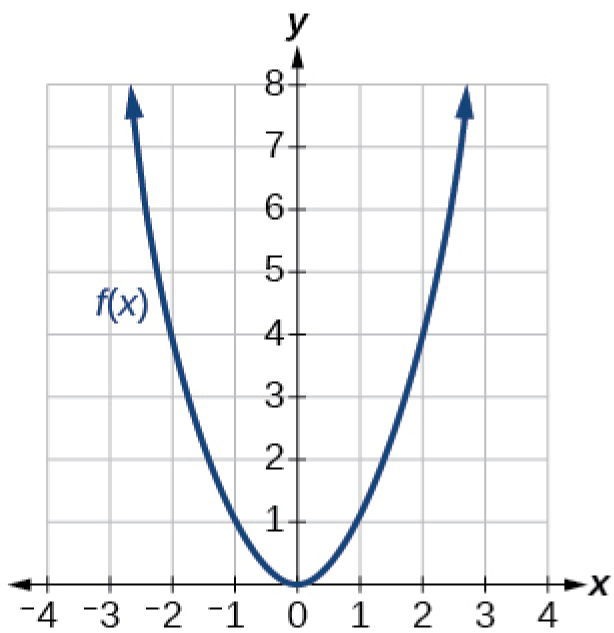
Exercise:

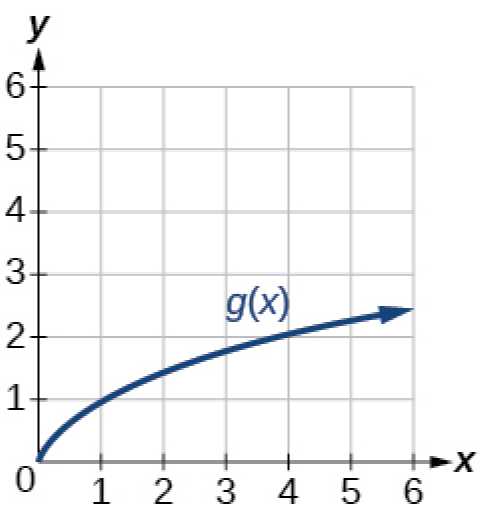
Problem: $g\left(g(0)\right)$

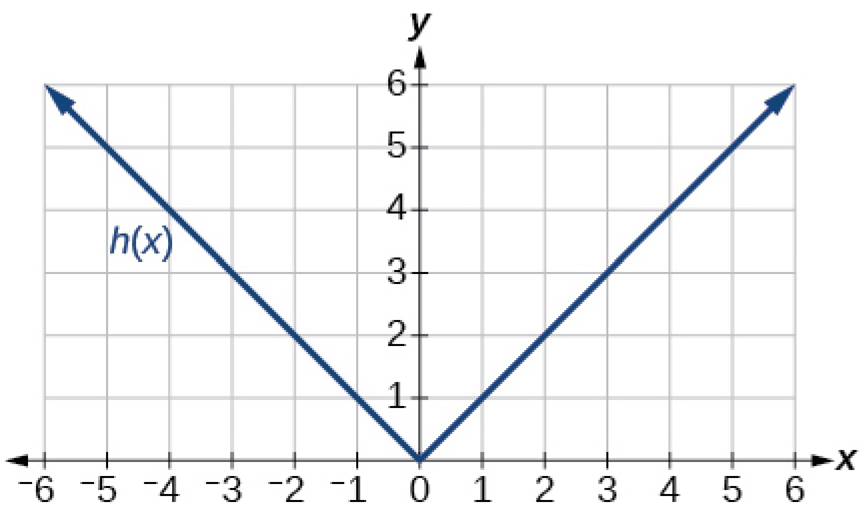
Solution:

0

For the following exercises, use graphs of $f(x)$,shown in [link], $g(x)$,shown in [link], and $h(x)$,shown in [link], to evaluate the expressions.


y
x
f(x)
8
7
6
5
4
3
2
1
−4
−3
−2
−1
0
1
2
3
4


y
x
g(x)
6
5
4
3
2
1
0
1
2
3
4
5
6



Exercise:

Problem: $g\left(f\left(1\right)\right)$

Exercise:

Problem: $g\left(f\left(2\right)\right)$

Solution:

2

Exercise:

Problem: $f\left(g\left(4\right)\right)$

Exercise:

Problem: $f\left(g\left(1\right)\right)$

Solution:

1

Exercise:

Problem: $f\left(h\left(2\right)\right)$

Exercise:

Problem: $h\left(f\left(2\right)\right)$

Solution:

4

Exercise:

Problem: $f\left(g\left(h\left(4\right)\right)\right)$

Exercise:

Problem: $f\left(g\left(f\left(-2\right)\right)\right)$

Solution:

4

Numeric

For the following exercises, use the function values for f and g shown in [link] to evaluate each expression.

x	$f(x)$	$g(x)$
0	7	9
1	6	5

2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

Exercise:

Problem: $f\left(g\left(8\right)\right)$

Exercise:

Problem: $f\left(g\left(5\right)\right)$

Solution:

9

Exercise:

Problem: $g\left(f\left(5\right)\right)$

Exercise:

Problem: $g\left(f\left(3\right)\right)$

Solution:

4

Exercise:

Problem: $f\left(f\left(4\right)\right)$

Exercise:

Problem: $f\left(f\left(1\right)\right)$

Solution:

2

Exercise:

Problem: $g\left(g\left(2\right)\right)$

Exercise:

Problem: $g\left(g\left(6\right)\right)$

Solution:

3

For the following exercises, use the function values for f and g shown in [link] to evaluate the expressions.

x	$f(x)$	$g(x)$
-3	11	-8

-2	9	-3
-1	7	0
0	5	1
1	3	0
2	1	-3
3	-1	-8

Exercise:

Problem: $(f \circ g)(1)$

Exercise:

Problem: $(f \circ g)(2)$

Solution:

11

Exercise:

Problem: $(g \circ f)(2)$

Exercise:

Problem: $(g \circ f)(3)$

Solution:

0

Exercise:

Problem: $(g \circ g)(1)$

Exercise:

Problem: $(f \circ f)(3)$

Solution:

7

For the following exercises, use each pair of functions to find $f\left(g\left(0\right)\right)$ and $g\left(f(0)\right)$.

Exercise:

Problem: $f(x)=4x+8,\ g(x)=7-x^2$

Exercise:

Problem: $f(x)=5x+7,\ g(x)=4-2x^2$

Solution:

$f(g(0))=27,\ g\left(f(0)\right)=-94$

Exercise:

Problem: $f(x)=\sqrt{x+4},\ g(x)=12-x^3$

Exercise:

Problem: $f(x)=\frac{1}{x+2},\ g(x)=4x+3$

Solution:

$f(g(0))=\frac{1}{5},\ g(f(0))=5$

For the following exercises, use the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x + 5$ to evaluate or find the composite function as indicated.

Exercise:

Problem: $f\left(g(2)\right)$

Exercise:

Problem: $f\left(g(x)\right)$

Solution:

$18x^2 + 60x + 51$

Exercise:

Problem: $g\left(f(-3)\right)$

Exercise:

Problem: $(g \circ g)(x)$

Solution:

$g \circ g(x) = 9x + 20$

Extensions

For the following exercises, use $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x - 1}$.

Exercise:

Problem: Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Compare the two answers.

Exercise:

Problem: Find $(f \circ g)(2)$ and $(g \circ f)(2)$.

Solution:

2

Exercise:

Problem: What is the domain of $(g \circ f)(x)$?

Exercise:

Problem: What is the domain of $(f \circ g)(x)$?

Solution:

$(-\infty, \infty)$

Exercise:

Problem: Let $f(x) = \frac{1}{x}$.

a. Find $(f \circ f)(x)$.
b. Is $(f \circ f)(x)$ for any function f the same result as the answer to part (a) for any function? Explain.

For the following exercises, let $F(x) = (x+1)^5$, $f(x) = x^5$, and $g(x) = x + 1$.

Exercise:

Problem: True or False: $(g \circ f)(x) = F(x)$.

Solution:

False

Exercise:

Problem: True or False: $(f \circ g)(x) = F(x)$.

For the following exercises, find the composition when $f(x) = x^2 + 2$ for all $x \geq 0$ and $g(x) = \sqrt{x-2}$.

Exercise:

Problem: $(f \circ g)(6)$; $(g \circ f)(6)$

Solution:

$(f \circ g)(6) = 6$; $(g \circ f)(6) = 6$

Exercise:

Problem: $(g \circ f)(a)$; $(f \circ g)(a)$

Exercise:

Problem: $(f \circ g)(11)$; $(g \circ f)(11)$

Solution:

$(f \circ g)(11) = 11$, $(g \circ f)(11) = 11$

Real-World Applications

Exercise:

Problem:

The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost $C(x)$ is the cost of producing x items. To determine the cost of production when the price is $6, you would do which of the following?

a. Evaluate $D\left(C(6)\right)$.
b. Evaluate $C\left(D(6)\right)$.
c. Solve $D\left(C(x)\right) = 6$.
d. Solve $C\left(D(p)\right) = 6$.

Exercise:

Problem:

The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of 4?

a. Evaluate $A\left(m(4)\right)$.
b. Evaluate $m\left(A(4)\right)$.
c. Solve $A\left(m(t)\right) = 4$.
d. Solve $m\left(A(d)\right) = 4$.

Solution:

c

Exercise:

Problem:

A store offers customers a 30% discount on the price x of selected items. Then, the store takes off an additional 15% at the cash register. Write a price function $P(x)$ that computes the final price of the item in terms of the original price $x.$ (Hint: Use function composition to find your answer.)

Exercise:

Problem:

A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t)=25\sqrt{t+2},$ find the area of the ripple as a function of time. Find the area of the ripple at $t=2.$

Solution:

$A(t)=\pi\left(25\sqrt{t+2}\right)^2$ and $A(2)=\pi\left(25\sqrt{4}\right)^2=2500\pi$ square inches

Exercise:

Problem:

A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t)=2t+1,$ express the area burned as a function of time, t (minutes).

Exercise:

Problem:

Use the function you found in the previous exercise to find the total area burned after 5 minutes.

Solution:

$A(5) = \pi(2(5) + 1)^2 = 121\pi$ square units

Exercise:

Problem:

The radius r, in inches, of a spherical balloon is related to the volume, V, by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t) = 10 + 20t$.

a. Find the composite function $r\left(V(t)\right)$.
b. Find the *exact* time when the radius reaches 10 inches.

Exercise:

Problem:

The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1,\ 3 < T < 33$, where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where t is the time in hours.

a. Find the composite function $N\left(T(t)\right)$.
b. Find the time (round to two decimal places) when the bacteria count reaches 6752.

Solution:

a. $N(T(t)) = 23(5t + 1.5)^2 - 56(5t + 1.5) + 1;$ b. 3.38 hours

Glossary

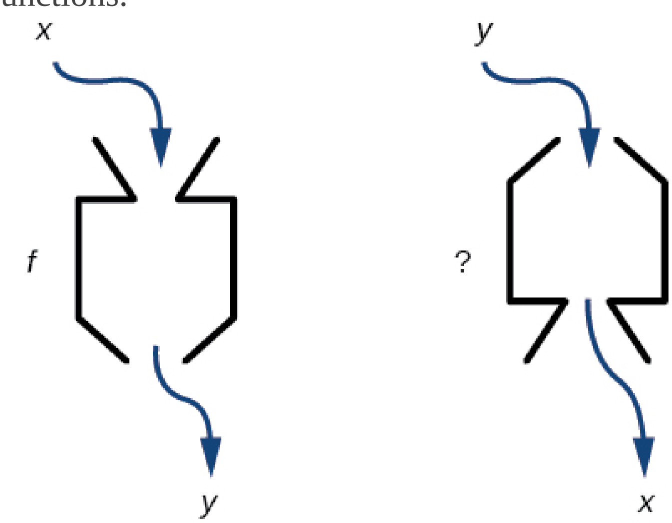
composite function
: the new function formed by function composition, when the output of one function is used as the input of another

Inverse Functions
In this section, you will:

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. [link] provides a visual representation of this question. In this section, we will consider the reverse nature of functions.



Can a function "machine" operate in reverse?

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the Celsius scale. To get an idea of

how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees Fahrenheit to degrees Celsius. She finds the formula

Equation:

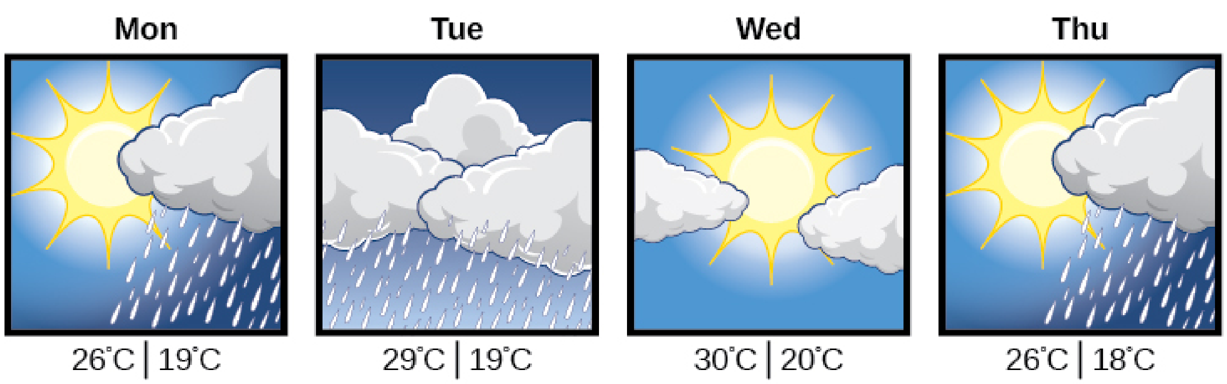
$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

Equation:

$$\frac{5}{9}(75 - 32) \approx 24^\circ\text{C}.$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from [link] for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.



At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

Equation:

$$\begin{aligned} 26 &= \tfrac{5}{9}(F - 32) \\ 26 \cdot \tfrac{9}{5} &= F - 32 \\ F &= 26 \cdot \tfrac{9}{5} + 32 \approx 79 \end{aligned}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x),$ we represent its inverse as $f^{-1}(x),$ read as "f inverse of $x.$" The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number $a,$ so $f^{-1} \circ f$ equals the identity function, that is,
Equation:

$$\left(f^{-1} \circ f\right)(x) = f^{-1}\left(f(x)\right) = f^{-1}\left(y\right) = x$$

This holds for all x in the domain of $f.$ Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

Given a function $f(x),$ we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.
Equation:

$$\left(f^{-1} \circ f\right)(x) = f^{-1}\left(4x\right) = \frac{1}{4}(4x) = x$$

and
Equation:

$$\left(f \circ f^{-1}\right)(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x$$

A few coordinate pairs from the graph of the function $y = 4x$ are (−2, −8), (0, 0), and (2, 8). A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are (−8, −2), (0, 0), and (8, 2). If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Note:
Inverse Function
For any one-to-one function $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.
The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that
Equation:

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

and not all functions have inverses.

Example:
Exercise:

Problem:
Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution:

The inverse function reverses the input and output quantities, so if
Equation:

$$f(2) = 4,\ \text{then}\ f^{-1}(4) = 2;$$
$$f\left(5\right) = 12,\ \text{then}\ \text{f}^{-1}\left(12\right) = 5.$$

Alternatively, if we want to name the inverse function $g,$ then $g(4) = 2$ and $g(12) = 5.$

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See [link].

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Note:
Exercise:

Problem:

Given that $h^{-1}(6) = 2,$ what are the corresponding input and output values of the original function h?

Solution:

$h(2) = 6$

Note:
Given two functions $f(x)$ and $g(x),$ test whether the functions are inverses of each other.

1. Determine whether $f(g(x)) = x$ or $g(f(x)) = x$.
2. If both statements are true, then $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

Example:
Exercise:

Problem:
Testing Inverse Relationships Algebraically

If $f\left(x\right) = \frac{1}{x+2}$ and $g\left(x\right) = \frac{1}{x} - 2$, is $g = f^{-1}$?

Solution:
Equation:

$$\begin{aligned} g(f(x)) &= \frac{1}{\left(\frac{1}{x+2}\right)} - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

so
Equation:

$$g = f^{-1} \text{ and } f = g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.
Equation:

$$\begin{aligned} f(g(x)) &= \frac{1}{\frac{1}{x} - 2 + 2} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned}$$

Analysis

Notice the inverse operations are in reverse order of the operations from the original function.

Note:
Exercise:

Problem: If $f\left(x\right)=x^{3}-4$ and $g\left(x\right)=\sqrt[3]{x+4}$, is $g=f^{-1}$?

Solution:

Yes

Example:
Exercise:

Problem:
Determining Inverse Relationships for Power Functions

If $f(x)=x^{3}$ (the cube function) and $g(x)=\frac{1}{3}x$, is $g=f^{-1}$?

Solution:
Equation:

$$f\left(g\left(x\right)\right)=\frac{x^{3}}{27}\neq x$$

No, the functions are not inverses.

Analysis

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x}=x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

Note:
Exercise:

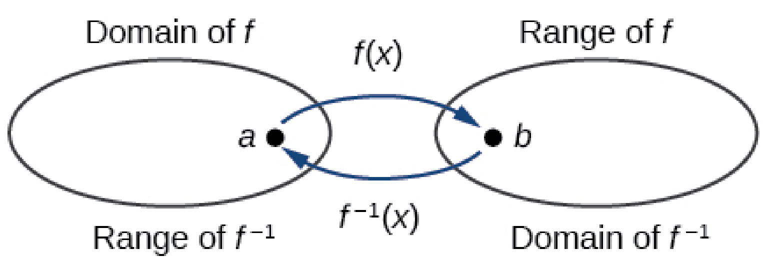
Problem: If $f\left(x\right)={\left(x-1\right)}^{3}$ and $g\left(x\right)=\sqrt[3]{x}+1,$ is $g={f}^{-1}?$

Solution:

Yes

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to ${f}^{-1},$ so the range of f is also the domain of ${f}^{-1}.$ Likewise, because the inputs to f are the outputs of ${f}^{-1},$ the domain of f is the range of ${f}^{-1}.$ We can visualize the situation as in [link].



Domain and range of a function and its inverse

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x)=\sqrt{x}$ is ${f}^{-1}(x)={x}^{2},$ because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $\left[0,\infty \right),$ since that is the range of $f(x)=\sqrt{x}.$

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x)={x}^{2}.$ If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the

horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its domain limited to $[0, \infty),$ which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x - 1)^2$ on $[1, \infty),$ then the inverse function is $f^{-1}(x) = \sqrt{x} + 1.$

- The domain of f = range of f^{-1} = $[1, \infty).$
- The domain of f^{-1} = range of f = $[0, \infty).$

Note:
Is it possible for a function to have more than one inverse?
No. If two supposedly different functions, say, g and $h,$ both meet the definition of being inverses of another function $f,$ then you can prove that $g = h.$ We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Note:
Domain and Range of Inverse Functions
The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x).$
The domain of $f(x)$ is the range of $f^{-1}(x).$

Note:
Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Example:
Exercise:

Problem:
Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in [link]. We restrict the domain in such a fashion that the function assumes all *y*-values exactly once.

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	Cube root	Square root	Absolute value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \|x\|$	

Solution:

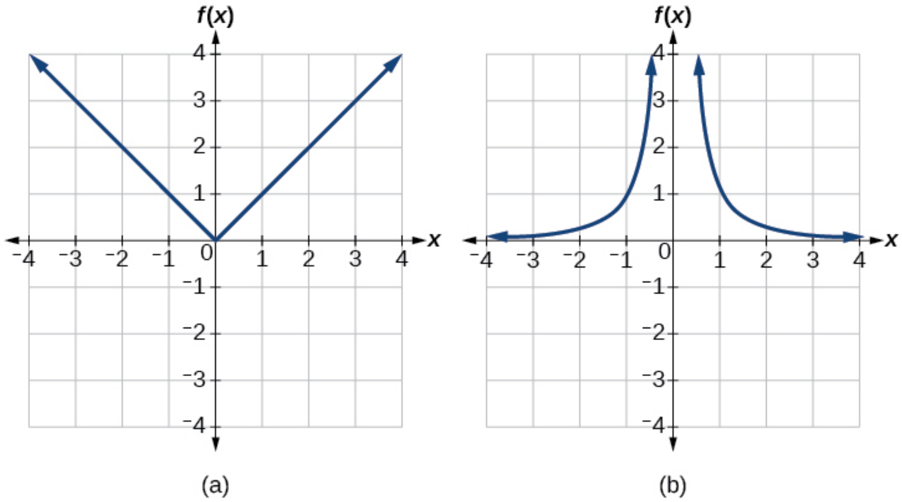
The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$,where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Analysis

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in [link]. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.



(a) Absolute value (b) Reciprocal squared

Note:
Exercise:

Problem:

The domain of function f is $(1,\infty)$ and the range of function f is $(-\infty,-2)$. Find the domain and range of the inverse function.

Solution:

The domain of function f^{-1} is $(-\infty,-2)$ and the range of function f^{-1} is $(1,\infty)$.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example:
Exercise:

Problem:
Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in [link], showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Solution:

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Note:
Exercise:

Problem: Using [link], find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Solution:

a. $f(60)=50$. In 60 minutes, 50 miles are traveled.
b. $f^{-1}(60)=70$. To travel 60 miles, it will take 70 minutes.

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in Functions and Function Notation that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

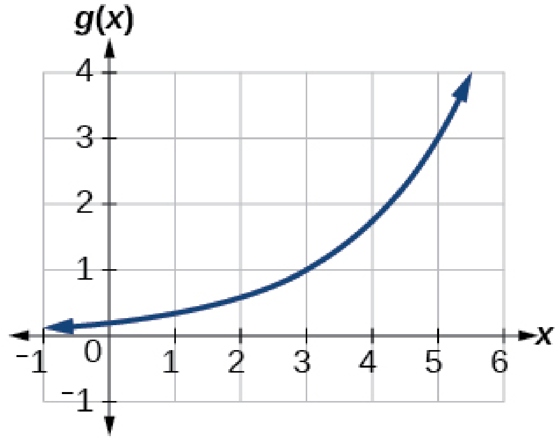
Note:
Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the *y*-axis of the given graph.
2. Read the inverse function's output from the *x*-axis of the given graph.

Example:
Exercise:

Problem:
Evaluating a Function and Its Inverse from a Graph at Specific Points

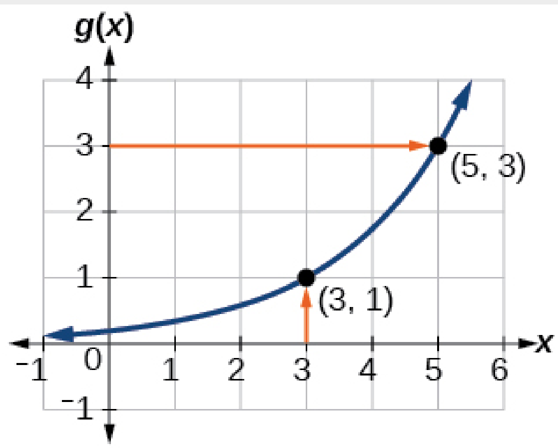
A function $g(x)$ is given in [link]. Find $g(3)$ and $g^{-1}(3)$.



Solution:

To evaluate $g(3)$, we find 3 on the *x*-axis and find the corresponding output value on the *y*-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of *x* for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See [link].



Note:
Exercise:

Problem: Using the graph in [link], (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

Solution:

a. 3; b. 5.6

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, y as a function of x— we can often find the inverse function by solving to obtain x as a function of y.

Note:
Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x.
3. Interchange x and y.

Example:
Exercise:

Problem:
Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.
Equation:

$$C = \frac{5}{9}(F - 32)$$

Solution:
Equation:

$$\begin{aligned} C &= \tfrac{5}{9}(F - 32) \\ C \cdot \tfrac{9}{5} &= F - 32 \\ F &= \tfrac{9}{5}C + 32 \end{aligned}$$

By solving in general, we have uncovered the inverse function. If
Equation:

$$C = h(F) = \frac{5}{9}(F - 32),$$

then
Equation:

$$F = h^{-1}(C) = \frac{9}{5}C + 32.$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Note:
Exercise:

Problem: Solve for x in terms of y given $y = \frac{1}{3}(x - 5)$

Solution:

$x = 3y + 5$

Example:
Exercise:

Problem:
Solving to Find an Inverse Function

Find the inverse of the function $f\left(x\right)=\frac{2}{x-3}+4$.

Solution:
Equation:

$$\begin{array}{rll} y & = \frac{2}{x-3}+4 & \text{Set up an equation.} \\ y-4 & = \frac{2}{x-3} & \text{Subtract 4 from both sides.} \\ x-3 & = \frac{2}{y-4} & \text{Multiply both sides by } x-3 \text{ and divide by } y-4. \\ x & = \frac{2}{y-4}+3 & \text{Add 3 to both sides.} \end{array}$$

So $f^{-1}\left(y\right)=\frac{2}{y-4}+3$ or $f^{-1}\left(x\right)=\frac{2}{x-4}+3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating [link].

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example:
Exercise:

Problem:
Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x)=2+\sqrt{x-4}$.

Solution:
Equation:

$$\begin{aligned} y &= 2+\sqrt{x-4} \\ (y-2)^2 &= x-4 \\ x &= (y-2)^2+4 \end{aligned}$$

So $f^{-1}(x)=(x-2)^2+4$.

The domain of f is $[4,\infty)$. Notice that the range of f is $[2,\infty)$, so this means that the domain of the inverse function f^{-1} is also $[2,\infty)$.

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2,\infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Note:
Exercise:

Problem:

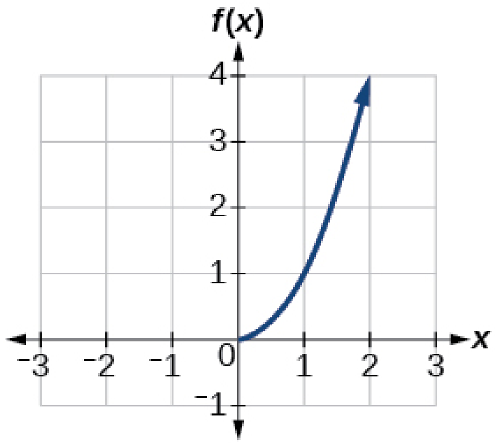
What is the inverse of the function $f(x)=2-\sqrt{x}$?State the domains of both the function and the inverse function.

Solution:

$f^{-1}(x)=(2-x)^2;\ \text{domain of } f:\ [0,\infty);\ \text{domain of } f^{-1}:\ (-\infty,2]$

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x)=x^2$ restricted to the domain $[0,\infty)$, on which this function is one-to-one, and graph it as in [link].

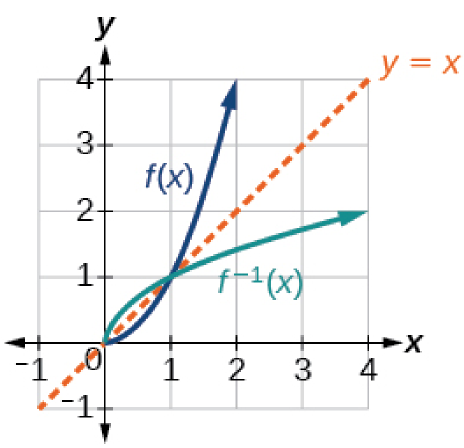


Quadratic function with domain restricted to [0, ∞).

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in [link].



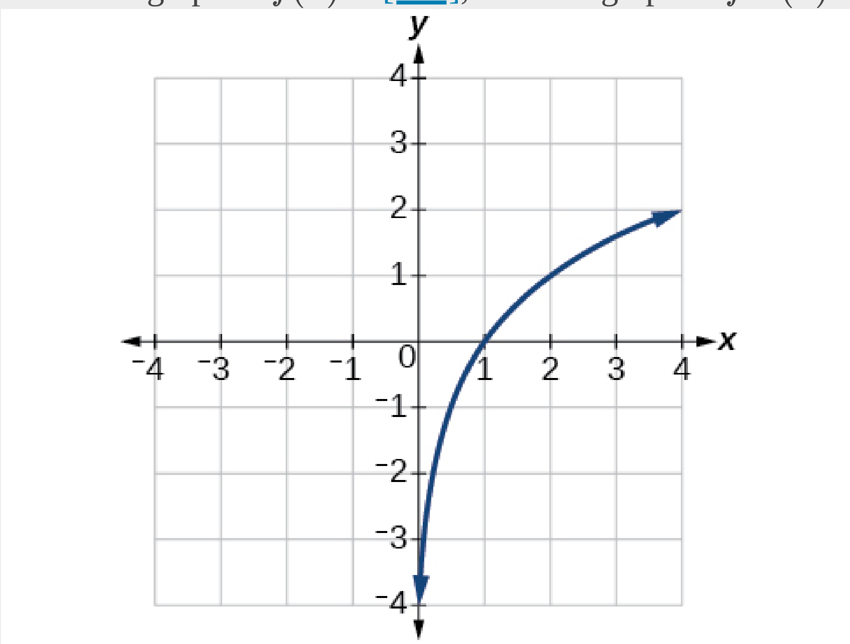
Square and square-root functions on the non-negative domain

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example:
Exercise:

Problem:
Finding the Inverse of a Function Using Reflection about the Identity Line

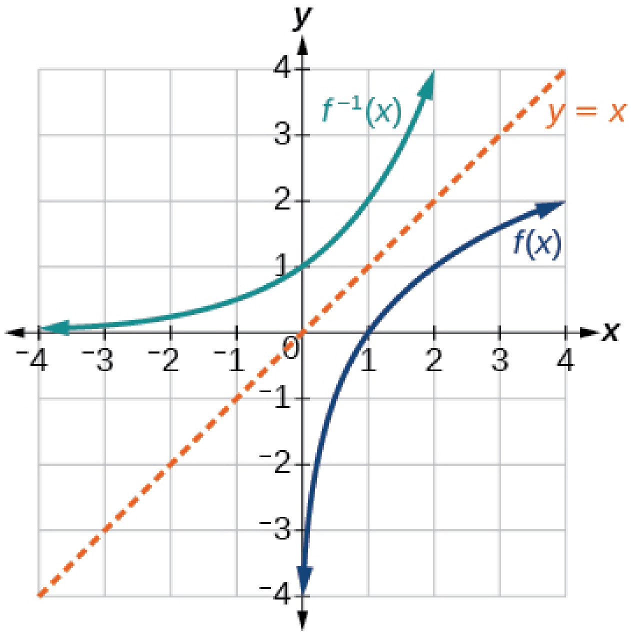
Given the graph of $f(x)$ in [link], sketch a graph of $f^{-1}(x)$.



Solution:

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives [link].

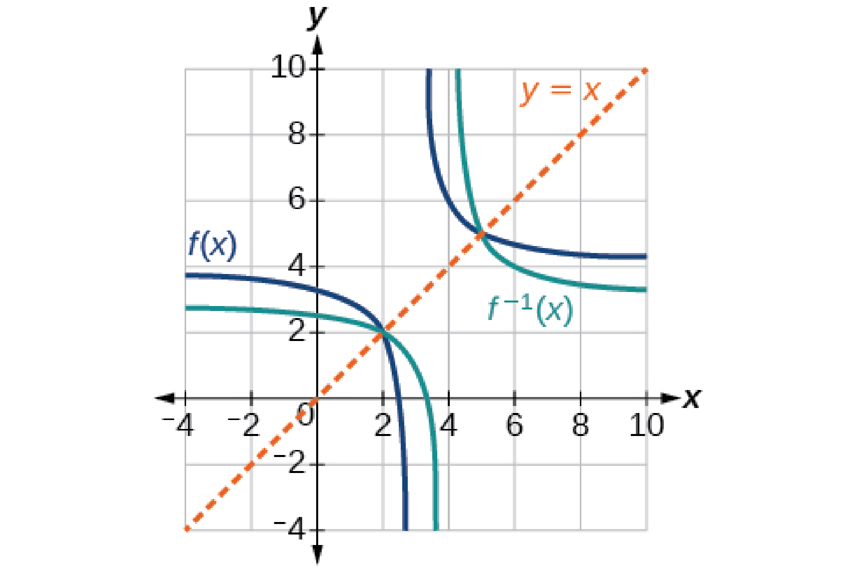


The function and its inverse, showing reflection about the identity line

Note:
Exercise:

Problem: Draw graphs of the functions f and $f^{-1}$from [link].

Solution:



Note:

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f\left(f\left(x\right)\right) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

Equation:

$$\frac{1}{\frac{1}{x}} = x$$

Any function $f\left(x\right) = c - x$, where c is a constant, is also equal to its own inverse.

Note:

Access these online resources for additional instruction and practice with inverse functions.

- [Inverse Functions]
- [Inverse Function Values Using Graph]
- [Restricting the Domain and Finding the Inverse]

Visit this website for additional practice questions from Learningpod.

Key Concepts

- If $g(x)$ is the inverse of $f(x),$ then $g(f(x))=f(g(x))=x.$ See [link], [link], and [link].
- Each of the toolkit functions has an inverse. See [link].
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse. See [link].
- The inverse of a function can be determined at specific points on its graph. See [link].
- To find the inverse of a formula, solve the equation $y=f(x)$ for x as a function of $y.$ Then exchange the labels x and $y.$ See [link], [link], and [link].
- The graph of an inverse function is the reflection of the graph of the original function across the line $y=x.$ See [link].

Section Exercises

Verbal

Exercise:

Problem:

Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?

Solution:

Each output of a function must have exactly one output for the function to be one-to-one. If any horizontal line crosses the graph of a function more than once, that means that y-values repeat and the function is not one-to-one. If no horizontal line crosses the graph of the function more than once, then no y-values repeat and the function is one-to-one.

Exercise:

Problem:

Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?

Exercise:

Problem: Can a function be its own inverse? Explain.

Solution:

Yes. For example, $f(x) = \frac{1}{x}$ is its own inverse.

Exercise:

Problem:

Are one-to-one functions either always increasing or always decreasing? Why or why not?

Exercise:

Problem: How do you find the inverse of a function algebraically?

Solution:

Given a function $y = f(x)$, solve for x in terms of y. Interchange the x and y. Solve the new equation for y. The expression for y is the inverse, $y = f^{-1}(x)$.

Algebraic

Exercise:

Problem:

Show that the function $f(x) = a - x$ is its own inverse for all real numbers a.

For the following exercises, find $f^{-1}(x)$ for each function.

Exercise:

Problem: $f(x) = x + 3$

Solution:

$f^{-1}(x) = x - 3$

Exercise:

Problem: $f(x) = x + 5$

Exercise:

Problem: $f(x) = 2 - x$

Solution:

$f^{-1}(x) = 2 - x$

Exercise:

Problem: $f(x) = 3 - x$

Exercise:

Problem: $f(x) = \frac{x}{x+2}$

Solution:

$f^{-1}(x) = \frac{-2x}{x-1}$

Exercise:

Problem: $f(x) = \frac{2x+3}{5x+4}$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

Exercise:

Problem: $f(x) = (x+7)^2$

Solution:

domain of $f(x):\ [-7, \infty);\ f^{-1}(x) = \sqrt{x} - 7$

Exercise:

Problem: $f(x) = (x-6)^2$

Exercise:

Problem: $f(x) = x^2 - 5$

Solution:

domain of $f(x):\ [0,\infty);\ f^{-1}(x) = \sqrt{x+5}$

Exercise:

Problem: Given $f\,(x) = \frac{x}{2+x}$ and $g(x) = \frac{2x}{1-x}$:

a. Find $f(g(x))$ and $g(f(x))$.
b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

Solution:

a. $f(g(x)) = x$ and $g(f(x)) = x.$ b. This tells us that f and g are inverse functions

For the following exercises, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

Exercise:

Problem: $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$

Solution:

$f(g(x)) = x,\ g(f(x)) = x$

Exercise:

Problem: $f(x) = -3x + 5$ and $g(x) = \frac{x-5}{-3}$

Graphical

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

Exercise:

Problem: $f(x) = \sqrt{x}$

Solution:

one-to-one

Exercise:

Problem: $f(x) = \sqrt[3]{3x + 1}$

Exercise:

Problem: $f(x) = -5x + 1$

Solution:

one-to-one

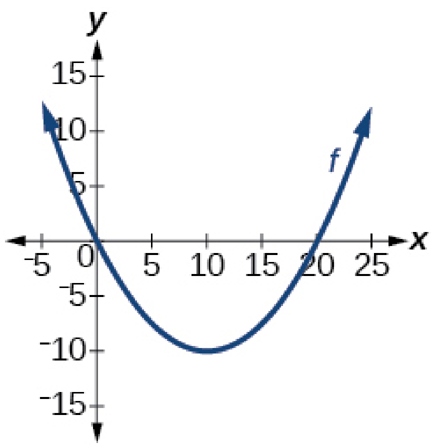
Exercise:

Problem: $f(x) = x^3 - 27$

For the following exercises, determine whether the graph represents a one-to-one function.

Exercise:

Problem:

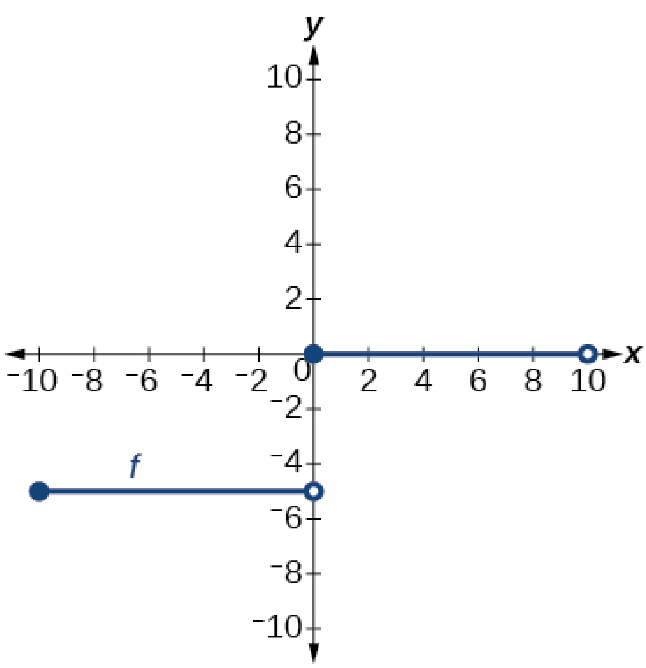


Solution:

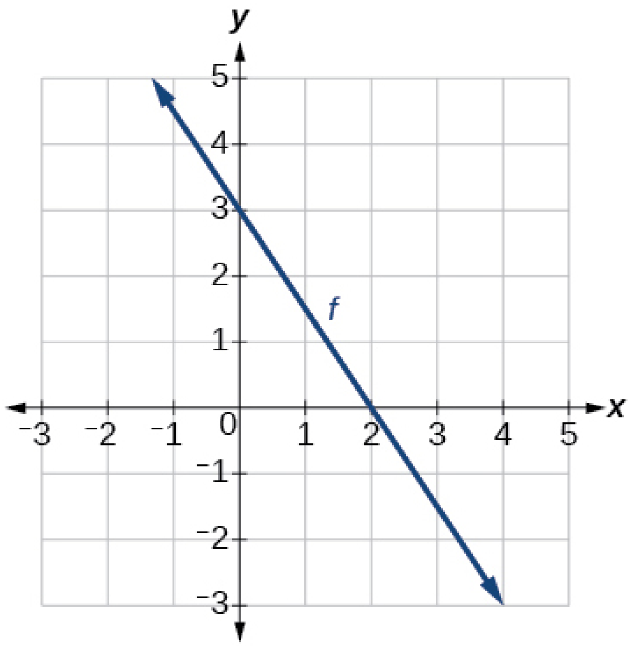
not one-to-one

Exercise:

Problem:



For the following exercises, use the graph of *f* shown in [link].



Exercise:

Problem: Find $f(0)$.

Solution:

3

Exercise:

Problem: Solve $f(x) = 0$.

Exercise:

Problem: Find $f^{-1}(0)$.

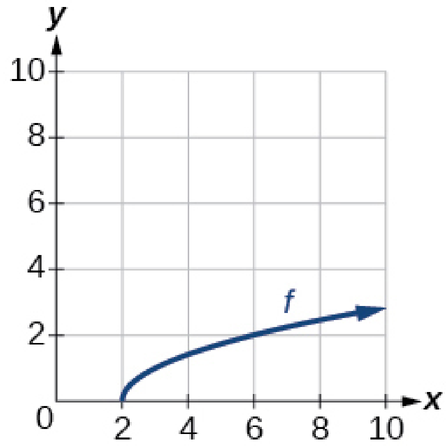
Solution:

2

Exercise:

Problem: Solve $f^{-1}(x) = 0$.

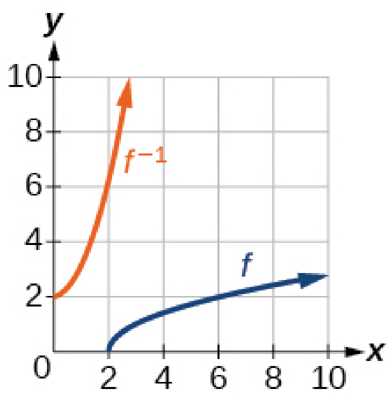
For the following exercises, use the graph of the one-to-one function shown in [link].



Exercise:

Problem: Sketch the graph of f^{-1}.

Solution:



Exercise:

Problem: Find $f(6)$ and $f^{-1}(2)$.

Exercise:

Problem: If the complete graph of f is shown, find the domain of f.

Solution:

$[2, 10]$

Exercise:

Problem: If the complete graph of f is shown, find the range of f.

Numeric

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

Exercise:

Problem: If $f(6) = 7,$ find $f^{-1}(7).$

Solution:

6

Exercise:

Problem: If $f(3) = 2,$ find $f^{-1}(2).$

Exercise:

Problem: If $f^{-1}(-4) = -8,$ find $f(-8).$

Solution:

-4

Exercise:

Problem: If $f^{-1}(-2) = -1,$ find $f(-1).$

For the following exercises, use the values listed in [link] to evaluate or solve.

x	$f(x)$

0	8
1	0
2	7
3	4
4	2
5	6
6	5
7	3
8	9
9	1

Exercise:

Problem: Find $f\left(1\right)$.

Solution:

0

Exercise:

Problem: Solve $f(x)=3$.

Exercise:

Problem: Find $f^{-1}\left(0\right)$.

Solution:

1

Exercise:

Problem: Solve $f^{-1}(x) = 7$.

Exercise:

Problem:

Use the tabular representation of f in [link] to create a table for $f^{-1}(x)$.

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

Solution:

x	1	4	7	12	16
$f^{-1}(x)$	3	6	9	13	14

Technology

For the following exercises, find the inverse function. Then, graph the function and its inverse.

Exercise:

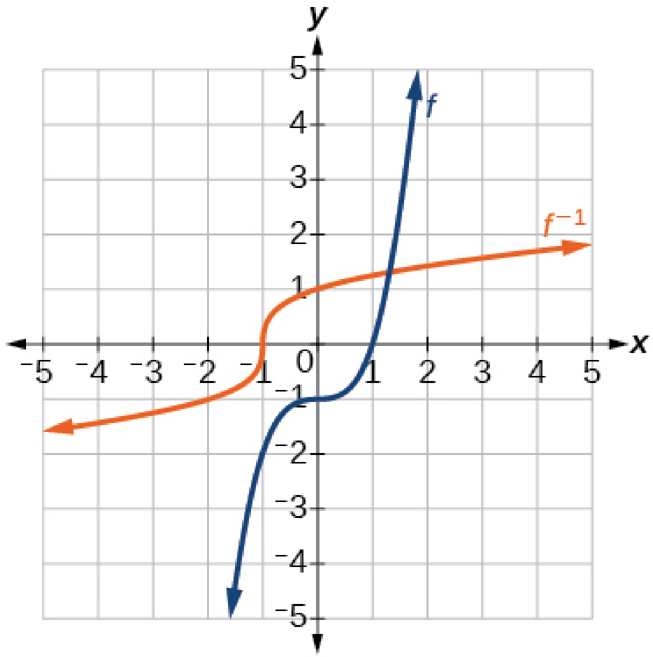
Problem: $f(x) = \frac{3}{x-2}$

Exercise:

Problem: $f(x) = x^3 - 1$

Solution:

$f^{-1}(x) = (1 + x)^{1/3}$



Exercise:

Problem:

Find the inverse function of $f(x) = \frac{1}{x-1}$. Use a graphing utility to find its domain and range. Write the domain and range in interval notation.

Real-World Applications

Exercise:

Problem:

To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \frac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.

Solution:

$f^{-1}(x) = \frac{5}{9}(x - 32).$ Given the Fahrenheit temperature, $x,$ this formula allows you to calculate the Celsius temperature.

Exercise:

Problem:

The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r.$ Express the radius of a circle as a function of its circumference. Call this function $r(C).$ Find $r(36\pi)$ and interpret its meaning.

Exercise:

Problem:

A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, $t,$ in hours given by $d(t) = 50t.$ Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d).$ Find $t(180)$ and interpret its meaning.

Solution:

$t(d) = \frac{d}{50},$ $t(180) = \frac{180}{50}.$ The time for the car to travel 180 miles is 3.6 hours.

Chapter Review Exercises

Functions and Function Notation

For the following exercises, determine whether the relation is a function.

Exercise:

Problem: $\{(a, b), (c, d), (e, d)\}$

Solution:

function

Exercise:

Problem: $\{(5, 2), (6, 1), (6, 2), (4, 8)\}$

Exercise:

Problem:

$y^2 + 4 = x,$ for x the independent variable and y the dependent variable

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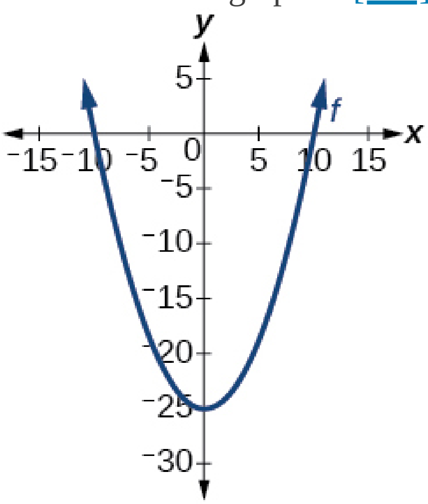
Solution:

not a function

Exercise:

Problem: Is the graph in [link] a function?



For the following exercises, evaluate the function at the indicated values:
$f(-3);\ f(2);\ f(-a);\ -f(a);\ f(a+h).$

Exercise:

Problem: $f(x) = -2x^2 + 3x$

Solution:

$f(-3) = -27; f(2) = -2; f(-a) = -2a^2 - 3a;$
$-f(a) = 2a^2 - 3a; f(a+h) = -2a^2 + 3a - 4ah + 3h - 2h^2$

Exercise:

Problem: $f(x) = 2\left|3x - 1\right|$

For the following exercises, determine whether the functions are one-to-one.

Exercise:

Problem: $f(x) = -3x + 5$

Solution:

one-to-one

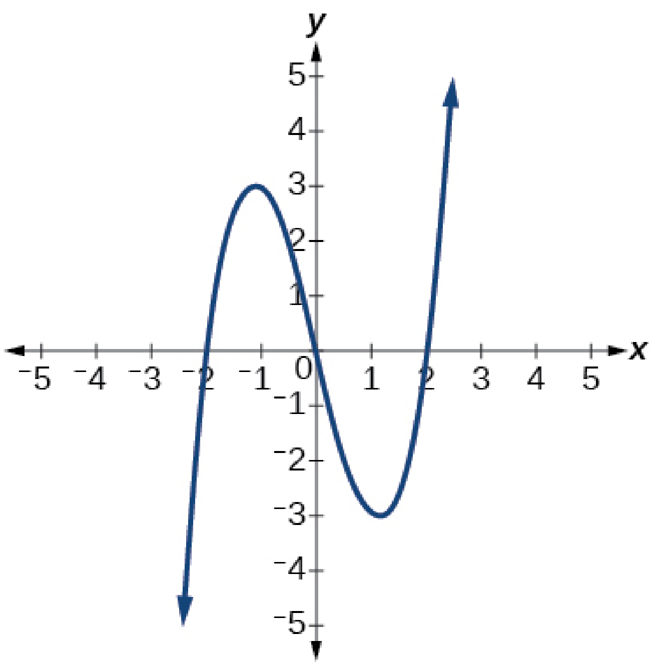
Exercise:

Problem: $f(x) = |x - 3|$

For the following exercises, use the vertical line test to determine if the relation whose graph is provided is a function.

Exercise:

Problem:

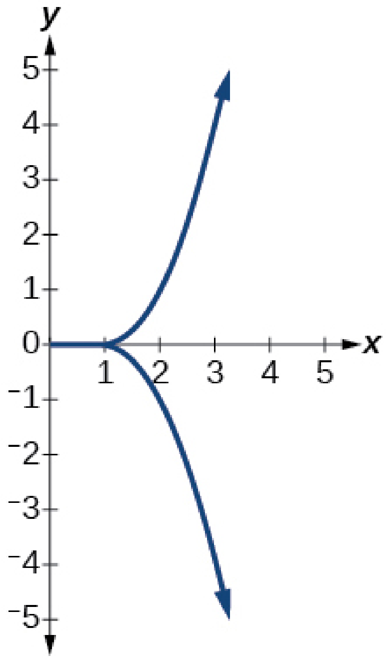


Solution:

function

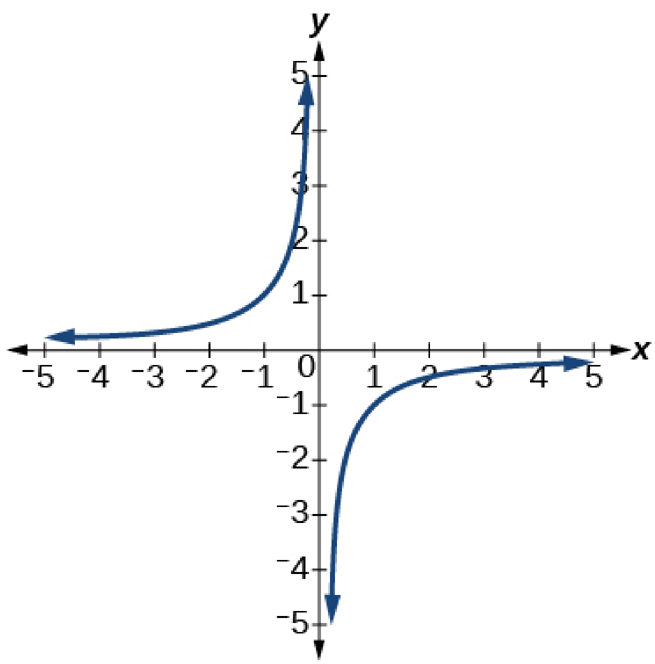
Exercise:

Problem:



Exercise:

Problem:



Solution:

function

For the following exercises, graph the functions.

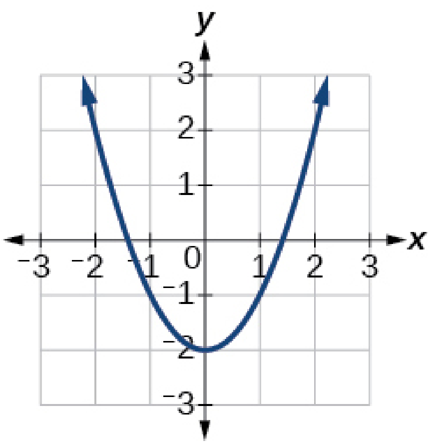
Exercise:

Problem: $f(x) = |x + 1|$

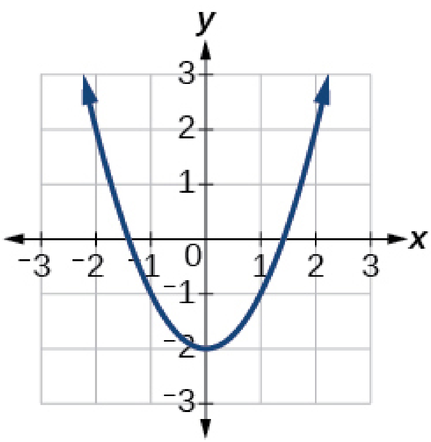
Exercise:

Problem: $f(x) = x^2 - 2$

Solution:



For the following exercises, use [link] to approximate the values.



Exercise:

Problem: $f(2)$

Exercise:

Problem: $f(-2)$

Solution:

2

Exercise:

Problem: If $f(x) = -2,$ then solve for $x.$

Exercise:

Problem: If $f(x) = 1,$ then solve for $x.$

Solution:

$x = -1.8$ or or $x = 1.8$

For the following exercises, use the function $h(t) = -16t^2 + 80t$ to find the values.

Exercise:

Problem: $\frac{h(2)-h(1)}{2-1}$

Exercise:

Problem: $\frac{h(a)-h(1)}{a-1}$

Solution:

$\frac{-64+80a-16a^2}{-1+a} = -16a + 64$

Domain and Range

For the following exercises, find the domain of each function, expressing answers using interval notation.

Exercise:

Problem: $f(x)=\frac{2}{3x+2}$

Exercise:

Problem: $f(x)=\frac{x-3}{x^2-4x-12}$

Solution:

$(-\infty,-2)\cup(-2,6)\cup(6,\infty)$

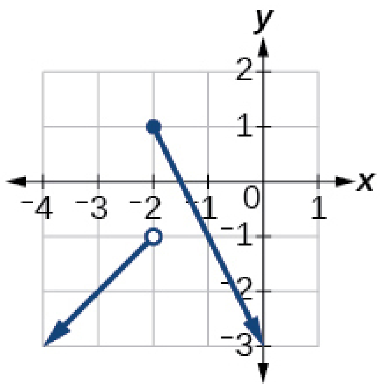
Exercise:

Problem: $f(x)=\frac{\sqrt{x-6}}{\sqrt{x-4}}$

Exercise:

Problem: Graph this piecewise function: $f(x)=\begin{cases}x+1 & x<-2\\ -2x-3 & x\geq -2\end{cases}$

Solution:



Rates of Change and Behavior of Graphs

For the following exercises, find the average rate of change of the functions from $x=1$ to $x=2$.

Exercise:

Problem: $f(x) = 4x - 3$

Exercise:

Problem: $f(x) = 10x^2 + x$

Solution:

31

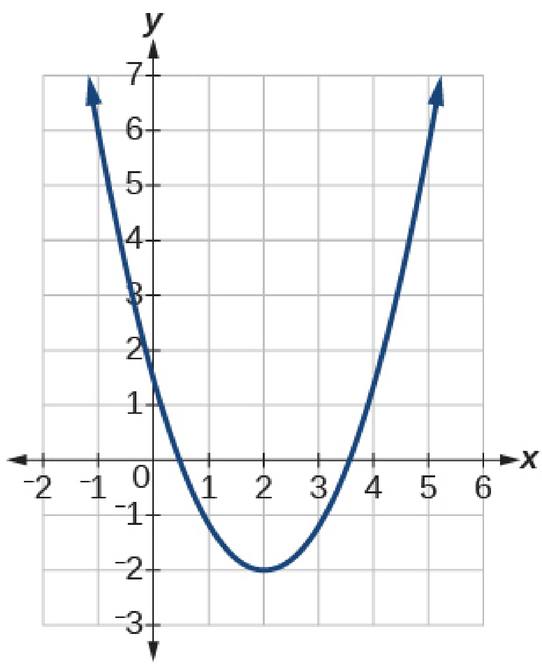
Exercise:

Problem: $f(x) = -\frac{2}{x^2}$

For the following exercises, use the graphs to determine the intervals on which the functions are increasing, decreasing, or constant.

Exercise:

Problem:

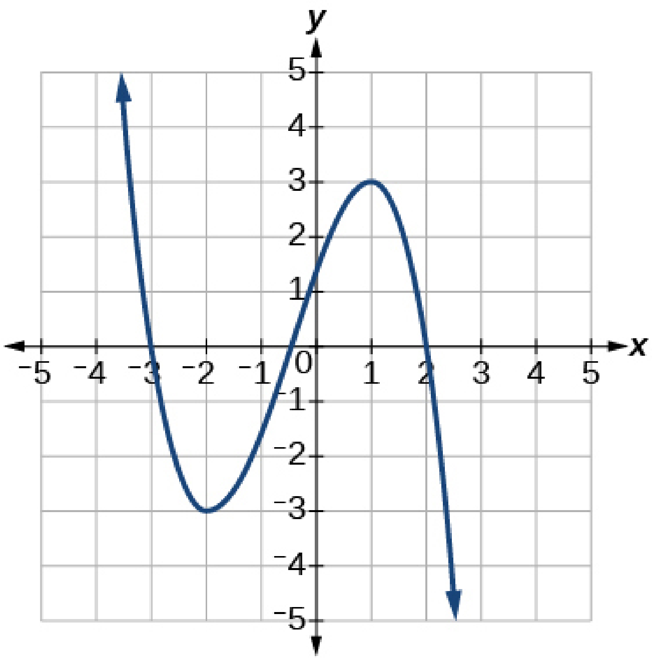


Solution:

increasing $(2, \infty)$; decreasing $(-\infty, 2)$

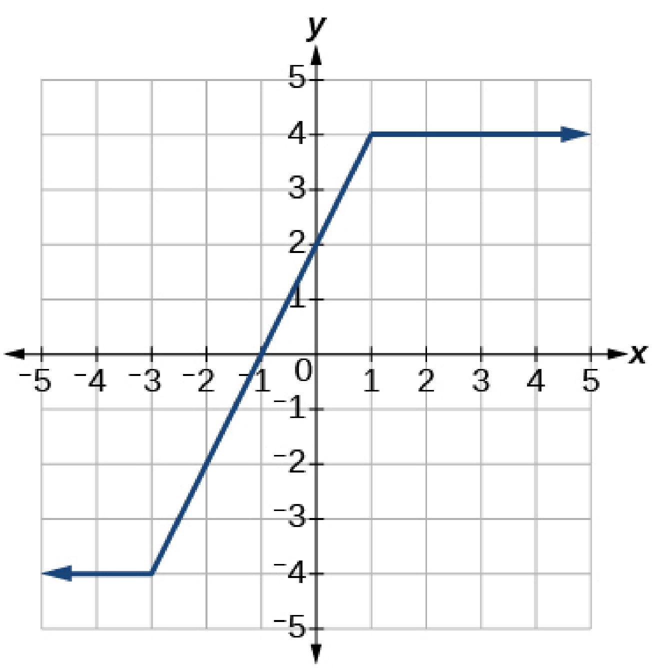
Exercise:

Problem:



Exercise:

Problem:



Solution:

increasing$(-3, 1)$; constant $(-\infty, -3) \cup (1, \infty)$

Exercise:

Problem: Find the local minimum of the function graphed in [link].

Exercise:

Problem: Find the local extrema for the function graphed in [link].

Solution:

local minimum $(-2, -3)$; local maximum $(1, 3)$

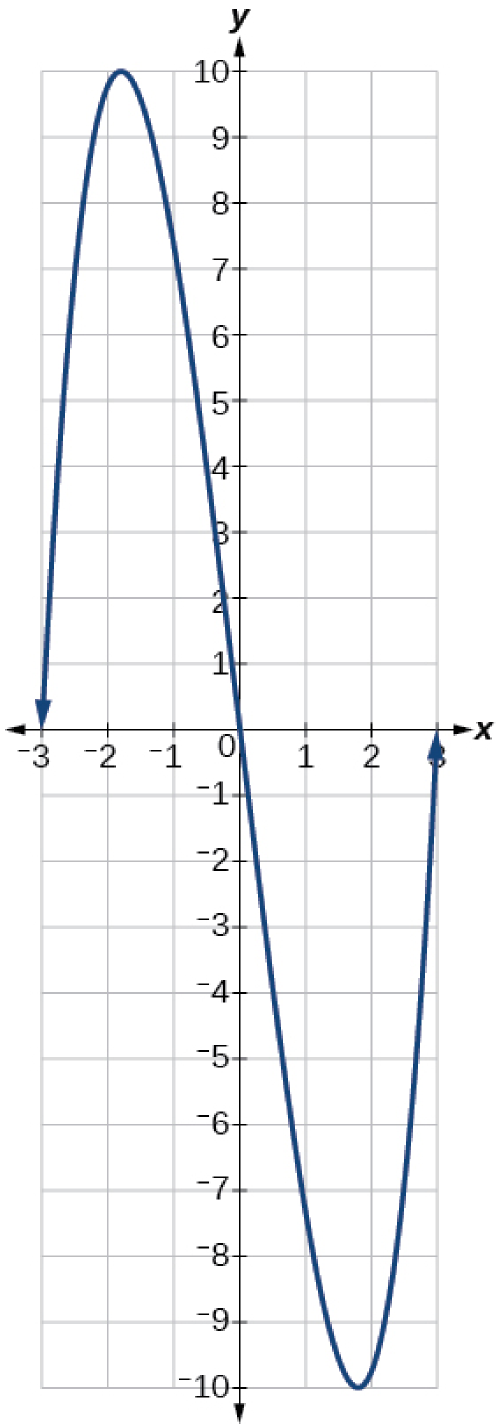
Exercise:

Problem:

For the graph in [link], the domain of the function is $[-3, 3]$.The range is $[-10, 10]$. Find the absolute minimum of the function on this interval.

Exercise:

Problem: Find the absolute maximum of the function graphed in [link].



Solution:

$(-1.8, 10)$

Composition of Functions

For the following exercises, find $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions.

Exercise:

Problem: $f(x) = 4 - x,\ g(x) = -4x$

Exercise:

Problem: $f(x) = 3x + 2,\ g(x) = 5 - 6x$

Solution:

$(f \circ g)(x) = 17 - 18x;\ (g \circ f)(x) = -7 - 18x$

Exercise:

Problem: $f(x) = x^2 + 2x,\ g(x) = 5x + 1$

Exercise:

Problem: $f(x) = \sqrt{x+2},\ g(x) = \frac{1}{x}$

Solution:

$(f \circ g)(x) = \sqrt{\frac{1}{x} + 2};\ (g \circ f)(x) = \frac{1}{\sqrt{x+2}}$

Exercise:

Problem: $f(x) = \frac{x+3}{2},\ g(x) = \sqrt{1-x}$

For the following exercises, find $(f \circ g)$ and the domain for $(f \circ g)(x)$ for each pair of functions.

Exercise:

Problem: $f(x) = \frac{x+1}{x+4},\ g(x) = \frac{1}{x}$

Solution:

$(f \circ g)(x) = \frac{1+x}{1+4x},\ x \neq 0,\ x \neq -\frac{1}{4}$

Exercise:

Problem: $f(x) = \frac{1}{x+3}, \ g(x) = \frac{1}{x-9}$

Exercise:

Problem: $f(x) = \frac{1}{x}, \ g(x) = \sqrt{x}$

Solution:

$(f \circ g)(x) = \frac{1}{\sqrt{x}}, \ x > 0$

Exercise:

Problem: $f(x) = \frac{1}{x^2-1}, \ g(x) = \sqrt{x+1}$

For the following exercises, express each function H as a composition of two functions f and g where $H(x) = (f \circ g)(x)$.

Exercise:

Problem: $H(x) = \sqrt{\frac{2x-1}{3x+4}}$

Solution:

sample: $g(x) = \frac{2x-1}{3x+4}; \ f(x) = \sqrt{x}$

Exercise:

Problem: $H(x) = \frac{1}{(3x^2-4)^{-3}}$

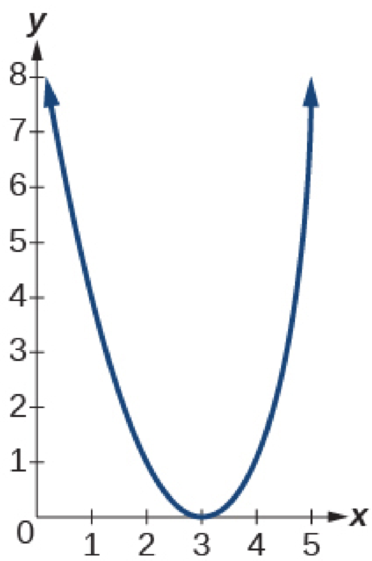
Transformation of Functions

For the following exercises, sketch a graph of the given function.

Exercise:

Problem: $f(x) = (x-3)^2$

Solution:



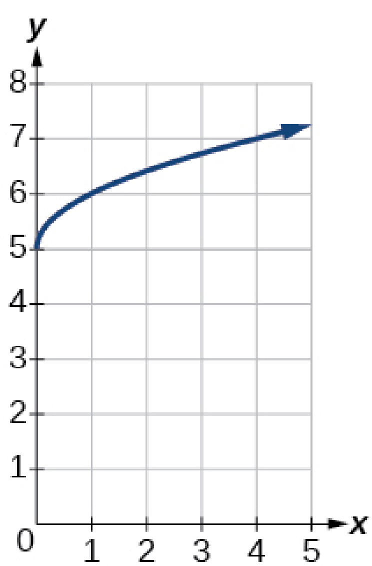
Exercise:

Problem: $f(x) = (x + 4)^3$

Exercise:

Problem: $f(x) = \sqrt{x} + 5$

Solution:



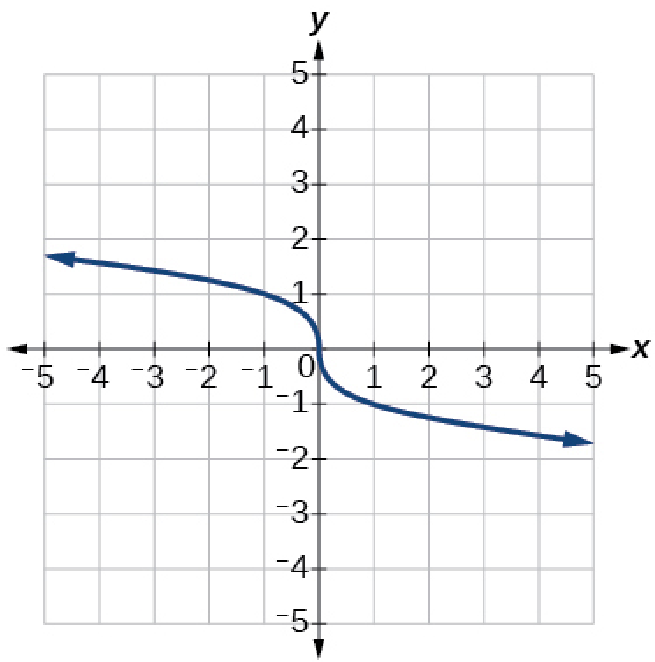
Exercise:

Problem: $f(x) = -x^3$

Exercise:

Problem: $f(x) = \sqrt[3]{-x}$

Solution:



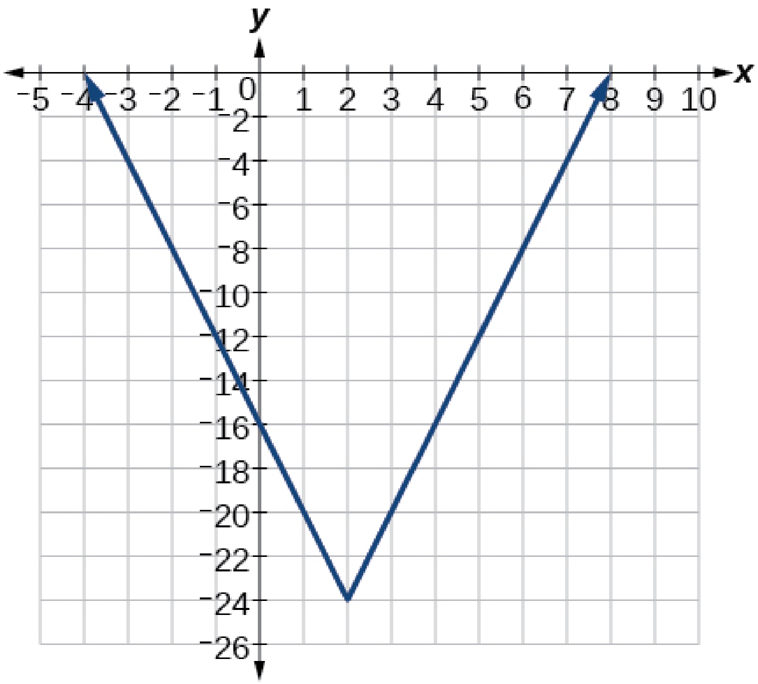
Exercise:

Problem: $f(x) = 5\sqrt{-x} - 4$

Exercise:

Problem: $f(x) = 4\left[|x-2| - 6\right]$

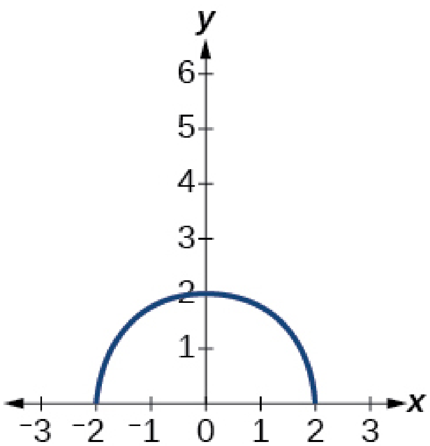
Solution:



Exercise:

Problem: $f(x) = -(x+2)^2 - 1$

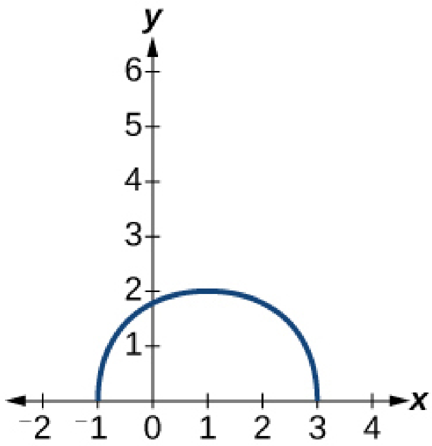
For the following exercises, sketch the graph of the function g if the graph of the function f is shown in [link].



Exercise:

Problem: $g(x) = f(x-1)$

Solution:



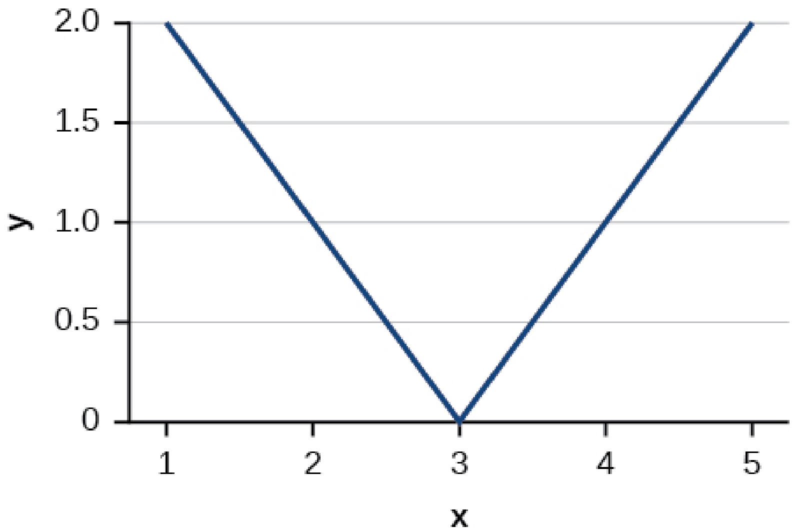
Exercise:

Problem: $g(x) = 3f(x)$

For the following exercises, write the equation for the standard function represented by each of the graphs below.

Exercise:

Problem:

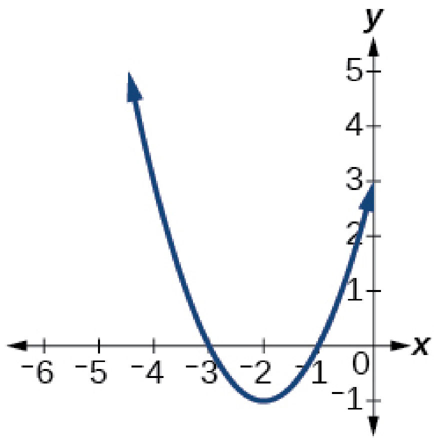


Solution:

$f(x) = |x - 3|$

Exercise:

Problem:



For the following exercises, determine whether each function below is even, odd, or neither.

Exercise:

Problem: $f(x) = 3x^4$

Solution:

even

Exercise:

Problem: $g(x) = \sqrt{x}$

Exercise:

Problem: $h(x) = \frac{1}{x} + 3x$

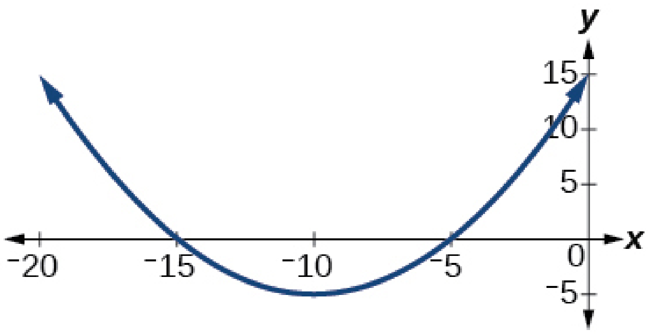
Solution:

odd

For the following exercises, analyze the graph and determine whether the graphed function is even, odd, or neither.

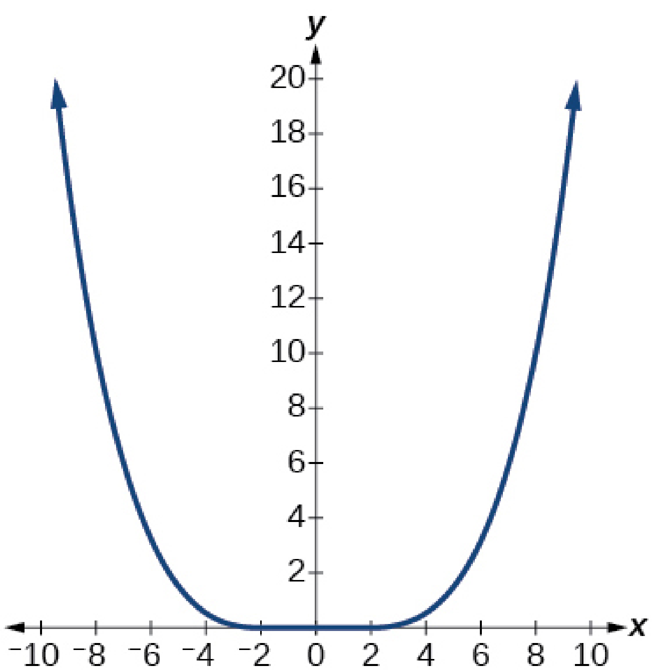
Exercise:

Problem:



Exercise:

Problem:

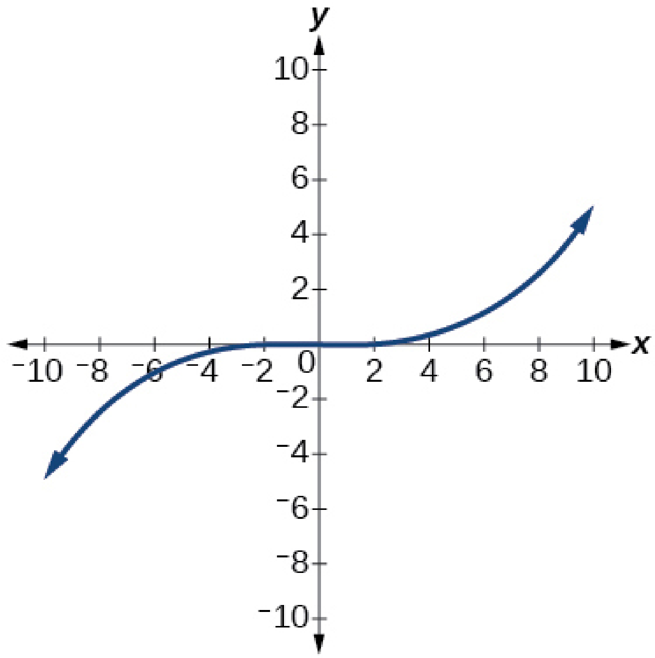


Solution:

even

Exercise:

Problem:

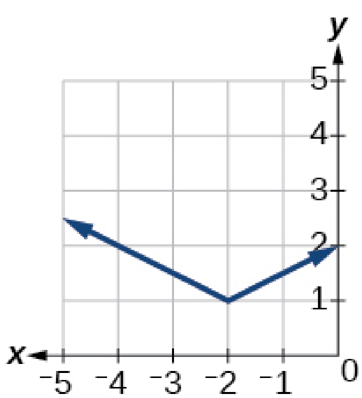


Absolute Value Functions

For the following exercises, write an equation for the transformation of $f(x) = |x|$.

Exercise:

Problem:

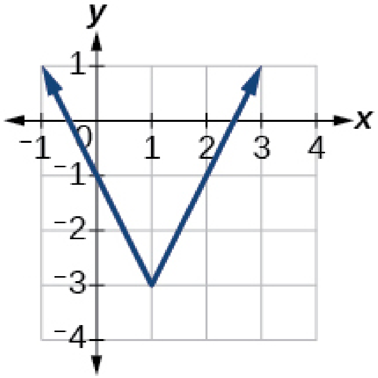


Solution:

$f(x) = \frac{1}{2}|x + 2| + 1$

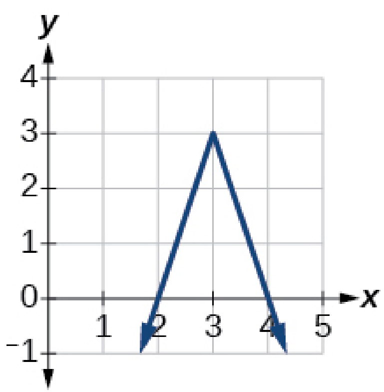
Exercise:

Problem:



Exercise:

Problem:



Solution:

$$f(x) = -3\left|x - 3\right| + 3$$

For the following exercises, graph the absolute value function.

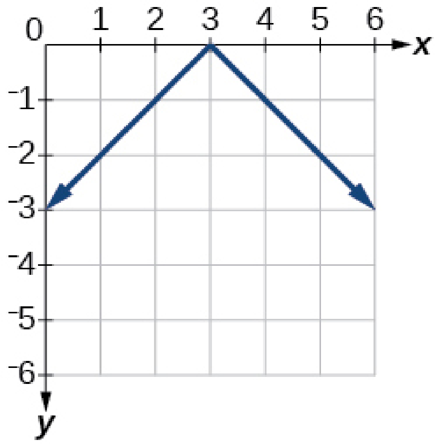
Exercise:

Problem: $f(x) = \left|x - 5\right|$

Exercise:

Problem: $f(x) = -\left|x - 3\right|$

Solution:



Exercise:

Problem: $f(x) = |2x - 4|$

For the following exercises, solve the absolute value equation.

Exercise:

Problem: $|x + 4| = 18$

Solution:

$x = -22,\ x = 14$

Exercise:

Problem: $\left|\frac{1}{3}x + 5\right| = \left|\frac{3}{4}x - 2\right|$

For the following exercises, solve the inequality and express the solution using interval notation.

Exercise:

Problem: $|3x - 2| < 7$

Solution:

$\left(-\frac{5}{3}, 3\right)$

Exercise:

Problem: $\left|\frac{1}{3}x - 2\right| \leq 7$

Inverse Functions

For the following exercises, find $f^{-1}(x)$ for each function.

Exercise:

Problem: $f(x) = 9 + 10x$

Exercise:

Problem: $f(x) = \frac{x}{x+2}$

Solution:

$f^{-1}(x) = \frac{-2x}{x-1}$

For the following exercise, find a domain on which the function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

Exercise:

Problem: $f(x) = x^2 + 1$

Exercise:

Problem: Given $f\left(x\right) = x^3 - 5$ and $g(x) = \sqrt[3]{x+5}$:

a. Find $f(g(x))$ and $g(f(x))$.
b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

Solution:

a. $f(g(x)) = x$ and $g(f(x)) = x$.
b. This tells us that f and g are inverse functions

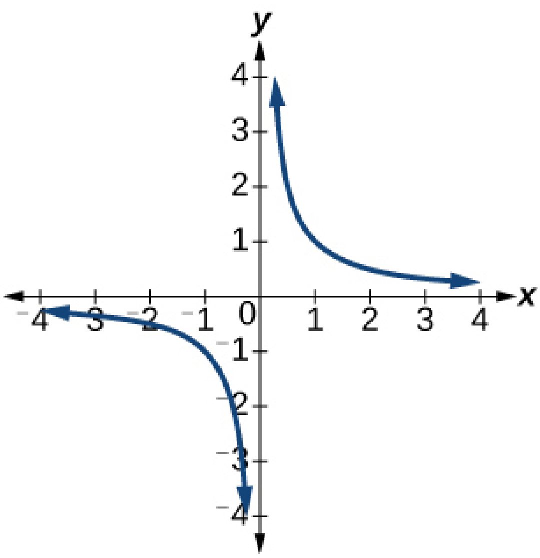
For the following exercises, use a graphing utility to determine whether each function is one-to-one.

Exercise:

Problem: $f(x) = \frac{1}{x}$

Solution:

The function is one-to-one.

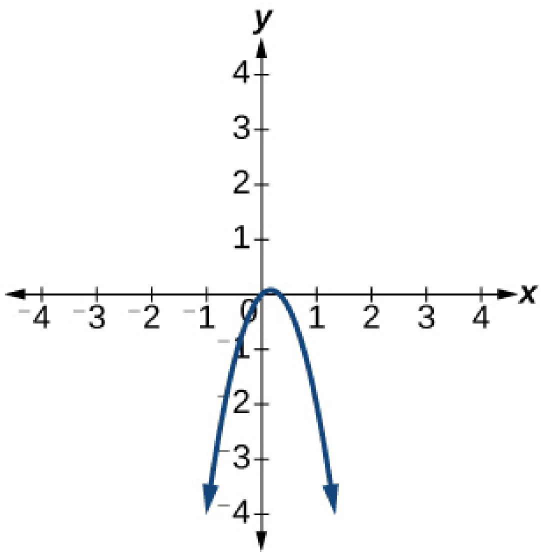


Exercise:

Problem: $f(x) = -3x^2 + x$

Solution:

The function is not one-to-one.



Exercise:

Problem: If $f(5) = 2$, find $f^{-1}(2)$.

Solution:

5

Exercise:

Problem: If $f(1) = 4$, find $f^{-1}(4)$.

Practice Test

For the following exercises, determine whether each of the following relations is a function.

Exercise:

Problem: $y = 2x + 8$

Solution:

The relation is a function.

Exercise:

Problem: $\{(2,1),(3,2),(-1,1),(0,-2)\}$

For the following exercises, evaluate the function $f(x)=-3x^2+2x$ at the given input.

Exercise:

Problem: $f(-2)$

Solution:

−16

Exercise:

Problem: $f(a)$

Exercise:

Problem: Show that the function $f(x)=-2(x-1)^2+3$ is not one-to-one.

Solution:

The graph is a parabola and the graph fails the horizontal line test.

Exercise:

Problem: Write the domain of the function $f(x)=\sqrt{3-x}$ in interval notation.

Exercise:

Problem: Given $f(x)=2x^2-5x,$ find $f(a+1)-f(1)$.

Solution:

$2a^2-a$

Exercise:

Problem: Graph the function $f(x)=\begin{cases} x+1 & \text{if} \quad -2<x<3 \\ -x & \text{if} \quad x\geq 3 \end{cases}$

Exercise:

Problem:

Find the average rate of change of the function $f(x) = 3 - 2x^2 + x$ by finding $\frac{f(b)-f(a)}{b-a}$.

Solution:

$-2(a+b)+1$

For the following exercises, use the functions $f(x) = 3 - 2x^2 + x$ and $g(x) = \sqrt{x}$ to find the composite functions.

Exercise:

Problem: $(g \circ f)(x)$

Exercise:

Problem: $(g \circ f)(1)$

Solution:

$\sqrt{2}$

Exercise:

Problem:

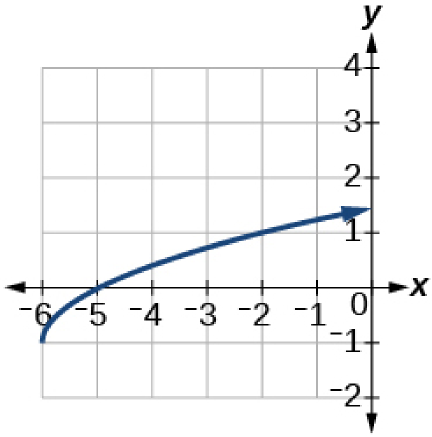
Express $H(x) = \sqrt[3]{5x^2 - 3x}$ as a composition of two functions, f and g, where $(f \circ g)(x) = H(x)$.

For the following exercises, graph the functions by translating, stretching, and/or compressing a toolkit function.

Exercise:

Problem: $f(x) = \sqrt{x+6} - 1$

Solution:



Exercise:

Problem: $f(x)=\frac{1}{x+2}-1$

For the following exercises, determine whether the functions are even, odd, or neither.

Exercise:

Problem: $f(x)=-\frac{5}{x^2}+9x^6$

Solution:

even

Exercise:

Problem: $f(x)=-\frac{5}{x^3}+9x^5$

Exercise:

Problem: $f(x)=\frac{1}{x}$

Solution:

odd

Exercise:

Problem: Graph the absolute value function $f(x)=-2\left|x-1\right|+3$.

Exercise:

Problem: Solve $|2x-3|=17$.

Solution:

$x=-7$ and $x=10$

Exercise:

Problem: Solve $-\left|\frac{1}{3}x-3\right|\geq 17$. Express the solution in interval notation.

For the following exercises, find the inverse of the function.

Exercise:

Problem: $f(x)=3x-5$

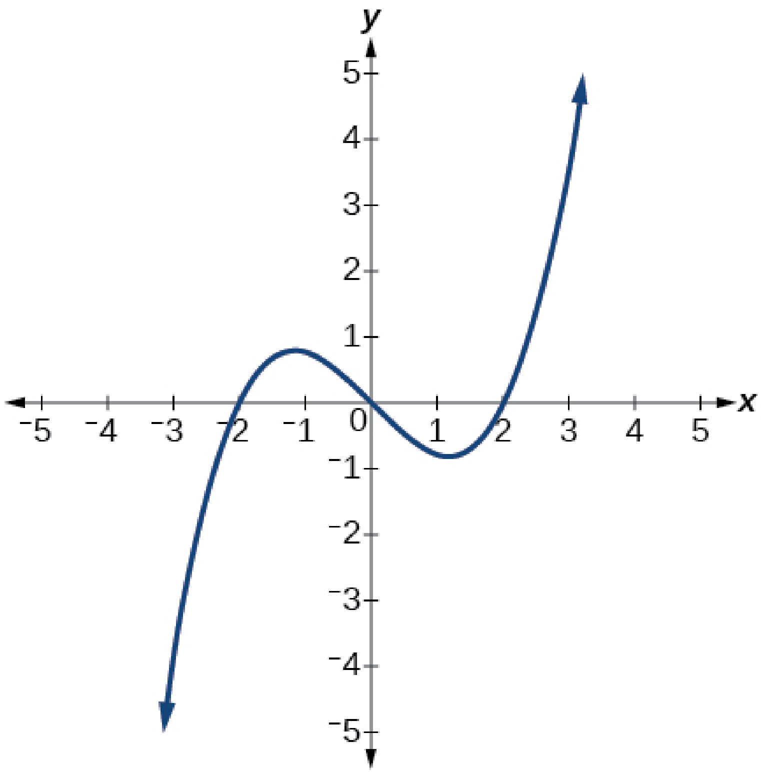
Solution:

$f^{-1}(x)=\frac{x+5}{3}$

Exercise:

Problem: $f(x)=\frac{4}{x+7}$

For the following exercises, use the graph of g shown in [link].



Exercise:

Problem: On what intervals is the function increasing?

Solution:

$(-\infty, -1.1)$ and $(1.1, \infty)$

Exercise:

Problem: On what intervals is the function decreasing?

Exercise:

Problem:

Approximate the local minimum of the function. Express the answer as an ordered pair.

Solution:

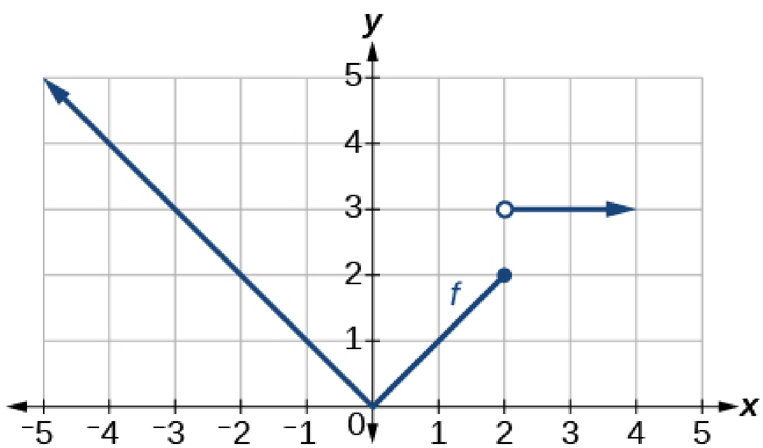
$(1.1, -0.9)$

Exercise:

Problem:

Approximate the local maximum of the function. Express the answer as an ordered pair.

For the following exercises, use the graph of the piecewise function shown in [link].



Exercise:

Problem: Find $f(2)$.

Solution:

$f(2) = 2$

Exercise:

Problem: Find $f(-2)$.

Exercise:

Problem: Write an equation for the piecewise function.

Solution:

$$f(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$$

For the following exercises, use the values listed in [link].

x	$F(x)$
0	1
1	3
2	5
3	7
4	9
5	11
6	13
7	15
8	17

Exercise:

Problem: Find $F(6)$.

Exercise:

Problem: Solve the equation $F(x) = 5$.

Solution:

$x = 2$

Exercise:

Problem: Is the graph increasing or decreasing on its domain?

Exercise:

Problem: Is the function represented by the graph one-to-one?

Solution:

yes

Exercise:

Problem: Find $F^{-1}(15)$.

Exercise:

Problem: Given $f(x) = -2x + 11,$ find $f^{-1}(x)$.

Solution:

$f^{-1}(x) = -\frac{x-11}{2}$

Glossary

inverse function
: for any one-to-one function $f(x),$ the inverse is a function $f^{-1}(x)$ such that $f^{-1}\left(f\left(x\right)\right) = x$ for all x in the domain of f; this also implies that $f\left(f^{-1}\left(x\right)\right) = x$ for all x in the domain of f^{-1}

Exponential Functions
In this section, you will:

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about 1.2% each year[footnote]. If this rate continues, the population of India will exceed China's population by the year 2031.When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.
http://www.worldometers.info/world-population/. Accessed February 24, 2014.

Identifying Exponential Functions

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$,the slope tells us the output increases by 3 each time the input increases by 1. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2021.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of exponential growth, let us contrast exponential growth with linear growth. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs. See [link].

x	$f(x)=2^x$	$g(x)=2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

From [link] we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

Apparently, the difference between “the same percentage” and “the same amount” is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

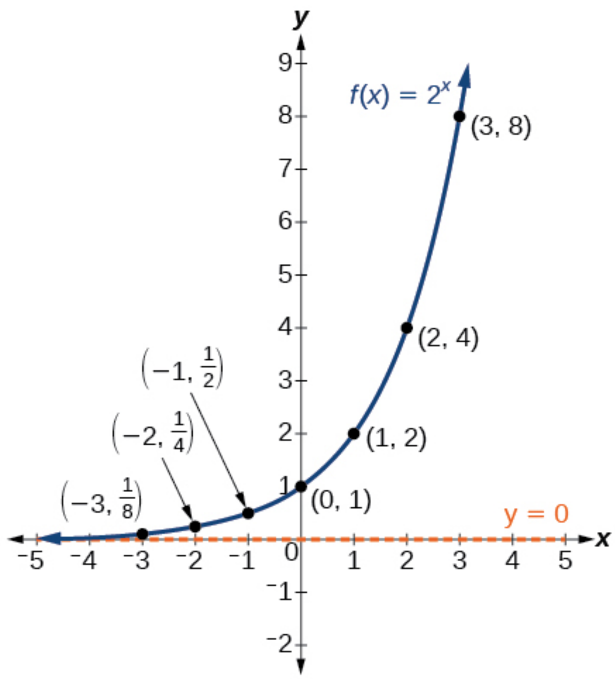
The general form of the exponential function is $f(x)=ab^x,$ where a is any nonzero number, b is a positive real number not equal to 1.

- If $b>1$,the function grows at a rate proportional to its size.
- If $0<b<1$, the function decays at a rate proportional to its size.

Let’s look at the function $f(x)=2^x$ from our example. We will create a table ([link]) to determine the corresponding outputs over an interval in the domain from -3 to 3.

x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$2^{-3}=\frac{1}{8}$	$2^{-2}=\frac{1}{4}$	$2^{-1}=\frac{1}{2}$	$2^0=1$	$2^1=2$	$2^2=4$	$2^3=8$

Let us examine the graph of f by plotting the ordered pairs we observe on the table in [link], and then make a few observations.



Let’s define the behavior of the graph of the exponential function $f(x) = 2^x$ and highlight some its key characteristics.

- the domain is $(-\infty, \infty)$,
- the range is $(0, \infty)$,
- as $x \to \infty$, $f(x) \to \infty$,
- as $x \to -\infty$, $f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the *y*-intercept is 1.

Note:
Exponential Function
For any real number x,an exponential function is a function with the form
Equation:

$$f(x) = ab^x$$

where

- a is a non-zero real number called the initial value and
- b is any positive real number such that $b \neq 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a > 0$.
- The range of f is all negative real numbers if $a < 0$.
- The *y*-intercept is $(0, a)$,and the horizontal asymptote is $y = 0$.

Example:
Exercise:

Problem:
Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution:

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Note:
Exercise:

Problem: Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Solution:

$g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions.

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base b to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{\frac{1}{2}} = \sqrt{-9}$, which is not a real number.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1 :

- Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x.

To evaluate an exponential function with the form $f(x) = b^x$,we simply substitute x with the given value, and calculate the resulting power. For example:

Let $f(x) = 2^x$. What is $f(3)$?
Equation:

$$\begin{aligned} f(x) &= 2^x & \\ f(3) &= 2^3 & \text{Substitute } x = 3. \\ &= 8 & \text{Evaluate the power.} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x) = 30(2)^x$. What is $f(3)$?
Equation:

$$\begin{aligned} f(x) &= 30(2)^x & \\ f(3) &= 30(2)^3 & \text{Substitute } x = 3. \\ &= 30(8) & \text{Simplify the power first.} \\ &= 240 & \text{Multiply.} \end{aligned}$$

Note that if the order of operations were not followed, the result would be incorrect:
Equation:

$$f(3) = 30(2)^3 \neq 60^3 = 216{,}000$$

Example:
Exercise:

Problem:
Evaluating Exponential Functions

Let $f(x) = 5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

Solution:

Follow the order of operations. Be sure to pay attention to the parentheses.
Equation:

$$\begin{aligned} f(x) &= 5(3)^{x+1} & \\ f(2) &= 5(3)^{2+1} & \text{Substitute } x = 2. \\ &= 5(3)^3 & \text{Add the exponents.} \\ &= 5(27) & \text{Simplify the power.} \\ &= 135 & \text{Multiply.} \end{aligned}$$

Note:
Exercise:

Problem: Let $f\left(x\right)=8{\left(1.2\right)}^{x-5}.$ Evaluate $f\left(3\right)$ using a calculator. Round to four decimal places.

Solution:

5.5556

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term “exponential growth” is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

Note:
Exponential Growth
A function that models **exponential growth** grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b\ne 1$,an exponential growth function has the form
Equation:

$$f(x)=a{b}^{x}$$

where

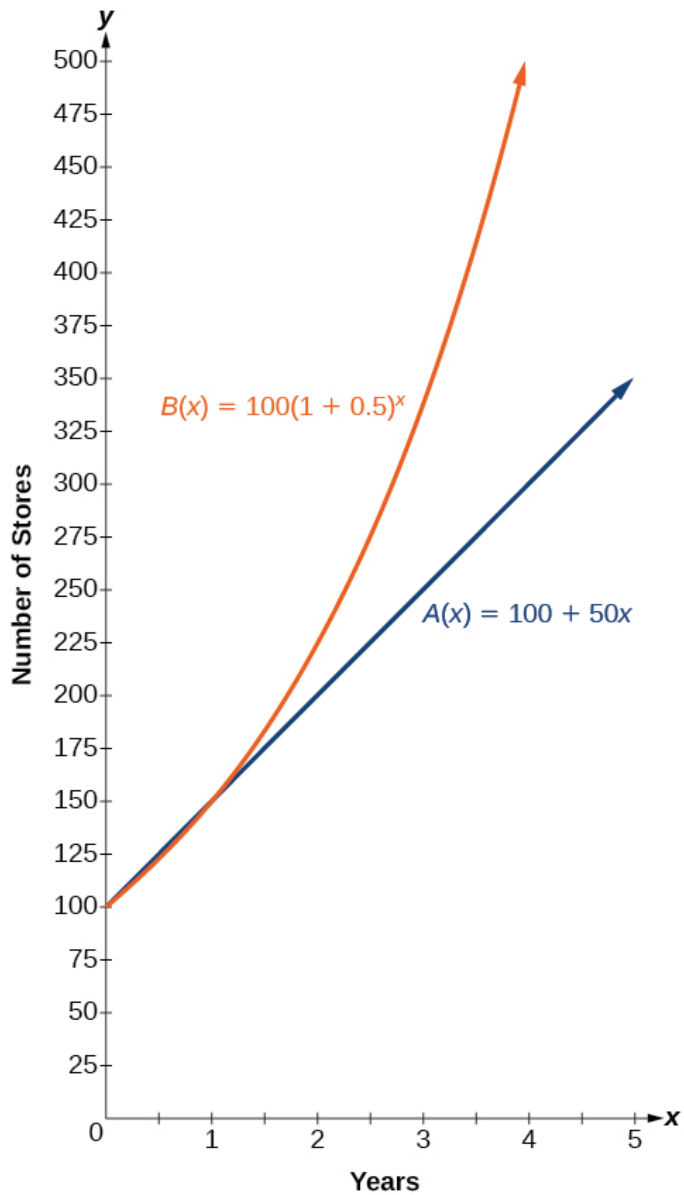
- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x.

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let’s consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A\left(x\right)=100+50x.$ Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x)=100{(1+0.5)}^{x}.$

A few years of growth for these companies are illustrated in [link].

Year, x	Stores, Company A	Stores, Company B
0	$100+50\left(0\right)=100$	$100{\left(1+0.5\right)}^{0}=100$
1	$100+50\left(1\right)=150$	$100{\left(1+0.5\right)}^{1}=150$
2	$100+50\left(2\right)=200$	$100{\left(1+0.5\right)}^{2}=225$
3	$100+50\left(3\right)=250$	$100{\left(1+0.5\right)}^{3}=337.5$
x	$A\left(x\right)=100+50x$	$B\left(x\right)=100{\left(1+0.5\right)}^{x}$

The graphs comparing the number of stores for each company over a five-year period are shown in [link]. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.



The graph shows the numbers of stores Companies A and B opened over a five-year period.

Notice that the domain for both functions is $[0, \infty)$,and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the growth rate, and $1 + 0.5 = 1.5$ represents the growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$,where 100 is the initial value, 1.5 is called the *base*, and x is called the *exponent*.

Example:
Exercise:

Problem:
Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about $1.2\%.$ This situation is represented by the growth function $P(t)=1.25(1.012)^t$, where t is the number of years since $2013.$ To the nearest thousandth, what will the population of India be in $2031?$

Solution:

To estimate the population in 2031, we evaluate the models for $t=18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

Equation:

$$P(18)=1.25(1.012)^{18}\approx 1.549$$

There will be about 1.549 billion people in India in the year 2031.

Note:
Exercise:

Problem:

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about $0.6\%.$ This situation is represented by the growth function $P(t)=1.39(1.006)^t$, where t is the number of years since $2013.$To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in [link]?

Solution:

About 1.548 billion people; by the year 2031, India's population will exceed China's by about 0.001 billion, or 1 million people.

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b,and evaluate the function.

Note:
Given two data points, write an exponential model.

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Example:
Exercise:

Problem:
Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population (N) of deer over time t.

Solution:

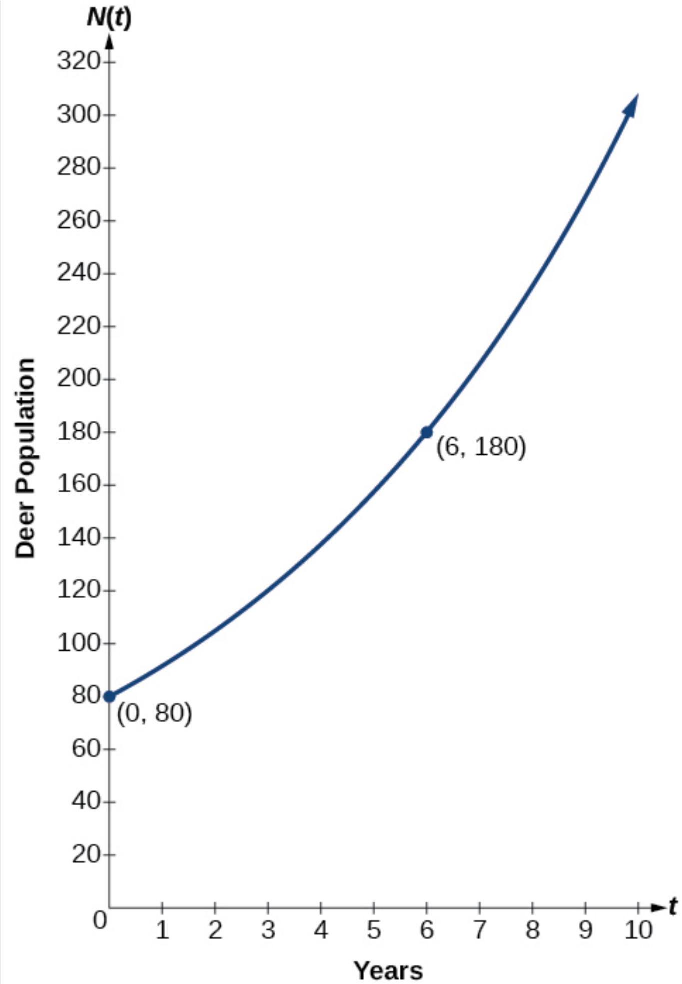
We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find b :
Equation:

$$\begin{array}{rll} N(t) & = 80b^t & \\ 180 & = 80b^6 & \text{Substitute using point } (6,\ 180). \\ \frac{9}{4} & = b^6 & \text{Divide and write in lowest terms.} \\ b & = \left(\frac{9}{4}\right)^{\frac{1}{6}} & \text{Isolate } b \text{ using properties of exponents.} \\ b & \approx 1.1447 & \text{Round to 4 decimal places.} \end{array}$$

NOTE: *Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.*

The exponential model for the population of deer is $N(t) = 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in [link] passes through the initial points given in the problem, $(0,\ 80)$ and $(6,\ 180)$. We can also see that the domain for the function is $[0, \infty)$,and the range for the function is $[80, \infty)$.



Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, t years after 2006

Note:
Exercise:

Problem:

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Solution:

$(0, 129)$ and $(2, 236)$; $N(t) = 129(1.3526)^t$

Example:
Exercise:

Problem:
Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

Solution:

Because we don’t have the initial value, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = ab^{-2}$
- Substituting $(2, 1)$ gives $1 = ab^2$

Use the first equation to solve for a in terms of b :

$6 = ab^{-2}$	
$\frac{6}{b^{-2}} = a$	Divide.
$a = 6b^2$	Use properties of exponents to rewrite the denominator.

Substitute a in the second equation, and solve for b :

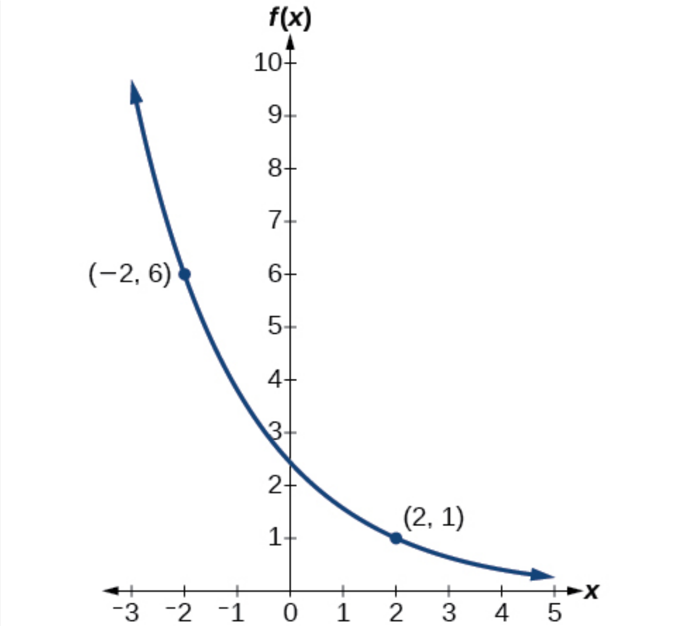
$1 = ab^2$	
$1 = 6b^2b^2 = 6b^4$	Substitute a.
$b = \left(\frac{1}{6}\right)^{\frac{1}{4}}$	Use properties of exponents to isolate b.
$b \approx 0.6389$	Round 4 decimal places.

Use the value of b in the first equation to solve for the value of a :

$$a = 6b^2 \approx 6(0.6389)^2 \approx 2.4492$$

Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in [link] passes through the initial points given in the problem, $(-2,\ 6)$ and $(2,\ 1)$. The graph is an example of an exponential decay function.



The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.

Note:
Exercise:

Problem:

Given the two points $(1, 3)$ and $(2, 4.5)$,find the equation of the exponential function that passes through these two points.

Solution:

$f(x) = 2(1.5)^x$

Note:
Do two points always determine a unique exponential function?
Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x*, which in many real world cases involves time.*

Note:

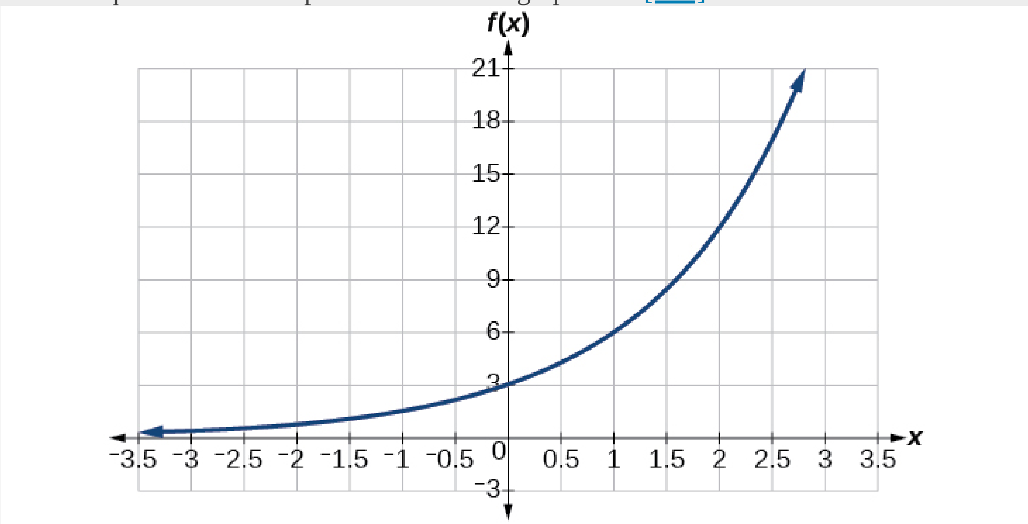
Given the graph of an exponential function, write its equation.

1. First, identify two points on the graph. Choose the y-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the y-intercept $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
3. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
4. Write the exponential function, $f(x) = a(b)^x$.

Example:
Exercise:

Problem:
Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in [link].



Solution:

We can choose the y-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

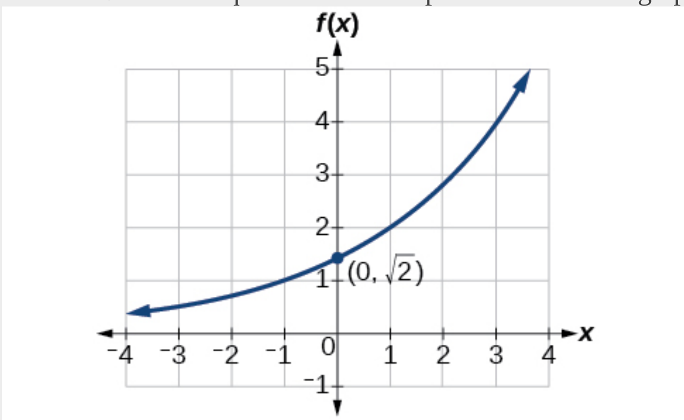
Equation:

$$\begin{array}{rl} y = ab^x & \text{Write the general form of an exponential equation.} \\ y = 3b^x & \text{Substitute the initial value 3 for } a. \\ 12 = 3b^2 & \text{Substitute in 12 for } y \text{ and 2 for } x. \\ 4 = b^2 & \text{Divide by 3.} \\ b = \pm 2 & \text{Take the square root.} \end{array}$$

Because we restrict ourselves to positive values of b, we will use $b = 2$. Substitute a and b into the standard form to yield the equation $f(x) = 3(2)^x$.

Note:
Exercise:

Problem: Find an equation for the exponential function graphed in [link].



Solution:

$f(x) = \sqrt{2}\left(\sqrt{2}\right)^x$. Answers may vary due to round-off error. The answer should be very close to $1.4142(1.4142)^x$.

Note:
Given two points on the curve of an exponential function, use a graphing calculator to find the equation.

1. Press **[STAT].**
2. Clear any existing entries in columns **L1** or **L2.**
3. In **L1**, enter the *x*-coordinates given.
4. In **L2**, enter the corresponding *y*-coordinates.
5. Press **[STAT]** again. Cursor right to **CALC**, scroll down to **ExpReg (Exponential Regression)**, and press **[ENTER].**
6. The screen displays the values of a and b in the exponential equation $y = a \cdot b^x$.

Example:
Exercise:

Problem:
Using a Graphing Calculator to Find an Exponential Function

Use a graphing calculator to find the exponential equation that includes the points $(2, 24.8)$ and $(5, 198.4)$.

Solution:

Follow the guidelines above. First press **[STAT]**, **[EDIT]**, **[1: Edit…],** and clear the lists **L1** and **L2**. Next, in the **L1** column, enter the *x*-coordinates, 2 and 5. Do the same in the **L2** column for the *y*-coordinates, 24.8 and 198.4.

Now press **[STAT]**, **[CALC]**, **[0: ExpReg]** and press **[ENTER]**. The values $a = 6.2$ and $b = 2$ will be displayed. The exponential equation is $y = 6.2 \cdot 2^x$.

Note:
Exercise:

Problem:

Use a graphing calculator to find the exponential equation that includes the points (3, 75.98) and (6, 481.07).

Solution:

$y \approx 12 \cdot 1.85^x$

Applying the Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, APR r, and number of compounding periods in a year n :
Equation:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe [link], which shows the result of investing $1,000 at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

Note:
The Compound Interest Formula
Compound interest can be calculated using the formula
Equation:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where

- $A(t)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Example:
Exercise:

Problem:
Calculating Compound Interest

If we invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution:

Because we are starting with $3,000, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$,the value when $t = 10$.
Equation:

$$\begin{aligned} A(t) &= P\left(1 + \tfrac{r}{n}\right)^{nt} && \text{Use the compound interest formula.} \\ A(10) &= 3000\left(1 + \tfrac{0.03}{4}\right)^{4 \cdot 10} && \text{Substitute using given values.} \\ &\approx \$4045.05 && \text{Round to two decimal places.} \end{aligned}$$

The account will be worth about \$4,045.05 in 10 years.

Note:
Exercise:

Problem:

An initial investment of \$100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

Solution:

about \$3,644,675.88

Example:
Exercise:

Problem:
Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to \$40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Solution:

The nominal interest rate is 6%, so $r = 0.06$. Interest is compounded twice a year, so $k = 2$.

We want to find the initial investment, P, needed so that the value of the account will be worth \$40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for P.
Equation:

$$\begin{aligned} A(t) &= P\left(1+\tfrac{r}{n}\right)^{nt} && \text{Use the compound interest formula.} \\ 40{,}000 &= P\left(1+\tfrac{0.06}{2}\right)^{2(18)} && \text{Substitute using given values } A,\ r,\ n, \text{ and } t. \\ 40{,}000 &= P(1.03)^{36} && \text{Simplify.} \\ \tfrac{40{,}000}{(1.03)^{36}} &= P && \text{Isolate } P. \\ P &\approx \$13,801 && \text{Divide and round to the nearest dollar.} \end{aligned}$$

Lily will need to invest \$13,801 to have \$40,000 in 18 years.

Note:
Exercise:

Problem:

Refer to [link]. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

Solution:

$13,693

Evaluating Functions with Base *e*

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. [link] shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $1 invested at 100% interest for 1 year, compounded at various frequencies, listed in [link].

Frequency	$A(n)=\left(1+\frac{1}{n}\right)^n$	**Value**
Annually	$\left(1+\frac{1}{1}\right)^1$	$2
Semiannually	$\left(1+\frac{1}{2}\right)^2$	$2.25
Quarterly	$\left(1+\frac{1}{4}\right)^4$	$2.441406
Monthly	$\left(1+\frac{1}{12}\right)^{12}$	$2.613035
Daily	$\left(1+\frac{1}{365}\right)^{365}$	$2.714567
Hourly	$\left(1+\frac{1}{8760}\right)^{8760}$	$2.718127
Once per minute	$\left(1+\frac{1}{525600}\right)^{525600}$	$2.718279
Once per second	$\left(1+\frac{1}{31536000}\right)^{31536000}$	$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

Note:
The Number e
The letter e represents the irrational number
Equation:

$$\left(1+\frac{1}{n}\right)^n, \text{as } n \text{ increases without bound}$$

The letter e is used as a base for many real-world exponential models. To work with base e, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Example:
Exercise:

Problem:
Using a Calculator to Find Powers of *e*

Calculate $e^{3.14}$. Round to five decimal places.

Solution:

On a calculator, press the button labeled $[e^x]$. The window shows $[e\ \hat{}\ (\ ]$. Type 3.14 and then close parenthesis, $[)]$. Press [ENTER]. Rounding to 5 decimal places, $e^{3.14} \approx 23.10387$. Caution: Many scientific calculators have an "Exp" button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

Note:
Exercise:

Problem: Use a calculator to find $e^{-0.5}$. Round to five decimal places.

Solution:

$e^{-0.5} \approx 0.60653$

Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, e is used as the base for exponential functions. Exponential models that use e as the base are called *continuous growth or decay models*. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

Note:
The Continuous Growth/Decay Formula

For all real numbers t,and all positive numbers a and r,continuous growth or decay is represented by the formula
Equation:

$$A(t) = ae^{rt}$$

where

- a is the initial value,
- r is the continuous growth rate per unit time,
- and t is the elapsed time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.
For business applications, the continuous growth formula is called the continuous compounding formula and takes the form
Equation:

$$A(t) = Pe^{rt}$$

where

- P is the principal or the initial invested,
- r is the growth or interest rate per unit time,
- and t is the period or term of the investment.

Note:
Given the initial value, rate of growth or decay, and time t, solve a continuous growth or decay function.

1. Use the information in the problem to determine a, the initial value of the function.
2. Use the information in the problem to determine the growth rate r.
   a. If the problem refers to continuous growth, then $r > 0$.
   b. If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time t.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

Example:
Exercise:

Problem:
Calculating Continuous Growth

A person invested $1,000 in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

Solution:

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $1,000, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

Equation:

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Use the continuous compounding formula.} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P,\ r,\ \text{and } t. \\ &\approx 1105.17 && \text{Use a calculator to approximate.} \end{aligned}$$

The account is worth $1,105.17 after one year.

Note:
Exercise:

Problem:

A person invests $100,000 at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

Solution:

$3,659,823.44

Example:
Exercise:

Problem:
Calculating Continuous Decay

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Solution:

Since the substance is decaying, the rate, 17.3%, is negative. So, $r = -0.173$. The initial amount of radon-222 was 100 mg, so $a = 100$. We use the continuous decay formula to find the value after $t = 3$ days:

Equation:

$$\begin{aligned} A(t) &= ae^{rt} && \text{Use the continuous growth formula.} \\ &= 100e^{-0.173(3)} && \text{Substitute known values for } a,\ r,\ \text{and } t. \\ &\approx 59.5115 && \text{Use a calculator to approximate.} \end{aligned}$$

So 59.5115 mg of radon-222 will remain.

Note:
Exercise:

Problem: Using the data in [link], how much radon-222 will remain after one year?

Solution:

3.77E-26 (This is calculator notation for the number written as 3.77×10^{-26} in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

Note:
Access these online resources for additional instruction and practice with exponential functions.

- Exponential Growth Function
- Compound Interest

Key Equations

definition of the exponential function	$f(x) = b^x, \text{ where } b > 0,\ b \neq 1$
definition of exponential growth	$f(x) = ab^x, \text{ where } a > 0, b > 0, b \neq 1$
compound interest formula	$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}, \text{ where}$ $A(t)$ is the account value at time t t is the number of years P is the initial investment, often called the principal r is the annual percentage rate (APR), or nominal rate n is the number of compounding periods in one year
continuous growth formula	$A(t) = ae^{rt}, \text{ where}$ $t$is the number of unit time periods of growth $a$is the starting amount (in the continuous compounding formula a is replaced with P, the principal) $e$is the mathematical constant, $e \approx 2.718282$

Key Concepts

- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent. See [link].
- A function is evaluated by solving at a specific value. See [link] and [link].
- An exponential model can be found when the growth rate and initial value are known. See [link].
- An exponential model can be found when the two data points from the model are known. See [link].
- An exponential model can be found using two data points from the graph of the model. See [link].
- An exponential model can be found using two data points from the graph and a calculator. See [link].
- The value of an account at any time t can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known. See [link].
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known. See [link].
- The number e is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282.$
- Scientific and graphing calculators have the key $[e^x]$ or $[\exp(x)]$ for calculating powers of $e.$ See [link].
- Continuous growth or decay models are exponential models that use e as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.

Solution:

Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

Exercise:

Problem:

Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

Exercise:

Problem:

The Oxford Dictionary defines the word *nominal* as a value that is “stated or expressed but not necessarily corresponding exactly to the real value.”[footnote] Develop a reasonable argument for why the term *nominal rate* is used to describe the annual percentage rate of an investment account that compounds interest.
Oxford Dictionary. http://oxforddictionaries.com/us/definition/american_english/nomina.

Solution:

When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not

necessarily correspond to the real interest earned, which is the very definition of *nominal*.

Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

Exercise:

Problem: The average annual population increase of a pack of wolves is 25.

Exercise:

Problem: A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.

Solution:

exponential; the population decreases by a proportional rate. .

Exercise:

Problem: The value of a coin collection has increased by 3.25% annually over the last 20 years.

Exercise:

Problem:

For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.

Solution:

not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function. .

Exercise:

Problem: The height of a projectile at time t is represented by the function $h(t) = -4.9t^2 + 18t + 40$.

For the following exercises, consider this scenario: For each year t,the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)

Exercise:

Problem: Which forest's population is growing at a faster rate?

Solution:

The forest represented by the function $B(t) = 82(1.029)^t$.

Exercise:

Problem: Which forest had a greater number of trees initially? By how many?

Exercise:

Problem:

Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?

Solution:

After $t=20$ years, forest A will have 43 more trees than forest B.

Exercise:

Problem:

Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?

Exercise:

Problem:

Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

Solution:

Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

Exercise:

Problem: $y=300(1-t)^5$

Exercise:

Problem: $y=220(1.06)^x$

Solution:

exponential growth; The growth factor, 1.06, is greater than 1.

Exercise:

Problem: $y=16.5(1.025)^{\frac{1}{x}}$

Exercise:

Problem: $y=11,701(0.97)^t$

Solution:

exponential decay; The decay factor, 0.97, is between 0 and 1.

For the following exercises, find the formula for an exponential function that passes through the two points given.

Exercise:

Problem: $(0, 6)$ and $(3, 750)$

Exercise:

Problem: $(0, 2000)$ and $(2, 20)$

Solution:

$f(x) = 2000(0.1)^x$

Exercise:

Problem: $\left(-1, \frac{3}{2}\right)$ and $(3, 24)$

Exercise:

Problem: $(-2, 6)$ and $(3, 1)$

Solution:

$f(x) = \left(\frac{1}{6}\right)^{-\frac{3}{5}} \left(\frac{1}{6}\right)^{\frac{x}{5}} \approx 2.93(0.699)^x$

Exercise:

Problem: $(3, 1)$ and $(5, 4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

Exercise:

Problem:

x	1	2	3	4
$f(x)$	70	40	10	-20

Solution:

Linear

Exercise:

Problem:

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

Exercise:

Problem:

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

Solution:

Neither

Exercise:

Problem:

x	1	2	3	4
$f(x)$	10	20	40	80

Exercise:

Problem:

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

Solution:

Linear

For the following exercises, use the compound interest formula, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.

Exercise:

Problem:

After a certain number of years, the value of an investment account is represented by the equation $10,250\left(1 + \frac{0.04}{12}\right)^{120}$. What is the value of the account?

Exercise:

Problem: What was the initial deposit made to the account in the previous exercise?

Solution:

$\$10,250$

Exercise:

Problem: How many years had the account from the previous exercise been accumulating interest?

Exercise:

Problem:

An account is opened with an initial deposit of \$6,500 and earns 3.6% interest compounded semi-annually. What will the account be worth in 20 years?

Solution:

$\$13,268.58$

Exercise:

Problem:

How much more would the account in the previous exercise have been worth if the interest were compounding weekly?

Exercise:

Problem: Solve the compound interest formula for the principal, P.

Solution:

$P = A(t) \cdot \left(1 + \frac{r}{n}\right)^{-nt}$

Exercise:

Problem:

Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)

Exercise:

Problem:

How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?

Solution:

$\$4{,}572.56$

Exercise:

Problem:

Use properties of rational exponents to solve the compound interest formula for the interest rate, r.

Exercise:

Problem:

Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $9,000 and was worth $13,373.53 after 10 years.

Solution:

4%

Exercise:

Problem:

Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $5,500, and was worth $38,455 after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

Exercise:

Problem: $y = 3742(e)^{0.75t}$

Solution:

continuous growth; the growth rate is greater than 0.

Exercise:

Problem: $y = 150(e)^{\frac{3.25}{t}}$

Exercise:

Problem: $y = 2.25(e)^{-2t}$

Solution:

continuous decay; the growth rate is less than 0.

Exercise:

Problem:

Suppose an investment account is opened with an initial deposit of $\$12,000$ earning 7.2% interest compounded continuously. How much will the account be worth after 30 years?

Exercise:

Problem:

How much less would the account from Exercise 42 be worth after 30 years if it were compounded monthly instead?

Solution:

$\$669.42$

Numeric

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

Exercise:

Problem: $f(x) = 2(5)^x$, for $f(-3)$

Exercise:

Problem: $f(x) = -4^{2x+3}$, for $f(-1)$

Solution:

$f(-1) = -4$

Exercise:

Problem: $f(x) = e^x$, for $f(3)$

Exercise:

Problem: $f(x) = -2e^{x-1}$, for $f(-1)$

Solution:

$f(-1) \approx -0.2707$

Exercise:

Problem: $f(x) = 2.7(4)^{-x+1} + 1.5$, for $f\left(-2\right)$

Exercise:

Problem: $f(x) = 1.2e^{2x} - 0.3$, for $f\left(3\right)$

Solution:

$f(3) \approx 483.8146$

Exercise:

Problem: $f(x) = -\frac{3}{2}(3)^{-x} + \frac{3}{2}$, for $f\left(2\right)$

Technology

For the following exercises, use a graphing calculator to find the equation of an exponential function given the points on the curve.

Exercise:

Problem: $(0, 3)$ and $(3, 375)$

Solution:

$y = 3 \cdot 5^{x}$

Exercise:

Problem: $(3, 222.62)$ and $(10, 77.456)$

Exercise:

Problem: $(20, 29.495)$ and $(150, 730.89)$

Solution:

$y \approx 18 \cdot 1.025^{x}$

Exercise:

Problem: $(5, 2.909)$ and $(13, 0.005)$

Exercise:

Problem: $(11{,}310.035)$ and $(25{,}356.3652)$

Solution:

$y \approx 0.2 \cdot 1.95^{x}$

Extensions

Exercise:

Problem:

The *annual percentage yield* (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $\text{APY} = \left(1 + \frac{r}{12}\right)^{12} - 1.$

Exercise:

Problem:

Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds n times per year.

Solution:

$$\text{APY} = \frac{A(t)-a}{a} = \frac{a\left(1+\frac{r}{365}\right)^{365(1)}-a}{a} = \frac{a\left[\left(1+\frac{r}{365}\right)^{365}-1\right]}{a} = \left(1 + \frac{r}{365}\right)^{365} - 1; I(n) = \left(1 + \frac{r}{n}\right)^{n} - 1$$

Exercise:

Problem:

Recall that an exponential function is any equation written in the form $f(x) = a \cdot b^x$ such that a and b are positive numbers and $b \neq 1$. Any positive number b can be written as $b = e^n$ for some value of n. Use this fact to rewrite the formula for an exponential function that uses the number e as a base.

Exercise:

Problem:

In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b > 1$, the exponential decay function can be written as $f(x) = a \cdot \left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b = e^n$, to show that an exponential decay function takes the form $f(x) = a(e)^{-nx}$ for some positive number n .

Solution:

Let f be the exponential decay function $f(x) = a \cdot \left(\frac{1}{b}\right)^x$ such that $b > 1$. Then for some number $n > 0$, $f(x) = a \cdot \left(\frac{1}{b}\right)^x = a\left(b^{-1}\right)^x = a\left(\left(e^n\right)^{-1}\right)^x = a\left(e^{-n}\right)^x = a(e)^{-nx}$.

Exercise:

Problem:

The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A(t) = a(e)^{rt}$,where a is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage of interest earned to principal at any time t can be calculated with the formula $I(t) = e^{rt} - 1$.

Real-World Applications

Exercise:

Problem:

The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were 23,900 fox counted in the area. What is the fox population predicted to be in the year 2020?

Solution:

$47,622$ fox

Exercise:

Problem:

A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?

Exercise:

Problem:

In the year 1985, a house was valued at $110,000. By the year 2005, the value had appreciated to $145,000. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

Solution:

$1.39\%;\ \$155,368.09$

Exercise:

Problem:

A car was valued at $38,000 in the year 2007. By 2013, the value had depreciated to $11,000 If the car's value continues to drop by the same percentage, what will it be worth by 2017?

Exercise:

Problem:

Jamal wants to save $54,000 for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?

Solution:

$\$35,838.76$

Exercise:

Problem:

Kyoko has $10,000 that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $15,000 by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (*Hint*: solve the compound interest formula for the interest rate.)

Exercise:

Problem:

Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $13,500. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

Solution:

$\$82,247.78;\ \$449.75$

Exercise:

Problem:

An investment account with an annual interest rate of 7% was opened with an initial deposit of $4,000 Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

Glossary

annual percentage rate (APR)
: the yearly interest rate earned by an investment account, also called *nominal rate*

compound interest
: interest earned on the total balance, not just the principal

exponential growth
: a model that grows by a rate proportional to the amount present

nominal rate
: the yearly interest rate earned by an investment account, also called *annual percentage rate*

Logarithmic Functions
In this section, you will:

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes[footnote]. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings,[footnote] like those shown in [link]. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a

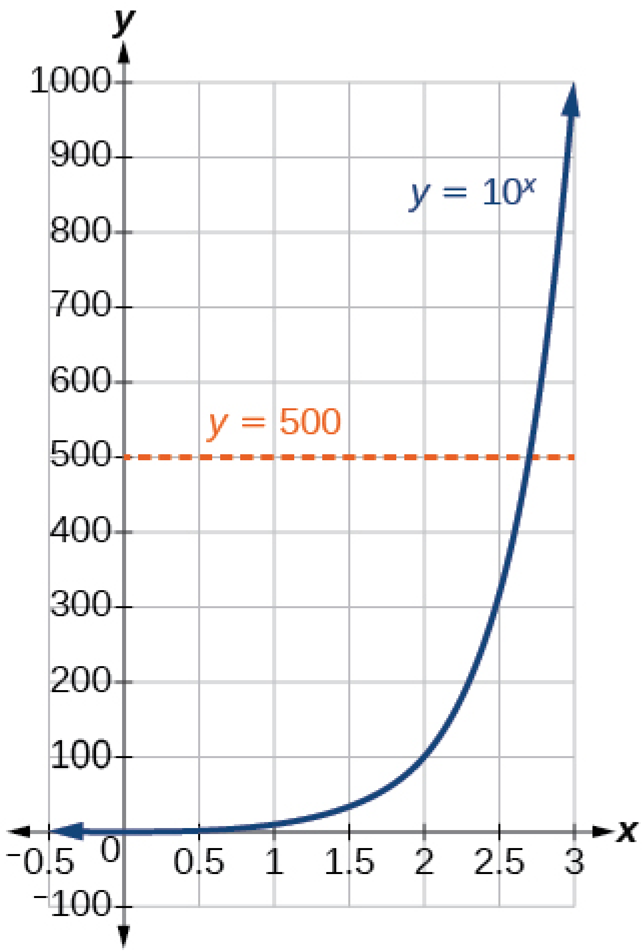
7.0 on the Richter Scale[footnote] whereas the Japanese earthquake registered a 9.0.[footnote]
http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/#summary. Accessed 3/4/2013.
http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#summary. Accessed 3/4/2013.
http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/. Accessed 3/4/2013.
http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#details. Accessed 3/4/2013.

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4} = 10^4 = 10{,}000$ times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in [link], to better estimate the solution.



Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in [link] passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

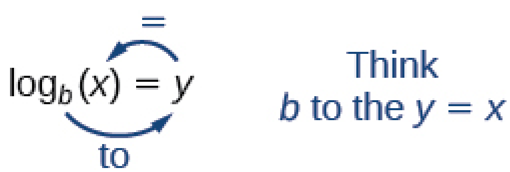
We read a logarithmic expression as, "The logarithm with base b of x is equal to y," or, simplified, "log base b of x is y." We can also say, "b raised to the power of y is x," because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

Equation:

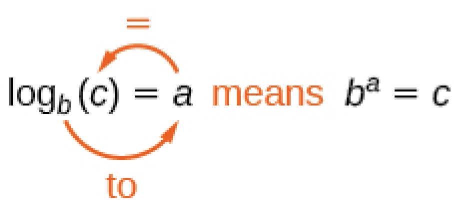
$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1$$

Note that the base b is always positive.



Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:



Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log_b(x)$ and $y = b^x$ are inverse functions.

Note:
Definition of the Logarithmic Function
A **logarithm** base b of a positive number x satisfies the following definition.

For $x > 0, b > 0, b \neq 1,$
Equation:

$$y = \log_b(x) \text{ is equivalent to } b^y = x$$

where,

- we read $\log_b(x)$ as, “the logarithm with base b of x” or the “log base b of x.″
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

Note:
Can we take the logarithm of a negative number?
No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

Note:
Given an equation in logarithmic form $\log_b(x) = y$, **convert it to exponential form.**

1. Examine the equation $y = \log_b(x)$ and identify $b, y,$ and x.
2. Rewrite $\log_b(x) = y$ as $b^y = x$.

Example:
Exercise:

Problem:
Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$
b. $\log_3(9) = 2$

Solution:

First, identify the values of $b, y, \text{and} x.$ Then, write the equation in the form $b^y = x$.

a. $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$

Here, $b = 6, y = \frac{1}{2}$, and $x = \sqrt{6}.$ Therefore, the equation $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$ is equivalent to

$$6^{\frac{1}{2}} = \sqrt{6}.$$

b. $\log_3(9) = 2$

Here, $b = 3, y = 2,$ and $x = 9.$ Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9.$

Note:
Exercise:

Problem:

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1{,}000{,}000) = 6$
b. $\log_5(25) = 2$

Solution:

a. $\log_{10}(1,000,000) = 6$ is equivalent to $10^6 = 1,000,000$
b. $\log_5(25) = 2$ is equivalent to $5^2 = 25$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b,exponent x,and output y. Then we write $x = \log_b(y)$.

Example:
Exercise:

Problem:
Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \frac{1}{10{,}000}$

Solution:

First, identify the values of $b, y,$ and x. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$

Here, $b = 2$, $x = 3$, and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$

Here, $b = 5$, $x = 2$, and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \frac{1}{10{,}000}$

Here, $b = 10$, $x = -4$, and $y = \frac{1}{10{,}000}$. Therefore, the equation $10^{-4} = \frac{1}{10{,}000}$ is equivalent to $\log_{10}\left(\frac{1}{10{,}000}\right) = -4$.

Note:
Exercise:

Problem:

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \frac{1}{2}$

Solution:

a. $3^2 = 9$ is equivalent to $\log_3(9) = 2$
b. $5^3 = 125$ is equivalent to $\log_5(125) = 3$
c. $2^{-1} = \frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right) = -1$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, “To what exponent must 2 be raised in order to get 8?” Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, “To what exponent must 7 be raised in order to get 49?” We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, “To what exponent must 3 be raised in order to get 27?” We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let’s evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, “To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$? ” We know $2^2 = 4$ and $3^2 = 9$,so $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, $\log_{\frac{2}{3}}\left(\frac{4}{9}\right) = 2$.

Note:
Given a logarithm of the form $y = \log_b(x)$,evaluate it mentally.

1. Rewrite the argument x as a power of b : $b^y = x$.
2. Use previous knowledge of powers of b identify y by asking, “To what exponent should b be raised in order to get x?”

Example:
Exercise:

Problem:
Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Solution:

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know
Equation:

$$4^3 = 64$$

Therefore,
Equation:

$$\log_4(64) = 3$$

Note:
Exercise:

Problem: Solve $y = \log_{121}(11)$ without using a calculator.

Solution:

$\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = (121)^{\frac{1}{2}} = 11$)

Example:

Exercise:

Problem:
Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Solution:

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

We know $3^3 = 27$,but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

Equation:

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

Note:
Exercise:

Problem: Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

Solution:

$\log_2\left(\frac{1}{32}\right) = -5$

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x).$ We call a base-10 logarithm a **common logarithm**. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

Note:
Definition of the Common Logarithm
A **common logarithm** is a logarithm with base $10.$ We write $\log_{10}(x)$ simply as $\log(x).$ The common logarithm of a positive number x satisfies the following definition.
For $x > 0,$
Equation:

$$y=\log(x)\text{ is equivalent to }10^y=x$$

We read $\log(x)$ as, "the logarithm with base 10 of x " or "log base 10 of $x.$ "
The logarithm y is the exponent to which 10 must be raised to get $x.$

Note:
Given a common logarithm of the form $y=\log(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of $10:\ 10^y=x.$
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get x?"

Example:
Exercise:

Problem:
Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution:

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, “To what exponent must 10 be raised in order to get 1000?” We know

Equation:

$$10^3 = 1000$$

Therefore, $\log\left(1000\right) = 3$.

Note:
Exercise:

Problem: Evaluate $y = \log(1{,}000{,}000)$.

Solution:

$\log(1,000,000) = 6$

Note:
Given a common logarithm with the form $y = \log\left(x\right)$, evaluate it using a calculator.

1. Press **[LOG]**.
2. Enter the value given for x,followed by **[ )]**.
3. Press **[ENTER]**.

Example:
Exercise:

Problem:
Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution:

- Press **[LOG]**.
- Enter 321, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:

Equation:

$$\begin{array}{ccccc} 100 & < & 321 & < & 1000 \\ 2 & < & 2.5065 & < & 3 \end{array}$$

Note:
Exercise:

Problem:

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Solution:

$\log(123) \approx 2.0899$

Example:
Exercise:

Problem:
Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution:

We begin by rewriting the exponential equation in logarithmic form.
Equation:

$$\begin{aligned} 10^x &= 500 \\ \log(500) &= x \qquad \text{Use the definition of the common log.} \end{aligned}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500,followed by **[ )]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes was about 2.699.

Note:
Exercise:

Problem:

The amount of energy released from one earthquake was $8{,}500$ times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution:

The difference in magnitudes was about 3.929.

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation, $\ln(x)$.

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

Note:
Definition of the Natural Logarithm

A **natural logarithm** is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

Equation:

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

Note:

Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator.

1. Press **[LN]**.
2. Enter the value given for x, followed by **[ )]**.
3. Press **[ENTER]**.

Example:

Exercise:

Problem:

Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution:

- Press **[LN]**.

- Enter 500,followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

Note:
Exercise:

Problem: Evaluate $\ln(-500)$.

Solution:

It is not possible to take the logarithm of a negative number in the set of real numbers.

Note:
Access this online resource for additional instruction and practice with logarithms.

- Introduction to Logarithms

Key Equations

Definition of the logarithmic function	For $x > 0, b > 0, b \neq 1$, $y = \log_b(x)$ if and only if $b^y = x$.

Definition of the common logarithm	For $x > 0, y = \log(x)$ if and only if $10^y = x$.
Definition of the natural logarithm	For $x > 0, y = \ln(x)$ if and only if $e^y = x$.

Key Concepts

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See [link].
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See [link].
- Logarithmic functions with base b can be evaluated mentally using previous knowledge of powers of b. See [link] and [link].
- Common logarithms can be evaluated mentally using previous knowledge of powers of 10. See [link].
- When common logarithms cannot be evaluated mentally, a calculator can be used. See [link].
- Real-world exponential problems with base 10 can be rewritten as a common logarithm and then evaluated using a calculator. See [link].
- Natural logarithms can be evaluated using a calculator [link].

Section Exercises

Verbal

Exercise:

Problem:

What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1$.

Solution:

A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given, the base b has the same value. The exponent, y,in the expression b^y can also be written as the logarithm, $\log_b x$,and the value of x is the result of raising b to the power of y.

Exercise:

Problem:

How is the logarithmic function $f(x) = \log_b x$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?

Exercise:

Problem:

How can the logarithmic equation $\log_b x = y$ be solved for x using the properties of exponents?

Solution:

Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$, and then properties of exponents can be applied to solve for x.

Exercise:

Problem:

Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Exercise:

Problem:

Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Solution:

The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e. Rather than notating the natural logarithm as $\log_e(x)$,the notation used is $\ln(x)$.

Algebraic

For the following exercises, rewrite each equation in exponential form.
Exercise:

Problem: $\log_4(q) = m$

Exercise:

Problem: $\log_a(b) = c$

Solution:

$a^c = b$

Exercise:

Problem: $\log_{16}(y) = x$

Exercise:

Problem: $\log_x(64) = y$

Solution:

$x^y = 64$

Exercise:

Problem: $\log_y(x) = -11$

Exercise:

Problem: $\log_{15}(a) = b$

Solution:

$15^b = a$

Exercise:

Problem: $\log_y(137) = x$

Exercise:

Problem: $\log_{13}(142) = a$

Solution:

$13^a = 142$

Exercise:

Problem: $\log(v) = t$

Exercise:

Problem: $\ln(w) = n$

Solution:

$e^n = w$

For the following exercises, rewrite each equation in logarithmic form.

Exercise:

Problem: $4^x = y$

Exercise:

Problem: $c^d = k$

Solution:

$\log_c(k) = d$

Exercise:

Problem: $m^{-7} = n$

Exercise:

Problem: $19^x = y$

Solution:

$\log_{19} y = x$

Exercise:

Problem: $x^{-\frac{10}{13}} = y$

Exercise:

Problem: $n^4 = 103$

Solution:

$\log_n (103) = 4$

Exercise:

Problem: $\left(\frac{7}{5}\right)^m = n$

Exercise:

Problem: $y^x = \frac{39}{100}$

Solution:

$\log_y\left(\frac{39}{100}\right) = x$

Exercise:

Problem: $10^a = b$

Exercise:

Problem: $e^k = h$

Solution:

$\ln(h) = k$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

Exercise:

Problem: $\log_3(x) = 2$

Exercise:

Problem: $\log_2(x) = -3$

Solution:

$x = 2^{-3} = \frac{1}{8}$

Exercise:

Problem: $\log_5(x) = 2$

Exercise:

Problem: $\log_3(x) = 3$

Solution:

$x = 3^3 = 27$

Exercise:

Problem: $\log_2(x) = 6$

Exercise:

Problem: $\log_9(x) = \frac{1}{2}$

Solution:

$x = 9^{\frac{1}{2}} = 3$

Exercise:

Problem: $\log_{18}(x) = 2$

Exercise:

Problem: $\log_6(x) = -3$

Solution:

$x = 6^{-3} = \frac{1}{216}$

Exercise:

Problem: $\log(x) = 3$

Exercise:

Problem: $\ln(x) = 2$

Solution:

$x = e^2$

For the following exercises, use the definition of common and natural logarithms to simplify.

Exercise:

Problem: $\log(100^8)$

Exercise:

Problem: $10^{\log(32)}$

Solution:

32

Exercise:

Problem: $2\log(.0001)$

Exercise:

Problem: $e^{\ln(1.06)}$

Solution:

1.06

Exercise:

Problem: $\ln\left(e^{-5.03}\right)$

Exercise:

Problem: $e^{\ln(10.125)} + 4$

Solution:

14.125

Numeric

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

Exercise:

Problem: $\log_3\left(\frac{1}{27}\right)$

Exercise:

Problem: $\log_6(\sqrt{6})$

Solution:

$\frac{1}{2}$

Exercise:

Problem: $\log_2\left(\frac{1}{8}\right) + 4$

Exercise:

Problem: $6\log_8(4)$

Solution:

4

For the following exercises, evaluate the common logarithmic expression without using a calculator.

Exercise:

Problem: $\log(10,000)$

Exercise:

Problem: $\log(0.001)$

Solution:

-3

Exercise:

Problem: $\log(1)+7$

Exercise:

Problem: $2\log(100^{-3})$

Solution:

-12

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

Exercise:

Problem: $\ln(e^{\frac{1}{3}})$

Exercise:

Problem: $\ln(1)$

Solution:

0

Exercise:

Problem: $\ln(e^{-0.225}) - 3$

Exercise:

Problem: $25\ln(e^{\frac{2}{5}})$

Solution:

10

Technology

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

Exercise:

Problem: $\log(0.04)$

Exercise:

Problem: $\ln(15)$

Solution:

2.708

Exercise:

Problem: $\ln\left(\frac{4}{5}\right)$

Exercise:

Problem: $\log(\sqrt{2})$

Solution:

0.151

Exercise:

Problem: $\ln(\sqrt{2})$

Extensions

Exercise:

Problem:

Is $x = 0$ in the domain of the function $f(x) = \log(x)$? If so, what is the value of the function when $x = 0$? Verify the result.

Solution:

No, the function has no defined value for $x = 0$. To verify, suppose $x = 0$ is in the domain of the function $f(x) = \log(x)$. Then there is some number n such that $n = \log(0)$. Rewriting as an exponential equation gives: $10^n = 0$, which is impossible since no such real number n exists. Therefore, $x = 0$ is *not* the domain of the function $f(x) = \log(x)$.

Exercise:

Problem:

Is $f(x) = 0$ in the range of the function $f(x) = \log(x)$? If so, for what value of x? Verify the result.

Exercise:

Problem:

Is there a number x such that $\ln x = 2$? If so, what is that number? Verify the result.

Solution:

Yes. Suppose there exists a real number x such that $\ln x = 2$. Rewriting as an exponential equation gives $x = e^2$, which is a real number. To verify, let $x = e^2$. Then, by definition, $\ln (x) = \ln \left(e^2\right) = 2$.

Exercise:

Problem: Is the following true: $\frac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)} = -1$? Verify the result.

Exercise:

Problem: Is the following true: $\frac{\ln\left(e^{1.725}\right)}{\ln(1)} = 1.725$? Verify the result.

Solution:

No; $\ln (1) = 0$, so $\frac{\ln\left(e^{1.725}\right)}{\ln(1)}$ is undefined.

Real-World Applications

Exercise:

Problem:

The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the “f-stop” setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?

Exercise:

Problem:

Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?

Solution:

2

Exercise:

Problem:

The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula $\log\frac{I_1}{I_2} = M_1 - M_2$ where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0.[footnote] How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number. http://earthquake.usgs.gov/earthquakes/world/historical.php. Accessed 3/4/2014.

Glossary

common logarithm

the exponent to which 10 must be raised to get x; $\log_{10}(x)$ is written simply as $\log(x)$.

logarithm
: the exponent to which b must be raised to get x; written $y = \log_b(x)$

natural logarithm
: the exponent to which the number e must be raised to get x; $\log_e(x)$ is written as $\ln(x)$.

Logarithmic Properties
In this section, you will:

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan)

In chemistry, pH is used as a measure of the acidity or alkalinity of a substance. The pH scale runs from 0 to 14. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 (at 25° C)
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where a is the concentration of hydrogen ion in the solution
Equation:

$$\begin{aligned} \text{pH} &= -\log([H^+]) \\ &= \log\left(\frac{1}{[H^+]}\right) \end{aligned}$$

The equivalence of $-\log\left([H^+]\right)$ and $\log\left(\frac{1}{[H^+]}\right)$ is one of the logarithm properties we will examine in this section.

Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

Equation:

$$\log_b 1 = 0$$
$$\log_b b = 1$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

Equation:

$$\log_b(b^x) = x$$
$$b^{\log_b x} = x, x > 0$$

For example, to evaluate $\log(100)$,we can rewrite the logarithm as $\log_{10}\left(10^2\right)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}\left(10^2\right) = 2$.

To evaluate $e^{\ln(7)}$,we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

Equation:

$$\log_b M = \log_b N \text{ if and only if } M = N$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x+5)$ for x. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for x :

Equation:

$$\begin{array}{ll} 3x = 2x + 5 & \text{Set the arguments equal.} \\ x = 5 & \text{Subtract } 2x. \end{array}$$

But what about the equation $\log_3(3x) + \log_3(2x+5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

Equation:

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that
Equation:

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for } M \text{ and } N. \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents.} \\ &= m + n && \text{Apply the inverse property of logs.} \\ &= \log_b(M) + \log_b(N) && \text{Substitute for } m \text{ and } n. \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:
Equation:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

Note:
The Product Rule for Logarithms
The **product rule for logarithms** can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.
Equation:

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0$$

Note:
Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms.

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Example:
Exercise:

Problem:
Using the Product Rule for Logarithms

Expand $\log_3(30x(3x+4))$.

Solution:

We begin by factoring the argument completely, expressing 30 as a product of primes.
Equation:

$$\log_3(30x(3x+4)) = \log_3(2 \cdot 3 \cdot 5 \cdot x \cdot (3x+4))$$

Next we write the equivalent equation by summing the logarithms of each factor.
Equation:

$$\log_3(30x(3x+4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x+4)$$

Note:
Exercise:

Problem: Expand $\log_b(8k)$.

Solution:

$\log_b 2 + \log_b 2 + \log_b 2 + \log_b k = 3\log_b 2 + \log_b k$

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $\frac{x^a}{x^b} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show
Equation:

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that
Equation:

$$\begin{aligned} \log_b\left(\tfrac{M}{N}\right) &= \log_b\left(\tfrac{b^m}{b^n}\right) && \text{Substitute for } M \text{ and } N. \\ &= \log_b\left(b^{m-n}\right) && \text{Apply the quotient rule for exponents.} \\ &= m - n && \text{Apply the inverse property of logs.} \\ &= \log_b(M) - \log_b(N) && \text{Substitute for } m \text{ and } n. \end{aligned}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,
Equation:

$$\begin{aligned} \log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{Factor the numerator and denominator.} \\ &= \log\left(\frac{2x}{3}\right) && \text{Cancel the common factors.} \end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.
Equation:

$$\begin{aligned}\log\left(\tfrac{2x}{3}\right) &= \log(2x) - \log(3) \\ &= \log(2) + \log(x) - \log(3)\end{aligned}$$

Note:
The Quotient Rule for Logarithms
The **quotient rule for logarithms** can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.
Equation:

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

Note:
Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms.

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Example:
Exercise:

Problem:
Using the Quotient Rule for Logarithms

Expand $\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

Solution:

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.
Equation:

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2\left(15x(x-1)\right) - \log_2\left((3x+4)(2-x)\right)$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.
Equation:

$$\begin{aligned}\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] - [\log_2(3x+4) \\ &= \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log\end{aligned}$$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$,$x > 1$,$x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

Note:
Exercise:

Problem: Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Solution:

$\log_3(x+3) - \log_3(x-1) - \log_3(x-2)$

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:
Equation:

$$\begin{aligned} \log_b\left(x^2\right) &= \log_b(x \cdot x) \\ &= \log_b x + \log_b x \\ &= 2\log_b x \end{aligned}$$

Notice that we used the product rule for logarithms to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,
Equation:

$$100 = 10^2 \qquad \sqrt{3} = 3^{\frac{1}{2}} \qquad \frac{1}{e} = e^{-1}$$

Note:
The Power Rule for Logarithms
The **power rule for logarithms** can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.
Equation:

$$\log_b(M^n) = n\log_b M$$

Note:

Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm.

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Example:
Exercise:

Problem:
Expanding a Logarithm with Powers

Expand $\log_2 x^5$.

Solution:

The argument is already written as a power, so we identify the exponent, 5, and the base, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.
Equation:

$$\log_2 \left(x^5\right) = 5\log_2 x$$

Note:
Exercise:

Problem: Expand $\ln x^2$.

Solution:

$2 \ln x$

Example:
Exercise:

Problem:
Rewriting an Expression as a Power before Using the Power Rule

Expand $\log_3 (25)$ using the power rule for logs.

Solution:

Expressing the argument as a power, we get $\log_3 (25) = \log_3 \left(5^2\right)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.
Equation:

$$\log_3 \left(5^2\right) = 2\log_3 (5)$$

Note:
Exercise:

Problem: Expand $\ln\left(\frac{1}{x^2}\right)$.

Solution:

$-2\ln(x)$

Example:
Exercise:

Problem:
Using the Power Rule in Reverse

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Solution:

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$,we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

Note:
Exercise:

Problem: Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Solution:

$\log_3 16$

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called “laws of logs.” Sometimes we apply more than one rule in order to simplify an expression. For example:
Equation:

$$\begin{aligned} \log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y \end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:
Equation:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b\left(AC^{-1}\right)\\ &= \log_b(A) + \log_b\left(C^{-1}\right)\\ &= \log_b A + (-1)\log_b C\\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Example:
Exercise:

Problem:
Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite $\ln\left(\frac{x^4y}{7}\right)$ as a sum or difference of logs.

Solution:

First, because we have a quotient of two expressions, we can use the quotient rule:
Equation:

$$\ln\left(\frac{x^4y}{7}\right) = \ln\left(x^4y\right) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:
Equation:

$$\ln\left(x^4y\right) - \ln(7) = \ln\left(x^4\right) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:
Equation:

$$\ln\left(x^4\right) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

Note:
Exercise:

Problem: Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

Solution:

$2\log x + 3\log y - 4\log z$

Example:
Exercise:

Problem:
Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log\left(\sqrt{x}\right)$.

Solution:
Equation:

$$\begin{aligned}\log\left(\sqrt{x}\right) &= \log x^{\left(\frac{1}{2}\right)}\\ &= \tfrac{1}{2}\log x\end{aligned}$$

Note:
Exercise:

Problem: Expand $\ln\left(\sqrt[3]{x^2}\right)$.

Solution:

$\frac{2}{3}\ln x$

Note:
Can we expand $\ln\left(x^2+y^2\right)$?
No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

Example:
Exercise:

Problem:
Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

Solution:

We can expand by applying the Product and Quotient Rules.
Equation:

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6 64+\log_6 x^3+\log_6(4x+1)-\log_6(2x-1) && \text{Apply the Quotient Rule.}\\ &= \log_6 2^6+\log_6 x^3+\log_6(4x+1)-\log_6(2x-1) && \text{Simplify by writing } 64 \text{ as } 2^6.\\ &= 6\log_6 2+3\log_6 x+\log_6(4x+1)-\log_6(2x-1) && \text{Apply the Power Rule.}\end{aligned}$$

Note:
Exercise:

Problem: Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$.

Solution:

$\frac{1}{2}\ln(x-1)+\ln(2x+1)-\ln(x+3)-\ln(x-3)$

Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

Note:
Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Example:
Exercise:

Problem:
Using the Product and Quotient Rules to Combine Logarithms

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

Solution:

Using the product and quotient rules
Equation:

$$\log_3(5)+\log_3(8)=\log_3(5\cdot 8)=\log_3(40)$$

This reduces our original expression to
Equation:

$$\log_3(40)-\log_3(2)$$

Then, using the quotient rule
Equation:

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

Note:
Exercise:

Problem: Condense $\log 3 - \log 4 + \log 5 - \log 6$.

Solution:

$\log\left(\frac{3 \cdot 5}{4 \cdot 6}\right)$; can also be written $\log\left(\frac{5}{8}\right)$ by reducing the fraction to lowest terms.

Example:
Exercise:

Problem:
Condensing Complex Logarithmic Expressions

Condense $\log_2\left(x^2\right) + \frac{1}{2}\log_2(x-1) - 3\log_2\left((x+3)^2\right)$.

Solution:

We apply the power rule first:
Equation:

$$\log_2\left(x^2\right) + \frac{1}{2}\log_2(x-1) - 3\log_2\left((x+3)^2\right) = \log_2\left(x^2\right) + \log_2\left(\sqrt{x-1}\right) - \log_2\left((x+3)^6\right)$$

Next we apply the product rule to the sum:
Equation:

$$\log_2\left(x^2\right) + \log_2\left(\sqrt{x-1}\right) - \log_2\left((x+3)^6\right) = \log_2\left(x^2\sqrt{x-1}\right) - \log_2\left((x+3)^6\right)$$

Finally, we apply the quotient rule to the difference:
Equation:

$$\log_2\left(x^2\sqrt{x-1}\right) - \log_2\left((x+3)^6\right) = \log_2\frac{x^2\sqrt{x-1}}{(x+3)^6}$$

Note:
Exercise:

Problem: Rewrite $\log(5) + 0.5\log(x) - \log(7x-1) + 3\log(x-1)$ as a single logarithm.

Solution:

$\log\left(\frac{5(x-1)^3\sqrt{x}}{(7x-1)}\right)$

Example:
Exercise:

Problem:
Rewriting as a Single Logarithm

Rewrite $2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5)$ as a single logarithm.

Solution:

We apply the power rule first:
Equation:

$$\log(x+5) + \frac{1}{x}\log(3x+5) = \log\left(x^2\right) - \log(x+5)^4 + \log\left((3x+5)^{x^{-1}}\right)$$

Next we rearrange and apply the product rule to the sum:
Equation:

$$\log\left(x^2\right) - \log(x+5)^4 + \log\left((3x+5)^{x^{-1}}\right)$$

Equation:

$$= \log\left(x^2\right) + \log\left((3x+5)^{x^{-1}}\right) - \log(x+5)^4$$

Equation:

$$= \log\left(x^2(3x+5)^{x^{-1}}\right) - \log(x+5)^4$$

Finally, we apply the quotient rule to the difference:
Equation:

$$= \log\left(x^2(3x+5)^{x^{-1}}\right) - \log(x+5)^4 = \log\frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4}$$

Note:
Exercise:

Problem: Condense $4(3\log(x) + \log(x+5) - \log(2x+3))$.

Solution:

$\log \frac{x^{12}(x+5)^4}{(2x+3)^4}$; this answer could also be written $\log \left(\frac{x^3(x+5)}{(2x+3)}\right)^4$.

Example:
Exercise:

Problem:
Applying of the Laws of Logs

Recall that, in chemistry, $\text{pH} = -\log[H^+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution:

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is
Equation:

$$\text{pH} = -\log(2C)$$

Using the product rule of logs
Equation:

$$\text{pH} = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$,the new pH is
Equation:

$$\text{pH} = P - \log(2) \approx P - 0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

Note:
Exercise:

Problem: How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Solution:

The pH increases by about 0.301.

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e,we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the one-to-one property and **power rule for logarithms**.

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,we show
Equation:

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M.$ By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M.$ It follows that
Equation:

$$\begin{array}{lll} \log_n(b^y) & = \log_n M & \text{Apply the one-to-one property.} \\ y\log_n b & = \log_n M & \text{Apply the power rule for logarithms.} \\ y & = \frac{\log_n M}{\log_n b} & \text{Isolate } y. \\ \log_b M & = \frac{\log_n M}{\log_n b} & \text{Substitute for } y. \end{array}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.
Equation:

$$\begin{array}{lll} \log_5 36 & = \frac{\log(36)}{\log(5)} & \text{Apply the change of base formula using base 10.} \\ & \approx 2.2266 & \text{Use a calculator to evaluate to 4 decimal places.} \end{array}$$

Note:
The Change-of-Base Formula
The **change-of-base formula** can be used to evaluate a logarithm with any base.
For any positive real numbers $M, b,$ and n, where $n \neq 1$ and $b \neq 1$,
Equation:

$$\log_b M = \frac{\log_n M}{\log_n b}.$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.
Equation:

$$\log_b M = \frac{\ln M}{\ln b}$$

and
Equation:

$$\log_b M = \frac{\log M}{\log b}$$

Note:
Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$.

1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$,has base e.
2. Rewrite the log as a quotient using the change-of-base formula

    a. The numerator of the quotient will be a logarithm with base n and argument M.
    b. The denominator of the quotient will be a logarithm with base n and argument b.

Example:
Exercise:

Problem:
Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.

Solution:

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.
Equation:

$$\begin{aligned} \log_b M &= \frac{\ln M}{\ln b} \\ \log_5 3 &= \frac{\ln 3}{\ln 5} \end{aligned}$$

Note:
Exercise:

Problem: Change $\log_{0.5} 8$ to a quotient of natural logarithms.

Solution:

$\frac{\ln 8}{\ln 0.5}$

Note:
Can we change common logarithms to natural logarithms?
Yes. Remember that $\log 9$ *means* $\log_{10} 9$. *So,* $\log 9 = \frac{\ln 9}{\ln 10}$.

Example:
Exercise:

Problem:
Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

Solution:

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.
Equation:

$$\begin{array}{rll} \log_2 10 = \frac{\ln 10}{\ln 2} & \text{Apply the change of base formula using base } e. \\ \approx 3.3219 & \text{Use a calculator to evaluate to 4 decimal places.} \end{array}$$

Note:
Exercise:

Problem: Evaluate $\log_5(100)$ using the change-of-base formula.

Solution:

$\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$

Note:
Access these online resources for additional instruction and practice with laws of logarithms.

- The Properties of Logarithms
- Expand Logarithmic Expressions
- Evaluate a Natural Logarithmic Expression

Key Equations

The Product Rule for Logarithms	$\log_b(MN) = \log_b(M) + \log_b(N)$
The Quotient Rule for Logarithms	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
The Power Rule for Logarithms	$\log_b(M^n) = n\log_b M$
The Change-of-Base Formula	$\log_b M = \frac{\log_n M}{\log_n b} \qquad n > 0, n \neq 1, b \neq 1$

Key Concepts

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See [link].

- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See [link].
- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See [link], [link], and [link].
- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See [link], [link], and [link].
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See [link], [link], [link], and [link].
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See [link].
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and e as the quotient of natural or common logs. That way a calculator can be used to evaluate. See [link].

Section Exercises

Verbal

Exercise:

Problem: How does the power rule for logarithms help when solving logarithms with the form $\log_b\left(\sqrt[n]{x}\right)$?

Solution:

Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log_b(x)$.

Exercise:

Problem: What does the change-of-base formula do? Why is it useful when using a calculator?

Algebraic

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

Exercise:

Problem: $\log_b\left(7x \cdot 2y\right)$

Solution:

$\log_b\left(2\right) + \log_b\left(7\right) + \log_b\left(x\right) + \log_b\left(y\right)$

Exercise:

Problem: $\ln\left(3ab \cdot 5c\right)$

Exercise:

Problem: $\log_b\left(\frac{13}{17}\right)$

Solution:

$\log_b\left(13\right) - \log_b\left(17\right)$

Exercise:

Problem: $\log_4\left(\frac{\frac{x}{z}}{w}\right)$

Exercise:

Problem: $\ln\left(\frac{1}{4^k}\right)$

Solution:

$-k\ln(4)$

Exercise:

Problem: $\log_2\left(y^x\right)$

For the following exercises, condense to a single logarithm if possible.

Exercise:

Problem: $\ln(7) + \ln(x) + \ln(y)$

Solution:

$\ln(7xy)$

Exercise:

Problem: $\log_3(2) + \log_3(a) + \log_3(11) + \log_3(b)$

Exercise:

Problem: $\log_b(28) - \log_b(7)$

Solution:

$\log_b(4)$

Exercise:

Problem: $\ln(a) - \ln(d) - \ln(c)$

Exercise:

Problem: $-\log_b\left(\frac{1}{7}\right)$

Solution:

$\log_b(7)$

Exercise:

Problem: $\frac{1}{3}\ln(8)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

Exercise:

Problem: $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$

Solution:

$15\log(x) + 13\log(y) - 19\log(z)$

Exercise:

Problem: $\ln\left(\frac{a^{-2}}{b^{-4}c^{5}}\right)$

Exercise:

Problem: $\log\left(\sqrt{x^3y^{-4}}\right)$

Solution:

$\frac{3}{2}\log(x) - 2\log(y)$

Exercise:

Problem: $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$

Exercise:

Problem: $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$

Solution:

$\frac{8}{3}\log(x) + \frac{14}{3}\log(y)$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

Exercise:

Problem: $\log\left(2x^4\right) + \log\left(3x^5\right)$

Exercise:

Problem: $\ln(6x^9) - \ln(3x^2)$

Solution:

$\ln(2x^7)$

Exercise:

Problem: $2\log(x) + 3\log(x+1)$

Exercise:

Problem: $\log(x) - \frac{1}{2}\log(y) + 3\log(z)$

Solution:

$\log\left(\frac{xz^3}{\sqrt{y}}\right)$

Exercise:

Problem: $4\log_7(c) + \frac{\log_7(a)}{3} + \frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

Exercise:

Problem: $\log_7(15)$ to base e

Solution:

$\log_7(15) = \frac{\ln(15)}{\ln(7)}$

Exercise:

Problem: $\log_{14}(55.875)$ to base 10

For the following exercises, suppose $\log_5(6) = a$ and $\log_5(11) = b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.

Exercise:

Problem: $\log_{11}(5)$

Solution:

$\log_{11}(5) = \frac{\log_5(5)}{\log_5(11)} = \frac{1}{b}$

Exercise:

Problem: $\log_6(55)$

Exercise:

Problem: $\log_{11}\left(\frac{6}{11}\right)$

Solution:

$\log_{11}\left(\frac{6}{11}\right) = \frac{\log_5\left(\frac{6}{11}\right)}{\log_5(11)} = \frac{\log_5(6) - \log_5(11)}{\log_5(11)} = \frac{a-b}{b} = \frac{a}{b} - 1$

Numeric

For the following exercises, use properties of logarithms to evaluate without using a calculator.

Exercise:

Problem: $\log_3\left(\frac{1}{9}\right) - 3\log_3(3)$

Exercise:

Problem: $6\log_8(2)+\frac{\log_8(64)}{3\log_8(4)}$

Solution:

3

Exercise:

Problem: $2\log_9(3)-4\log_9(3)+\log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

Exercise:

Problem: $\log_3(22)$

Solution:

2.81359

Exercise:

Problem: $\log_8(65)$

Exercise:

Problem: $\log_6(5.38)$

Solution:

0.93913

Exercise:

Problem: $\log_4\left(\frac{15}{2}\right)$

Exercise:

Problem: $\log_{\frac{1}{2}}(4.7)$

Solution:

-2.23266

Extensions

Exercise:

Problem:

Use the product rule for logarithms to find all x values such that $\log_{12}(2x+6)+\log_{12}(x+2)=2.$ Show the steps for solving.

Exercise:

Problem:

Use the quotient rule for logarithms to find all x values such that $\log_6(x+2) - \log_6(x-3) = 1$. Show the steps for solving.

Solution:

$x = 4$; By the quotient rule: $\log_6(x+2) - \log_6(x-3) = \log_6\left(\frac{x+2}{x-3}\right) = 1$.

Rewriting as an exponential equation and solving for x :

$$\begin{aligned} 6^1 &= \frac{x+2}{x-3} \\ 0 &= \frac{x+2}{x-3} - 6 \\ 0 &= \frac{x+2}{x-3} - \frac{6(x-3)}{(x-3)} \\ 0 &= \frac{x+2-6x+18}{x-3} \\ 0 &= \frac{x-4}{x-3} \\ x &= 4 \end{aligned}$$

Checking, we find that $\log_6(4+2) - \log_6(4-3) = \log_6(6) - \log_6(1)$ is defined, so $x = 4$.

Exercise:

Problem:

Can the power property of logarithms be derived from the power property of exponents using the equation $b^x = m$? If not, explain why. If so, show the derivation.

Exercise:

Problem: Prove that $\log_b(n) = \frac{1}{\log_n(b)}$ for any positive integers $b > 1$ and $n > 1$.

Solution:

Let b and n be positive integers greater than 1. Then, by the change-of-base formula, $\log_b(n) = \frac{\log_n(n)}{\log_n(b)} = \frac{1}{\log_n(b)}$.

Exercise:

Problem: Does $\log_{81}(2401) = \log_3(7)$? Verify the claim algebraically.

Glossary

change-of-base formula
: a formula for converting a logarithm with any base to a quotient of logarithms with any other base.

power rule for logarithms
: a rule of logarithms that states that the log of a power is equal to the product of the exponent and the log of its base

product rule for logarithms
: a rule of logarithms that states that the log of a product is equal to a sum of logarithms

quotient rule for logarithms
: a rule of logarithms that states that the log of a quotient is equal to a difference of logarithms

Exponential and Logarithmic Equations
In this section, you will:

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the “rabbit plague.” (credit: Richard Taylor, Flickr)

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers b, S, and T, where $b>0,\ b\neq 1$,$b^S=b^T$ if and only if $S=T$.

In other words, when an exponential equation has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7}=\frac{3^{2x}}{3}$. To solve for x, we use the division property of exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for x :
Equation:

$$\begin{aligned}
3^{4x-7} &= \frac{3^{2x}}{3} && \\
3^{4x-7} &= \frac{3^{2x}}{3^1} && \text{Rewrite 3 as } 3^1. \\
3^{4x-7} &= 3^{2x-1} && \text{Use the division property of exponents.} \\
4x-7 &= 2x-1 && \text{Apply the one-to-one property of exponents.} \\
2x &= 6 && \text{Subtract } 2x \text{ and add 7 to both sides.} \\
x &= 3 && \text{Divide by 3.}
\end{aligned}$$

Note:
Using the One-to-One Property of Exponential Functions to Solve Exponential Equations
For any algebraic expressions S and T, and any positive real number $b \neq 1$,
Equation:

$$b^S = b^T \text{ if and only if } S = T$$

Note:
Given an exponential equation with the form $b^S = b^T$, where S and T are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Example:
Exercise:

Problem:
Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

Solution:
Equation:

$$\begin{aligned}
2^{x-1} &= 2^{2x-4} && \text{The common base is } 2. \\
x-1 &= 2x-4 && \text{By the one-to-one property the exponents must be equal.} \\
x &= 3 && \text{Solve for } x.
\end{aligned}$$

Note:
Exercise:

Problem: Solve $5^{2x} = 5^{3x+2}$.

Solution:

$x = -2$

Rewriting Equations So All Powers Have the Same Base

Sometimes the common base for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for x :

Equation:

$$\begin{array}{rll} 256 &= 4^{x-5} & \\ 2^8 &= \left(2^2\right)^{x-5} & \text{Rewrite each side as a power with base 2.} \\ 2^8 &= 2^{2x-10} & \text{Use the one-to-one property of exponents.} \\ 8 &= 2x - 10 & \text{Apply the one-to-one property of exponents.} \\ 18 &= 2x & \text{Add 10 to both sides.} \\ x &= 9 & \text{Divide by 2.} \end{array}$$

Note:
Given an exponential equation with unlike bases, use the one-to-one property to solve it.

1. Rewrite each side in the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
3. Use the one-to-one property to set the exponents equal.
4. Solve the resulting equation, $S = T$, for the unknown.

Example:
Exercise:

Problem:
Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2} = 16^{x+1}$.

Solution:
Equation:

$$\begin{array}{rll} 8^{x+2} &= 16^{x+1} & \\ \left(2^3\right)^{x+2} &= \left(2^4\right)^{x+1} & \text{Write 8 and 16 as powers of 2.} \\ 2^{3x+6} &= 2^{4x+4} & \text{To take a power of a power, multiply exponents.} \\ 3x + 6 &= 4x + 4 & \text{Use the one-to-one property to set the exponents equal.} \\ x &= 2 & \text{Solve for } x. \end{array}$$

Note:
Exercise:

Problem: Solve $5^{2x} = 25^{3x+2}$.

Solution:

$x = -1$

Example:
Exercise:

Problem:
Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x} = \sqrt{2}$.

Solution:
Equation:

$$\begin{array}{rl} 2^{5x} = 2^{\frac{1}{2}} & \text{Write the square root of } 2 \text{ as a power of } 2. \\ 5x = \frac{1}{2} & \text{Use the one-to-one property.} \\ x = \frac{1}{10} & \text{Solve for } x. \end{array}$$

Note:
Exercise:

Problem: Solve $5^{x} = \sqrt{5}$.

Solution:

$x = \frac{1}{2}$

Note:
Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?
No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.

Example:
Exercise:

Problem:
Solving an Equation with Positive and Negative Powers

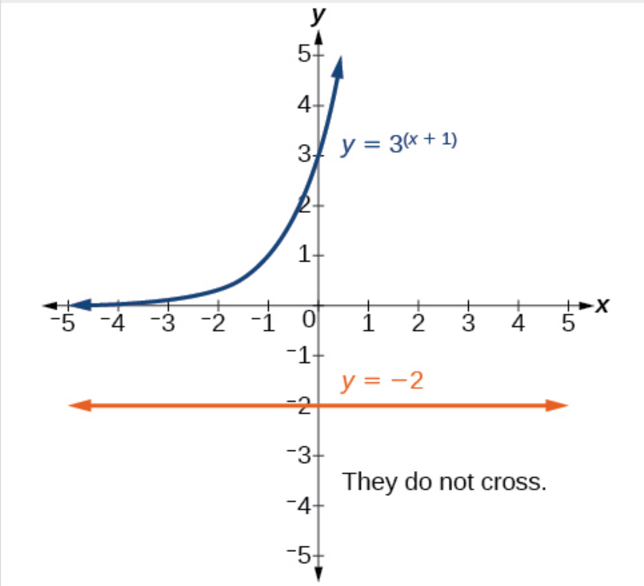
Solve $3^{x+1} = -2$.

Solution:

This equation has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Analysis

[link] shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.



Note:
Exercise:

Problem: Solve $2^x = -100$.

Solution:

The equation has no solution.

Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

Note:
Given an exponential equation in which a common base cannot be found, solve for the unknown.

1. Apply the logarithm of both sides of the equation.
   a. If one of the terms in the equation has base 10, use the common logarithm.

b. If none of the terms in the equation has base 10, use the natural logarithm.

2. Use the rules of logarithms to solve for the unknown.

Example:
Exercise:

Problem:
Solving an Equation Containing Powers of Different Bases

Solve $5^{x+2} = 4^x$.

Solution:
Equation:

$5^{x+2} = 4^x$	There is no easy way to get the powers to have the same base.
$\ln 5^{x+2} = \ln 4^x$	Take ln of both sides.
$(x+2)\ln 5 = x \ln 4$	Use laws of logs.
$x \ln 5 + 2 \ln 5 = x \ln 4$	Use the distributive law.
$x \ln 5 - x \ln 4 = -2 \ln 5$	Get terms containing x on one side, terms without x on the other.
$x(\ln 5 - \ln 4) = -2 \ln 5$	On the left hand side, factor out an x.
$x \ln\left(\frac{5}{4}\right) = \ln\left(\frac{1}{25}\right)$	Use the laws of logs.
$x = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}$	Divide by the coefficient of x.

Note:
Exercise:

Problem: Solve $2^x = 3^{x+1}$.

Solution:

$x = \frac{\ln 3}{\ln\left(2/3\right)}$

Note:
Is there any way to solve $2^x = 3^x$?
Yes. The solution is 0.

Equations Containing *e*

One common type of exponential equations are those with base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

Note:
Given an equation of the form $y = Ae^{kt}$, solve for t.

1. Divide both sides of the equation by A.
2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by k.

Example:
Exercise:

Problem:
Solve an Equation of the Form *y* = *Ae*^*kt*

Solve $100 = 20e^{2t}$.

Solution:
Equation:

$$\begin{array}{lll} 100 & = 20e^{2t} & \\ 5 & = e^{2t} & \text{Divide by the coefficient of the power.} \\ \ln 5 & = 2t & \text{Take ln of both sides. Use the fact that } \ln(x) \text{ and } e^x \text{ are inverse functions.} \\ t & = \frac{\ln 5}{2} & \text{Divide by the coefficient of } t. \end{array}$$

Analysis

Using laws of logs, we can also write this answer in the form $t = \ln\sqrt{5}$.If we want a decimal approximation of the answer, we use a calculator.

Note:
Exercise:

Problem: Solve $3e^{0.5t} = 11$.

Solution:

$t = 2\ln\left(\frac{11}{3}\right)$ or $\ln\left(\frac{11}{3}\right)^2$

Note:
Does every equation of the form $y = Ae^{kt}$ have a solution?
No. There is a solution when $k \neq 0$,and when y and A are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is $2 = -3e^t$.

Example:
Exercise:

Problem:

Solving an Equation That Can Be Simplified to the Form $y = Ae^{kt}$

Solve $4e^{2x}+5=12$.

Solution:
Equation:

$$\begin{aligned} 4e^{2x}+5 &= 12 & \\ 4e^{2x} &= 7 & \text{Combine like terms.} \\ e^{2x} &= \tfrac{7}{4} & \text{Divide by the coefficient of the power.} \\ 2x &= \ln\left(\tfrac{7}{4}\right) & \text{Take ln of both sides.} \\ x &= \tfrac{1}{2}\ln\left(\tfrac{7}{4}\right) & \text{Solve for } x. \end{aligned}$$

Note:
Exercise:

Problem: Solve $3+e^{2t}=7e^{2t}$.

Solution:

$t=\ln\left(\frac{1}{\sqrt{2}}\right)=-\frac{1}{2}\ln(2)$

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

Example:
Exercise:

Problem:
Solving Exponential Functions in Quadratic Form

Solve $e^{2x}-e^{x}=56$.

Solution:
Equation:

$e^{2x} - e^x$	$= 56$	
$e^{2x} - e^x - 56$	$= 0$	Get one side of the equation equal to zero.
$(e^x + 7)(e^x - 8)$	$= 0$	Factor by the FOIL method.
$e^x + 7$	$= 0$ or $e^x - 8 = 0$	If a product is zero, then one factor must be zero.
e^x	$= -7$ or $e^x = 8$	Isolate the exponentials.
e^x	$= 8$	Reject the equation in which the power equals a negative n
x	$= \ln 8$	Solve the equation in which the power equals a positive nu

Analysis

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^x = -7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

Note:
Exercise:

Problem: Solve $e^{2x} = e^x + 2$.

Solution:

$x = \ln 2$

Note:
Does every logarithmic equation have a solution?
No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x.$ We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3.$ To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x :
Equation:

$\log_2(2) + \log_2(3x - 5) = 3$	
$\log_2(2(3x - 5)) = 3$	Apply the product rule of logarithms.
$\log_2(6x - 10) = 3$	Distribute.
$2^3 = 6x - 10$	Apply the definition of a logarithm.
$8 = 6x - 10$	Calculate 2^3.
$18 = 6x$	Add 10 to both sides.
$x = 3$	Divide by 6.

Note:
Using the Definition of a Logarithm to Solve Logarithmic Equations
For any algebraic expression S and real numbers b and c,where $b > 0,\ b \neq 1$,
Equation:

$$\log_b(S) = c \text{ if and only if } b^c = S$$

Example:
Exercise:

Problem:
Using Algebra to Solve a Logarithmic Equation

Solve $2\ln x + 3 = 7$.

Solution:
Equation:

$$\begin{aligned} 2\ln x + 3 &= 7 & \\ 2\ln x &= 4 & \text{Subtract 3.} \\ \ln x &= 2 & \text{Divide by 2.} \\ x &= e^2 & \text{Rewrite in exponential form.} \end{aligned}$$

Note:
Exercise:

Problem: Solve $6 + \ln x = 10$.

Solution:

$x = e^4$

Example:
Exercise:

Problem:
Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2\ln(6x) = 7$.

Solution:
Equation:

$$\begin{aligned} 2\ln(6x) &= 7 \\ \ln(6x) &= \frac{7}{2} && \text{Divide by 2.} \\ 6x &= e^{\left(\frac{7}{2}\right)} && \text{Use the definition of } \ln. \\ x &= \frac{1}{6}e^{\left(\frac{7}{2}\right)} && \text{Divide by 6.} \end{aligned}$$

Note:
Exercise:

Problem: Solve $2\ln(x+1) = 10.$

Solution:

$x = e^5 - 1$

Example:
Exercise:

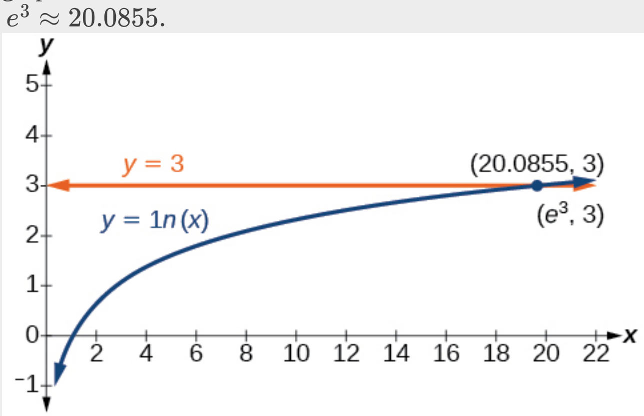
Problem:
Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x = 3.$

Solution:
Equation:

$$\begin{aligned} \ln x &= 3 \\ x &= e^3 && \text{Use the definition of the natural logarithm.} \end{aligned}$$

[link] represents the graph of the equation. On the graph, the *x*-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855.$



The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3)$, which is approximately (20.0855, 3).

Note:
Exercise:

Problem:

Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

Solution:

$x \approx 9.97$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

Equation:

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

Equation:

$$\text{If } \log_2(x-1) = \log_2(8), \text{then } x - 1 = 8.$$

So, if $x - 1 = 8$,then we can solve for x,and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x :

Equation:

$$\begin{array}{rl}
\log(3x-2) - \log(2) = \log(x+4) & \\
\log\left(\frac{3x-2}{2}\right) = \log(x+4) & \text{Apply the quotient rule of logarithms.} \\
\frac{3x-2}{2} = x+4 & \text{Apply the one to one property of a logarithm.} \\
3x - 2 = 2x + 8 & \text{Multiply both sides of the equation by 2.} \\
x = 10 & \text{Subtract } 2x \text{ and add 2.}
\end{array}$$

To check the result, substitute $x = 10$ into $\log(3x-2) - \log(2) = \log(x+4)$.

Equation:

$$\begin{aligned} \log(3(10)-2)-\log(2) &= \log((10)+4) \\ \log(28)-\log(2) &= \log(14) \\ \log\left(\tfrac{28}{2}\right) &= \log(14) \qquad \text{The solution checks.} \end{aligned}$$

Note:
Using the One-to-One Property of Logarithms to Solve Logarithmic Equations
For any algebraic expressions S and T and any positive real number b, where $b \neq 1$,
Equation:

$$\log_b S = \log_b T \text{ if and only if } S = T$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

Note:
Given an equation containing logarithms, solve it using the one-to-one property.

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b S = \log_b T$.
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Example:
Exercise:

Problem:
Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2) = \ln(2x+3)$.

Solution:
Equation:

$$\begin{aligned} \ln(x^2) &= \ln(2x+3) \\ x^2 &= 2x+3 && \text{Use the one-to-one property of the logarithm.} \\ x^2-2x-3 &= 0 && \text{Get zero on one side before factoring.} \\ (x-3)(x+1) &= 0 && \text{Factor using FOIL.} \\ x-3=0 \text{ or } x+1 &= 0 && \text{If a product is zero, one of the factors must be zero.} \\ x=3 \text{ or } x &= -1 && \text{Solve for } x. \end{aligned}$$

Analysis

There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

Note:

Exercise:

Problem: Solve $\ln(x^2) = \ln 1$.

Solution:

$x = 1$ or $x = -1$

Solving Applied Problems Using Exponential and Logarithmic Equations

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its half-life. [link] lists the half-life for several of the more common radioactive substances.

Substance	**Use**	**Half-life**
gallium-67	nuclear medicine	80 hours
cobalt-60	manufacturing	5.3 years
technetium-99m	nuclear medicine	6 hours
americium-241	construction	432 years
carbon-14	archeological dating	5,715 years
uranium-235	atomic power	703,800,000 years

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:
Equation:

$$\begin{aligned} A(t) &= A_0 e^{\frac{\ln(0.5)}{T}t} \\ A(t) &= A_0 e^{\ln(0.5)\frac{t}{T}} \\ A(t) &= A_0 (e^{\ln(0.5)})^{\frac{t}{T}} \\ A(t) &= A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}} \end{aligned}$$

where

- A_0 is the amount initially present

- T is the half-life of the substance
- t is the time period over which the substance is studied
- y is the amount of the substance present after time t

Example:
Exercise:

Problem:
Using the Formula for Radioactive Decay to Find the Quantity of a Substance

How long will it take for ten percent of a 1000-gram sample of uranium-235 to decay?

Solution:
Equation:

$$\begin{array}{rll} y &= 1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} & \\ 900 &= 1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} & \text{After 10\% decays, 900 grams are left.} \\ 0.9 &= e^{\frac{\ln(0.5)}{703{,}800{,}000}t} & \text{Divide by 1000.} \\ \ln(0.9) &= \ln\left(e^{\frac{\ln(0.5)}{703{,}800{,}000}t}\right) & \text{Take ln of both sides.} \\ \ln(0.9) &= \frac{\ln(0.5)}{703{,}800{,}000}t & \ln(e^M) = M \\ t &= 703{,}800{,}000 \times \frac{\ln(0.9)}{\ln(0.5)}\text{years} & \text{Solve for } t. \\ t &\approx 106{,}979{,}777 \text{ years} & \end{array}$$

Analysis

Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium-235 remaining is 900 grams.

Note:
Exercise:

Problem:

How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?

Solution:

$t = 703,800,000 \times \frac{\ln(0.8)}{\ln(0.5)} \text{ years } \approx 226,572,993 \text{ years.}$

Note:
Access these online resources for additional instruction and practice with exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Exponential Equations with Logarithms

Key Equations

One-to-one property for exponential functions	For any algebraic expressions S and T and any positive real number b, where $b^S = b^T$ if and only if $S = T$.
Definition of a logarithm	For any algebraic expression S and positive real numbers b and c, where $b \neq 1$, $\log_b(S) = c$ if and only if $b^c = S$.
One-to-one property for logarithmic functions	For any algebraic expressions S and T and any positive real number b, where $b \neq 1$, $\log_b S = \log_b T$ if and only if $S = T$.

Key Concepts

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown. See [link].
- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown. See [link], [link], and [link].
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side. See [link].
- We can solve exponential equations with base e,by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other. See [link] and [link].
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions. See [link].
- When given an equation of the form $\log_b(S) = c$, where S is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown. See [link] and [link].
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the *x*-value of the intersecting point. See [link].
- When given an equation of the form $\log_b S = \log_b T$, where S and T are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. See [link].
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations, whether the unknown is in an exponent or in the argument of a logarithm. See [link].

Section Exercises

Verbal

Exercise:

Problem: How can an exponential equation be solved?

Solution:

Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve.

Exercise:

Problem: When does an extraneous solution occur? How can an extraneous solution be recognized?

Exercise:

Problem:

When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

Solution:

The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base.

Algebraic

For the following exercises, use like bases to solve the exponential equation.

Exercise:

Problem: $4^{-3v-2} = 4^{-v}$

Exercise:

Problem: $64 \cdot 4^{3x} = 16$

Solution:

$x = -\frac{1}{3}$

Exercise:

Problem: $3^{2x+1} \cdot 3^{x} = 243$

Exercise:

Problem: $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$

Solution:

$n = -1$

Exercise:

Problem: $625 \cdot 5^{3x+3} = 125$

Exercise:

Problem: $\frac{36^{3b}}{36^{2b}} = 216^{2-b}$

Solution:

$b = \frac{6}{5}$

Exercise:

Problem: $\left(\frac{1}{64}\right)^{3n} \cdot 8 = 2^6$

For the following exercises, use logarithms to solve.

Exercise:

Problem: $9^{x-10} = 1$

Solution:

$x = 10$

Exercise:

Problem: $2e^{6x} = 13$

Exercise:

Problem: $e^{r+10} - 10 = -42$

Solution:

No solution

Exercise:

Problem: $2 \cdot 10^{9a} = 29$

Exercise:

Problem: $-8 \cdot 10^{p+7} - 7 = -24$

Solution:

$p = \log\left(\frac{17}{8}\right) - 7$

Exercise:

Problem: $7e^{3n-5} + 5 = -89$

Exercise:

Problem: $e^{-3k} + 6 = 44$

Solution:

$k = -\frac{\ln(38)}{3}$

Exercise:

Problem: $-5e^{9x-8} - 8 = -62$

Exercise:

Problem: $-6e^{9x+8} + 2 = -74$

Solution:

$x = \frac{\ln\left(\frac{38}{3}\right) - 8}{9}$

Exercise:

Problem: $2^{x+1} = 5^{2x-1}$

Exercise:

Problem: $e^{2x} - e^{x} - 132 = 0$

Solution:

$x = \ln 12$

Exercise:

Problem: $7e^{8x+8} - 5 = -95$

Exercise:

Problem: $10e^{8x+3} + 2 = 8$

Solution:

$x = \frac{\ln\left(\frac{3}{5}\right) - 3}{8}$

Exercise:

Problem: $4e^{3x+3} - 7 = 53$

Exercise:

Problem: $8e^{-5x-2} - 4 = -90$

Solution:

no solution

Exercise:

Problem: $3^{2x+1} = 7^{x-2}$

Exercise:

Problem: $e^{2x} - e^{x} - 6 = 0$

Solution:

$x = \ln(3)$

Exercise:

Problem: $3e^{3-3x} + 6 = -31$

For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.

Exercise:

Problem: $\log\left(\frac{1}{100}\right) = -2$

Solution:

$10^{-2} = \frac{1}{100}$

Exercise:

Problem: $\log_{324}(18) = \frac{1}{2}$

For the following exercises, use the definition of a logarithm to solve the equation.

Exercise:

Problem: $5\log_7 n = 10$

Solution:

$n = 49$

Exercise:

Problem: $-8\log_9 x = 16$

Exercise:

Problem: $4 + \log_2(9k) = 2$

Solution:

$k = \frac{1}{36}$

Exercise:

Problem: $2\log(8n + 4) + 6 = 10$

Exercise:

Problem: $10 - 4\ln(9 - 8x) = 6$

Solution:

$x = \frac{9-e}{8}$

For the following exercises, use the one-to-one property of logarithms to solve.

Exercise:

Problem: $\ln(10-3x)=\ln(-4x)$

Exercise:

Problem: $\log_{13}(5n-2)=\log_{13}(8-5n)$

Solution:

$n=1$

Exercise:

Problem: $\log(x+3)-\log(x)=\log(74)$

Exercise:

Problem: $\ln(-3x)=\ln\left(x^2-6x\right)$

Solution:

No solution

Exercise:

Problem: $\log_4(6-m)=\log_4 3m$

Exercise:

Problem: $\ln(x-2)-\ln(x)=\ln(54)$

Solution:

No solution

Exercise:

Problem: $\log_9\left(2n^2-14n\right)=\log_9\left(-45+n^2\right)$

Exercise:

Problem: $\ln\left(x^2-10\right)+\ln(9)=\ln(10)$

Solution:

$x=\pm\frac{10}{3}$

For the following exercises, solve each equation for x.

Exercise:

Problem: $\log(x+12)=\log(x)+\log(12)$

Exercise:

Problem: $\ln(x)+\ln(x-3)=\ln(7x)$

Solution:

$x = 10$

Exercise:

Problem: $\log_2(7x + 6) = 3$

Exercise:

Problem: $\ln(7) + \ln\left(2 - 4x^2\right) = \ln(14)$

Solution:

$x = 0$

Exercise:

Problem: $\log_8(x + 6) - \log_8(x) = \log_8(58)$

Exercise:

Problem: $\ln(3) - \ln(3 - 3x) = \ln(4)$

Solution:

$x = \frac{3}{4}$

Exercise:

Problem: $\log_3(3x) - \log_3(6) = \log_3(77)$

Graphical

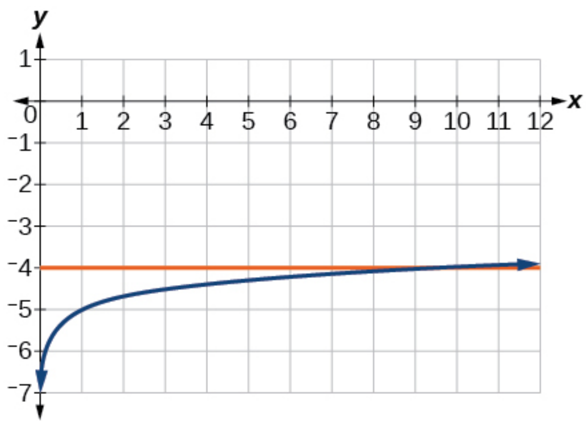
For the following exercises, solve the equation for x,if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.

Exercise:

Problem: $\log_9(x) - 5 = -4$

Solution:

$x = 9$



Exercise:

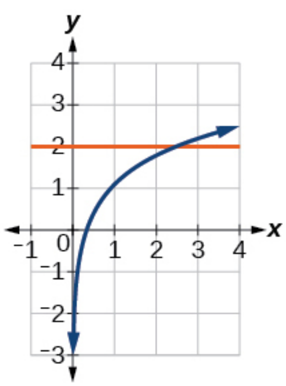
Problem: $\log_3(x) + 3 = 2$

Exercise:

Problem: $\ln(3x) = 2$

Solution:

$x = \frac{e^2}{3} \approx 2.5$



Exercise:

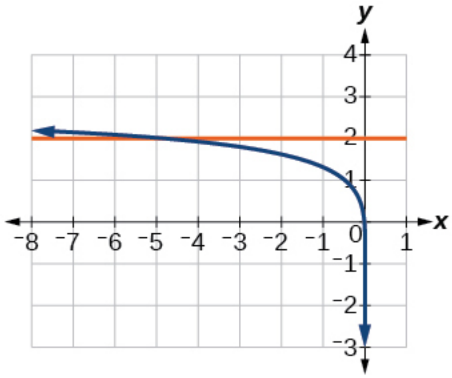
Problem: $\ln(x - 5) = 1$

Exercise:

Problem: $\log(4) + \log(-5x) = 2$

Solution:

$x = -5$



Exercise:

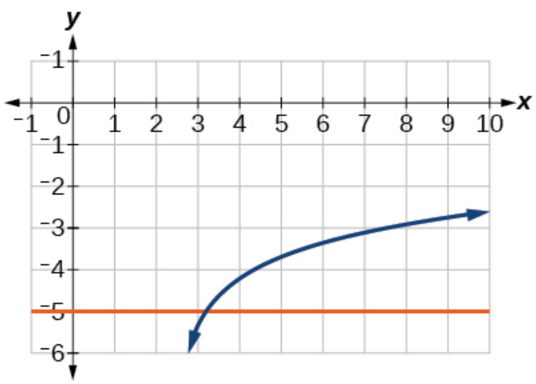
Problem: $-7 + \log_3(4 - x) = -6$

Exercise:

Problem: $\ln(4x - 10) - 6 = -5$

Solution:

$x = \frac{e+10}{4} \approx 3.2$



Exercise:

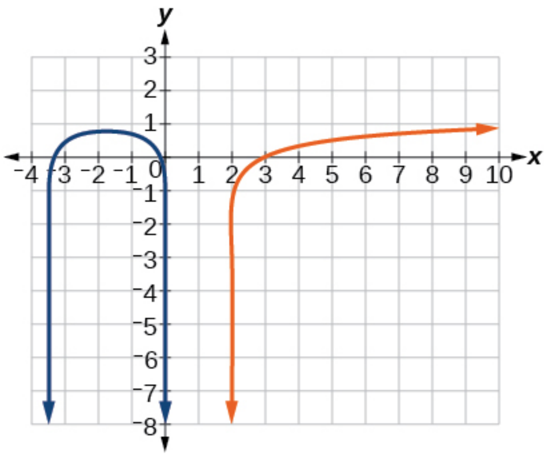
Problem: $\log(4 - 2x) = \log(-4x)$

Exercise:

Problem: $\log_{11}\left(-2x^2 - 7x\right) = \log_{11}(x - 2)$

Solution:

No solution



Exercise:

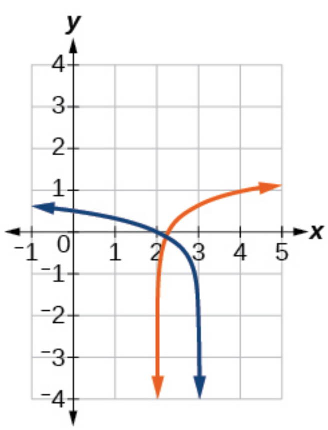
Problem: $\ln(2x+9) = \ln(-5x)$

Exercise:

Problem: $\log_9(3-x) = \log_9(4x-8)$

Solution:

$x = \frac{11}{5} \approx 2.2$



Exercise:

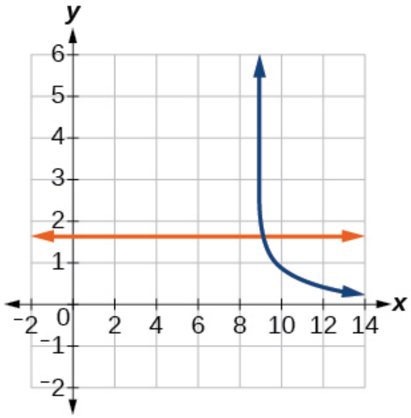
Problem: $\log\left(x^2+13\right) = \log(7x+3)$

Exercise:

Problem: $\frac{3}{\log_2(10)} - \log(x-9) = \log(44)$

Solution:

$x = \frac{101}{11} \approx 9.2$



Exercise:

Problem: $\ln(x) - \ln(x+3) = \ln(6)$

For the following exercises, solve for the indicated value, and graph the situation showing the solution point.

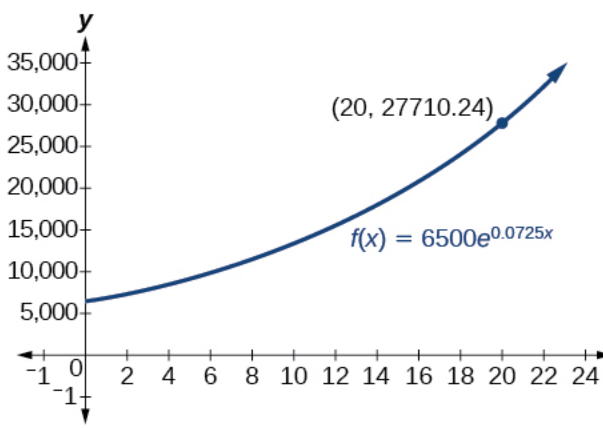
Exercise:

Problem:

An account with an initial deposit of $\$6{,}500$ earns 7.25% annual interest, compounded continuously. How much will the account be worth after 20 years?

Solution:

about $\$27,710.24$



Exercise:

Problem:

The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8.3 \cdot 10^2$ watts per square meter?

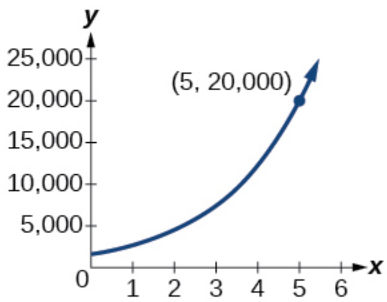
Exercise:

Problem:

The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach $20{,}000$?

Solution:

about 5 years



Technology

For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate the variable to 3 decimal places.

Exercise:

Problem: $1000(1.03)^t = 5000$ using the common log.

Exercise:

Problem: $e^{5x} = 17$ using the natural log

Solution:

$\frac{\ln(17)}{5} \approx 0.567$

Exercise:

Problem: $3(1.04)^{3t} = 8$ using the common log

Exercise:

Problem: $3^{4x-5} = 38$ using the common log

Solution:

$x = \frac{\log(38)+5\log(3)}{4\log(3)} \approx 2.078$

Exercise:

Problem: $50e^{-0.12t} = 10$ using the natural log

For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.

Exercise:

Problem: $7e^{3x-5} + 7.9 = 47$

Solution:

$x \approx 2.2401$

Exercise:

Problem: $\ln(3) + \ln(4.4x + 6.8) = 2$

Exercise:

Problem: $\log(-0.7x - 9) = 1 + 5\log(5)$

Solution:

$x \approx -44655.7143$

Exercise:

Problem:

Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch? (*Hint*: there are 5280 feet in a mile)

Exercise:

Problem:

The magnitude M of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules and $E_0 = 10^{4.4}$ is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

Solution:

about 5.83

Extensions

Exercise:

Problem:

Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b x} = x$.

Exercise:

Problem:

Recall the formula for continually compounding interest, $y = Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

Solution:

$t = \ln\left(\left(\frac{y}{A}\right)^{\frac{1}{k}}\right)$

Exercise:

Problem:

Recall the compound interest formula $A = a\left(1 + \frac{r}{k}\right)^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t.

Exercise:

Problem:

Newton's Law of Cooling states that the temperature T of an object at any time t can be described by the equation $T = T_s + (T_0 - T_s)e^{-kt}$, where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and $k$is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

Solution:

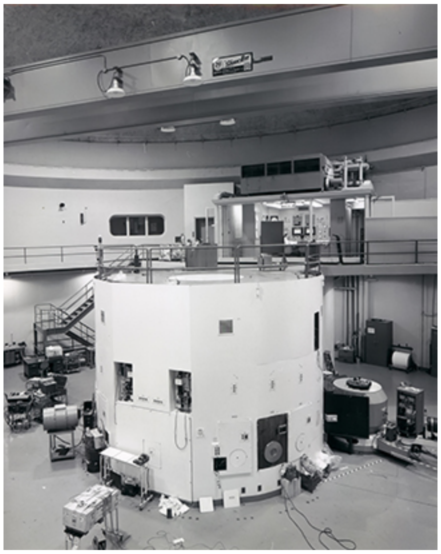
$t = \ln\left(\left(\frac{T - T_s}{T_0 - T_s}\right)^{-\frac{1}{k}}\right)$

Glossary

extraneous solution
: a solution introduced while solving an equation that does not satisfy the conditions of the original equation

Exponential and Logarithmic Models
In this section, you will:

- Model exponential growth and decay.
- Use Newton’s Law of Cooling.
- Use logistic-growth models.
- Choose an appropriate model for data.
- Express an exponential model in base e.



A nuclear research reactor inside the Neely Nuclear Research Center on the Georgia Institute of Technology campus (credit: Georgia Tech Research Institute)

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and Newton’s Law of Cooling.

Modeling Exponential Growth and Decay

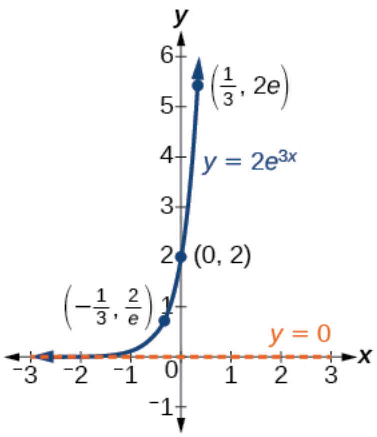
In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function:
Equation:

$$y = A_0 e^{kt}$$

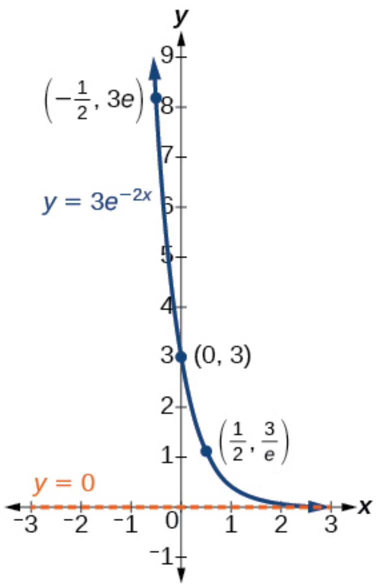
where A_0 is equal to the value at time zero, e is Euler’s constant, and k is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double. Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth based on a doubling time. In some applications, however, as we will see when we discuss the logistic equation, the logistic model sometimes fits the data better than the exponential model.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the **exponential decay** model. Again, we have the form $y = A_0 e^{kt}$ where A_0 is the starting value, and e is Euler's constant. Now k is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating **half-life**, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in [link] and [link]. It is important to remember that, although parts of each of the two graphs seem to lie on the *x*-axis, they are really a tiny distance above the *x*-axis.



A graph showing exponential growth. The equation is $y = 2e^{3x}$.



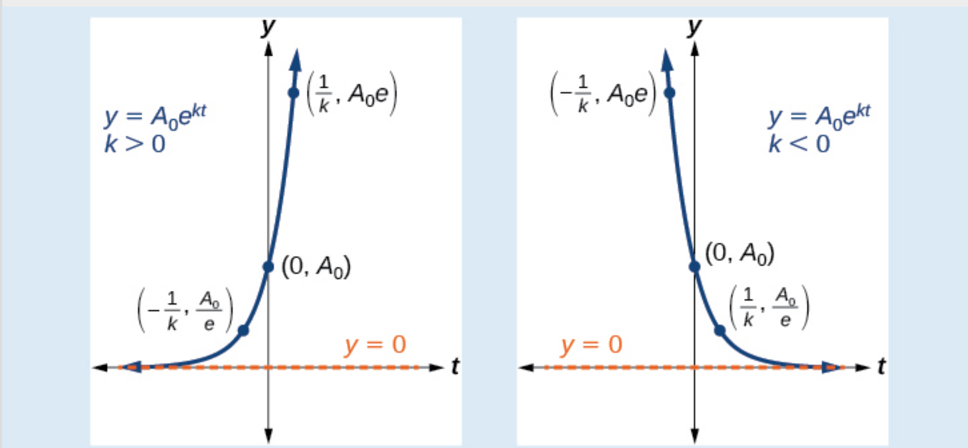
A graph showing exponential decay. The equation is

A graph showing exponential decay. The equation is $y = 3e^{-2x}$.

Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude. The **order of magnitude** is the power of ten, when the number is expressed in scientific notation, with one digit to the left of the decimal. For example, the distance to the nearest star, Proxima Centauri, measured in kilometers, is 40,113,497,200,000 kilometers. Expressed in scientific notation, this is $4.01134972 \times 10^{13}$. So, we could describe this number as having order of magnitude 10^{13}.

Note:
Characteristics of the Exponential Function, $y = A_0e^{kt}$
An exponential function with the form $y = A_0e^{kt}$ has the following characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x intercept: none
- y-intercept: $(0, A_0)$
- increasing if $k > 0$ (see [link])
- decreasing if $k < 0$ (see [link])



An exponential function models exponential growth when $k > 0$ and exponential decay when $k < 0$.

Example:
Exercise:

Problem:
Graphing Exponential Growth

A population of bacteria doubles every hour. If the culture started with 10 bacteria, graph the population as a function of time.

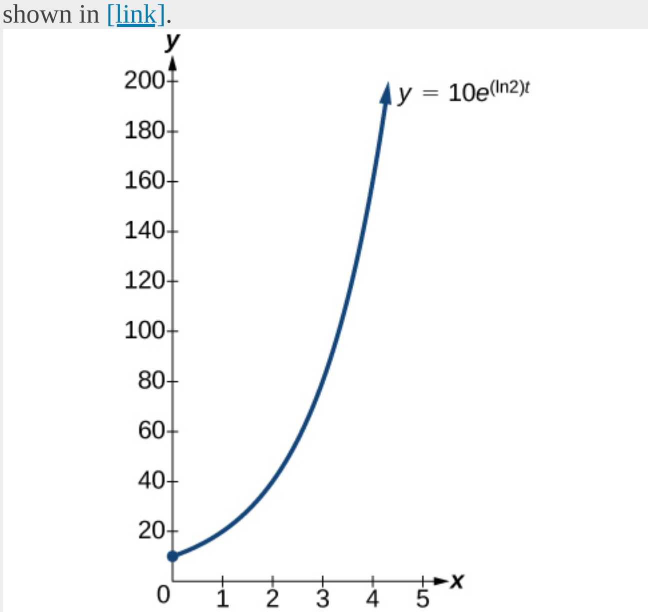
Solution:

When an amount grows at a fixed percent per unit time, the growth is exponential. To find A_0 we use the fact that A_0 is the amount at time zero, so $A_0 = 10$. To find k, use the fact that after one hour $(t = 1)$ the population doubles from 10 to 20. The formula is derived as follows

Equation:

$$\begin{aligned} 20 &= 10e^{k \cdot 1} \\ 2 &= e^k && \text{Divide by 10} \\ \ln 2 &= k && \text{Take the natural logarithm} \end{aligned}$$

so $k = \ln(2)$. Thus the equation we want to graph is $y = 10e^{(\ln 2)t} = 10\left(e^{\ln 2}\right)^t = 10 \cdot 2^t$. The graph is shown in [link].



The graph of $y = 10e^{(\ln 2)t}$

Analysis

The population of bacteria after ten hours is 10,240. We could describe this amount is being of the order of magnitude 10^4. The population of bacteria after twenty hours is 10,485,760 which is of the order of magnitude 10^7, so we could say that the population has increased by three orders of magnitude in ten hours.

Half-Life

We now turn to **exponential decay**. One of the common terms associated with exponential decay, as stated above, is **half-life**, the length of time it takes an exponentially decaying quantity to decrease to half its original amount. Every radioactive isotope has a half-life, and the process describing the exponential decay of an isotope is called radioactive decay.

To find the half-life of a function describing exponential decay, solve the following equation:

Equation:

$$\frac{1}{2}A_0 = A_o e^{kt}$$

We find that the half-life depends only on the constant k and not on the starting quantity A_0.

The formula is derived as follows
Equation:

$$\begin{aligned} \tfrac{1}{2}A_0 &= A_o e^{kt} & \\ \tfrac{1}{2} &= e^{kt} & \text{Divide by } A_0. \\ \ln\left(\tfrac{1}{2}\right) &= kt & \text{Take the natural log.} \\ -\ln(2) &= kt & \text{Apply laws of logarithms.} \\ -\tfrac{\ln(2)}{k} &= t & \text{Divide by } k. \end{aligned}$$

Since t, the time, is positive, k must, as expected, be negative. This gives us the half-life formula
Equation:

$$t = -\frac{\ln(2)}{k}$$

Note:
Given the half-life, find the decay rate.

1. Write $A = A_o e^{kt}$.
2. Replace A by $\frac{1}{2}A_0$ and replace t by the given half-life.
3. Solve to find k. Express k as an exact value (do not round).

Note: *It is also possible to find the decay rate using* $k = -\frac{\ln(2)}{t}$.

Example:
Exercise:

Problem:
Finding the Function that Describes Radioactive Decay

The half-life of carbon-14 is 5,730 years. Express the amount of carbon-14 remaining as a function of time, t.

Solution:

This formula is derived as follows.
Equation:

$$\begin{aligned}
A &= A_0 e^{kt} && \text{The continuous growth formula.} \\
0.5A_0 &= A_0 e^{k \cdot 5730} && \text{Substitute the half-life for } t \text{ and } 0.5A_0 \text{ for } f(t). \\
0.5 &= e^{5730k} && \text{Divide by } A_0. \\
\ln(0.5) &= 5730k && \text{Take the natural log of both sides.} \\
k &= \frac{\ln(0.5)}{5730} && \text{Divide by the coefficient of } k. \\
A &= A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t} && \text{Substitute for } r \text{ in the continuous growth formula.}
\end{aligned}$$

The function that describes this continuous decay is $f(t) = A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$. We observe that the coefficient of t, $\frac{\ln(0.5)}{5730} \approx -1.2097 \times 10^{-4}$ is negative, as expected in the case of exponential decay.

Note:
Exercise:

Problem:

The half-life of plutonium-244 is 80,000,000 years. Find function gives the amount of carbon-14 remaining as a function of time, measured in years.

Solution:

$f(t) = A_0 e^{-0.0000000087t}$

Radiocarbon Dating

The formula for radioactive decay is important in radiocarbon dating, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about 1% error for plants or animals that died within the last 60,000 years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of 5,730 years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12, which has an atomic weight of 12 and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last 60,000 years, using tree rings and other organic samples of known dates—although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is 5,730 years, the formula for the amount of carbon-14 remaining after t years is
Equation:

$$A \approx A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$$

where

- A is the amount of carbon-14 remaining
- A_0 is the amount of carbon-14 when the plant or animal began decaying.

This formula is derived as follows:
Equation:

$$\begin{aligned}
A &= A_0 e^{kt} && \text{The continuous growth formula.} \\
0.5A_0 &= A_0 e^{k \cdot 5730} && \text{Substitute the half-life for } t \text{ and } 0.5A_0 \text{ for } f(t). \\
0.5 &= e^{5730k} && \text{Divide by } A_0. \\
\ln(0.5) &= 5730k && \text{Take the natural log of both sides.} \\
k &= \frac{\ln(0.5)}{5730} && \text{Divide by the coefficient of } k. \\
A &= A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t} && \text{Substitute for } r \text{ in the continuous growth formula.}
\end{aligned}$$

To find the age of an object, we solve this equation for t :
Equation:

$$t = \frac{\ln\left(\frac{A}{A_0}\right)}{-0.000121}$$

Out of necessity, we neglect here the many details that a scientist takes into consideration when doing carbon-14 dating, and we only look at the basic formula. The ratio of carbon-14 to carbon-12 in the atmosphere is approximately 0.0000000001%. Let r be the ratio of carbon-14 to carbon-12 in the organic artifact or fossil to be dated, determined by a method called liquid scintillation. From the equation $A \approx A_0 e^{-0.000121t}$ we know the ratio of the percentage of carbon-14 in the object we are dating to the percentage of carbon-14 in the atmosphere is $r = \frac{A}{A_0} \approx e^{-0.000121t}$. We solve this equation for t, to get
Equation:

$$t = \frac{\ln(r)}{-0.000121}$$

Note:
Given the percentage of carbon-14 in an object, determine its age.

1. Express the given percentage of carbon-14 as an equivalent decimal, k.
2. Substitute for k in the equation $t = \frac{\ln(r)}{-0.000121}$ and solve for the age, t.

Example:
Exercise:

Problem:
Finding the Age of a Bone

A bone fragment is found that contains 20% of its original carbon-14. To the nearest year, how old is the bone?

Solution:

We substitute $20\% = 0.20$ for k in the equation and solve for t :
Equation:

$$\begin{aligned} t &= \frac{\ln(r)}{-0.000121} && \text{Use the general form of the equation.} \\ &= \frac{\ln(0.20)}{-0.000121} && \text{Substitute for } r. \\ &\approx 13301 && \text{Round to the nearest year.} \end{aligned}$$

The bone fragment is about 13,301 years old.

Analysis

The instruments that measure the percentage of carbon-14 are extremely sensitive and, as we mention above, a scientist will need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about 1%, so this age should be given as $13{,}301$ years $\pm\ 1\%$ or $13{,}301$ years $\pm\ 133$ years.

Note:
Exercise:

Problem:

Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Solution:

less than 230 years, 229.3157 to be exact

Calculating Doubling Time

For decaying quantities, we determined how long it took for half of a substance to decay. For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the **doubling time**.

Given the basic exponential growth equation $A = A_0e^{kt}$, doubling time can be found by solving for when the original quantity has doubled, that is, by solving $2A_0 = A_0e^{kt}$.

The formula is derived as follows:
Equation:

$$\begin{aligned} 2A_0 &= A_0e^{kt} \\ 2 &= e^{kt} && \text{Divide by } A_0. \\ \ln 2 &= kt && \text{Take the natural logarithm.} \\ t &= \frac{\ln 2}{k} && \text{Divide by the coefficient of } t. \end{aligned}$$

Thus the doubling time is
Equation:

$$t = \frac{\ln 2}{k}$$

Example:
Exercise:

Problem:
Finding a Function That Describes Exponential Growth

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

Solution:

The formula is derived as follows:
Equation:

$t = \frac{\ln 2}{k}$	The doubling time formula.
$2 = \frac{\ln 2}{k}$	Use a doubling time of two years.
$k = \frac{\ln 2}{2}$	Multiply by k and divide by 2.
$A = A_0 e^{\frac{\ln 2}{2} t}$	Substitute k into the continuous growth formula.

The function is $A_0 e^{\frac{\ln 2}{2} t}$.

Note:
Exercise:

Problem:

Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

Solution:

$f(t) = A_0 e^{\frac{\ln 2}{3} t}$

Using Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless the room temperature is zero, this will correspond to a vertical shift of the generic **exponential decay** function. This translation leads to **Newton's Law of Cooling**, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature
Equation:

$$T(t) = ae^{kt} + T_s$$

This formula is derived as follows:
Equation:

$$
\begin{array}{ll}
T(t) = Ab^{ct} + T_s & \\
T(t) = Ae^{\ln(b^{ct})} + T_s & \text{Laws of logarithms.} \\
T(t) = Ae^{ct \ln b} + T_s & \text{Laws of logarithms.} \\
T(t) = Ae^{kt} + T_s & \text{Rename the constant } c \ \ln \ b, \text{ calling it } k.
\end{array}
$$

Note:
Newton's Law of Cooling
The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula
Equation:

$$T(t) = Ae^{kt} + T_s$$

where

- t is time
- A is the difference between the initial temperature of the object and the surroundings
- k is a constant, the continuous rate of cooling of the object

Note:
Given a set of conditions, apply Newton's Law of Cooling.

1. Set T_s equal to the *y*-coordinate of the horizontal asymptote (usually the ambient temperature).
2. Substitute the given values into the continuous growth formula $T(t) = Ae^{kt} + T_s$ to find the parameters A and k.
3. Substitute in the desired time to find the temperature or the desired temperature to find the time.

Example:
Exercise:

Problem:
Using Newton's Law of Cooling

A cheesecake is taken out of the oven with an ideal internal temperature of $165^\circ\mathrm{F}$, and is placed into a $35^\circ\mathrm{F}$ refrigerator. After 10 minutes, the cheesecake has cooled to $150^\circ\mathrm{F}$. If we must wait until the cheesecake has cooled to $70^\circ\mathrm{F}$ before we eat it, how long will we have to wait?

Solution:

Because the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially toward 35, following the equation
Equation:

$$T(t) = Ae^{kt} + 35$$

We know the initial temperature was 165, so $T(0) = 165$.
Equation:

$$\begin{array}{ll} 165 = Ae^{k0} + 35 & \text{Substitute } (0, 165). \\ A = 130 & \text{Solve for } A. \end{array}$$

We were given another data point, $T(10) = 150$, which we can use to solve for k.
Equation:

$$\begin{array}{rl} 150 = 130e^{k10} + 35 & \text{Substitute (10, 150).} \\ 115 = 130e^{k10} & \text{Subtract 35.} \\ \frac{115}{130} = e^{10k} & \text{Divide by 130.} \\ \ln\left(\frac{115}{130}\right) = 10k & \text{Take the natural log of both sides.} \\ k = \frac{\ln\left(\frac{115}{130}\right)}{10} \approx -0.0123 & \text{Divide by the coefficient of } k. \end{array}$$

This gives us the equation for the cooling of the cheesecake: $T(t) = 130e^{-0.0123t} + 35$.

Now we can solve for the time it will take for the temperature to cool to 70 degrees.
Equation:

$$\begin{array}{rl} 70 = 130e^{-0.0123t} + 35 & \text{Substitute in 70 for } T(t). \\ 35 = 130e^{-0.0123t} & \text{Subtract 35.} \\ \frac{35}{130} = e^{-0.0123t} & \text{Divide by 130.} \\ \ln\left(\frac{35}{130}\right) = -0.0123t & \text{Take the natural log of both sides} \\ t = \frac{\ln\left(\frac{35}{130}\right)}{-0.0123} \approx 106.68 & \text{Divide by the coefficient of } t. \end{array}$$

It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool to $70^{\circ}\mathrm{F}$.

Note:
Exercise:

Problem:

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Solution:

6.026 hours

Using Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over 17 billion lines, or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an

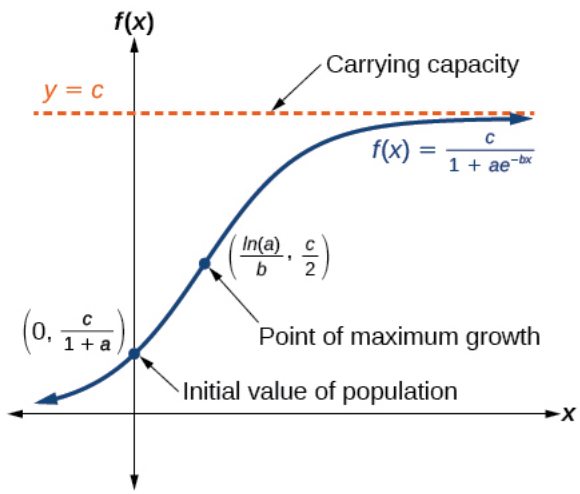
exponential growth model, though the exponential growth model is still useful over a short term, before approaching the limiting value.

The **logistic growth model** is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, called the **carrying capacity**. For constants $a,$ $b,$ and $c,$ the logistic growth of a population over time x is represented by the model

Equation:

$$f(x)=\frac{c}{1+ae^{-bx}}$$

The graph in [link] shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate, at which point the rate of increase decreases.



Note:
Logistic Growth
The logistic growth model is
Equation:

$$f(x)=\frac{c}{1+ae^{-bx}}$$

where

- $\frac{c}{1+a}$ is the initial value
- c is the *carrying capacity*, or *limiting value*
- b is a constant determined by the rate of growth.

Example:
Exercise:

Problem:

Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of 1,000 people who has the flu. So, in that community, at most 1,000 people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030$. Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

Solution:

We substitute the given data into the logistic growth model
Equation:

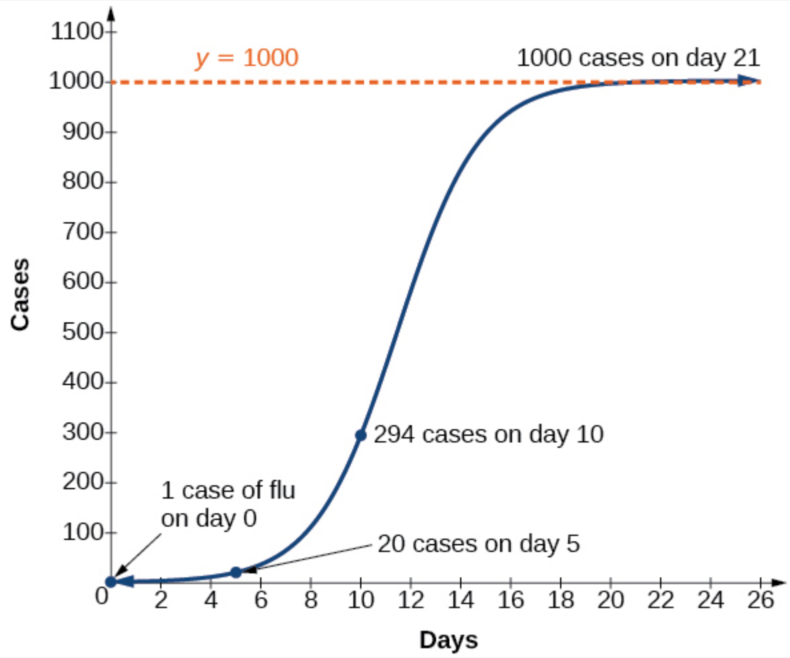
$$f(x) = \frac{c}{1 + ae^{-bx}}$$

Because at most 1,000 people, the entire population of the community, can get the flu, we know the limiting value is $c = 1000$. To find a,we use the formula that the number of cases at time $t = 0$ is $\frac{c}{1+a} = 1$, from which it follows that $a = 999$.This model predicts that, after ten days, the number of people who have had the flu is $f(x) = \frac{1000}{1+999e^{-0.6030x}} \approx 293.8$. Because the actual number must be a whole number (a person has either had the flu or not) we round to 294. In the long term, the number of people who will contract the flu is the limiting value, $c = 1000$.

Analysis

Remember that, because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values.

The graph in [link] gives a good picture of how this model fits the data.



The graph of $f(x)=\frac{1000}{1+999e^{-0.6030x}}$

Note:
Exercise:

Problem: Using the model in [link], estimate the number of cases of flu on day 15.

Solution:

895 cases on day 15

Choosing an Appropriate Model for Data

Now that we have discussed various mathematical models, we need to learn how to choose the appropriate model for the raw data we have. Many factors influence the choice of a mathematical model, among which are experience, scientific laws, and patterns in the data itself. Not all data can be described by elementary functions. Sometimes, a function is chosen that approximates the data over a given interval. For instance, suppose data were gathered on the number of homes bought in the United States from the years 1960 to 2013. After plotting these data in a scatter plot, we notice that the shape of the data from the years 2000 to 2013 follow a logarithmic curve. We could restrict the interval from 2000 to 2010, apply regression analysis using a logarithmic model, and use it to predict the number of home buyers for the year 2015.

Three kinds of functions that are often useful in mathematical models are linear functions, exponential functions, and logarithmic functions. If the data lies on a straight line, or seems to lie approximately along a straight line, a linear model may be best. If the data is non-linear, we often consider an exponential or logarithmic model, though other models, such as quadratic models, may also be considered.

In choosing between an exponential model and a logarithmic model, we look at the way the data curves. This is called the concavity. If we draw a line between two data points, and all (or most) of the data between those two points lies above that line, we say the curve is concave down. We can think of it as a bowl that bends downward and therefore cannot hold water. If all (or most) of the data between those two points lies below the line, we say the curve is concave up. In this case, we can think of a bowl that bends upward and can therefore hold water. An exponential curve, whether rising or falling, whether representing growth or decay, is always concave up away from its horizontal asymptote. A logarithmic curve is always concave away from its vertical asymptote. In the case of positive data, which is the most common case, an exponential curve is always concave up, and a logarithmic curve always concave down.

A logistic curve changes concavity. It starts out concave up and then changes to concave down beyond a certain point, called a point of inflection.

After using the graph to help us choose a type of function to use as a model, we substitute points, and solve to find the parameters. We reduce round-off error by choosing points as far apart as possible.

Example:
Exercise:

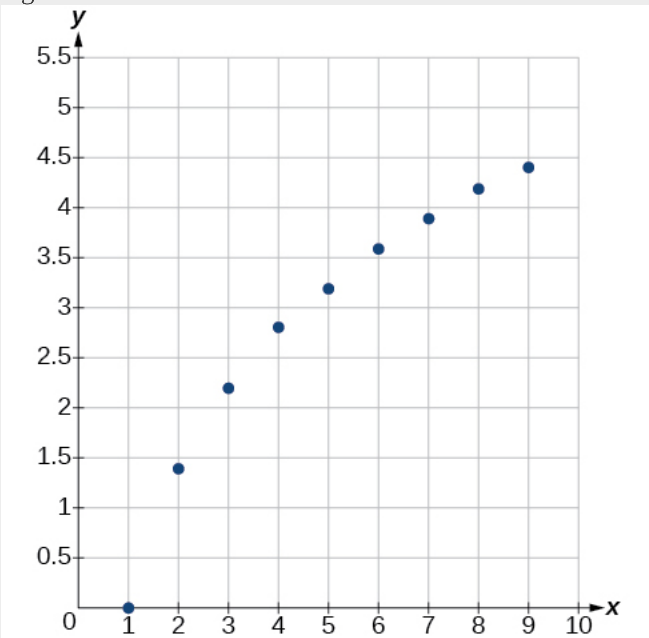
Problem:
Choosing a Mathematical Model

Does a linear, exponential, logarithmic, or logistic model best fit the values listed in [link]? Find the model, and use a graph to check your choice.

x	1	2	3	4	5	6	7	8	9
y	0	1.386	2.197	2.773	3.219	3.584	3.892	4.159	4.394

Solution:

First, plot the data on a graph as in [link]. For the purpose of graphing, round the data to two significant digits.



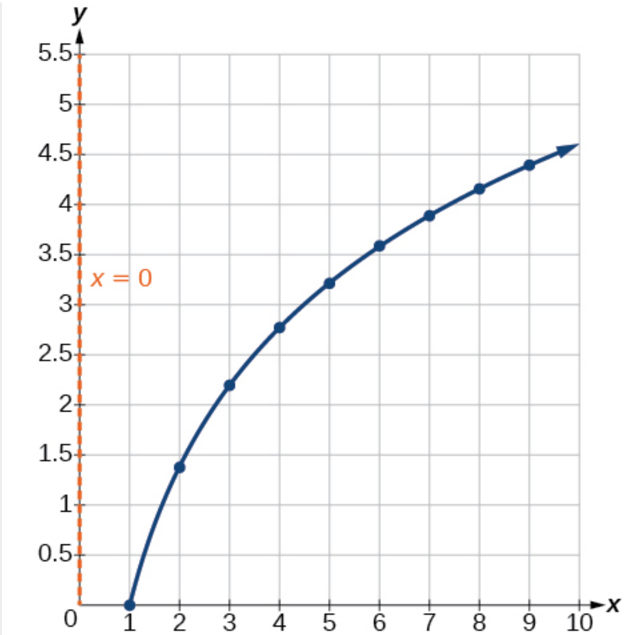
Clearly, the points do not lie on a straight line, so we reject a linear model. If we draw a line between any two of the points, most or all of the points between those two points lie above the line, so the graph is concave down, suggesting a logarithmic model. We can try $y = a\ln(bx)$. Plugging in the first point, $(1,0)$, gives $0 = a\ln b$. We reject the case that $a = 0$ (if it were, all outputs would be 0), so we know $\ln(b) = 0$. Thus $b = 1$ and $y = a\ln(\text{x})$. Next we can use the point $(9,4.394)$ to solve for a :

Equation:

$$\begin{aligned} y &= a\ln(x) \\ 4.394 &= a\ln(9) \\ a &= \frac{4.394}{\ln(9)} \end{aligned}$$

Because $a = \frac{4.394}{\ln(9)} \approx 2$,an appropriate model for the data is $y = 2\ln(x)$.

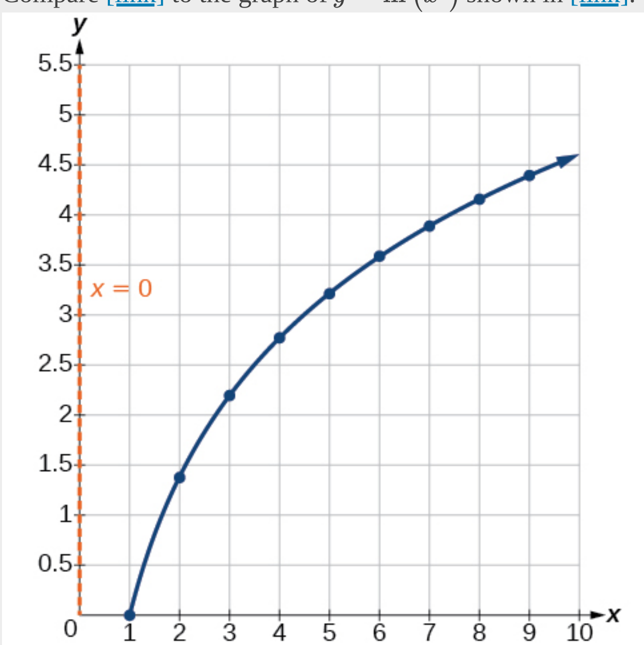
To check the accuracy of the model, we graph the function together with the given points as in [link].



The graph of $y = 2\ln x$.

We can conclude that the model is a good fit to the data.

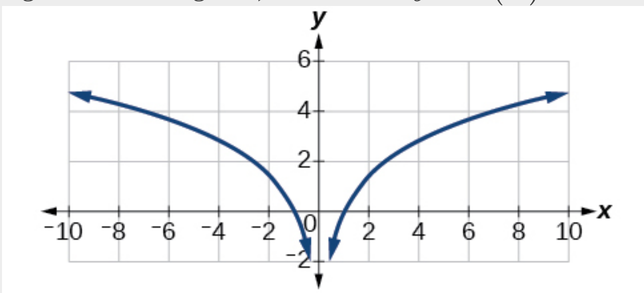
Compare [link] to the graph of $y = \ln\left(x^2\right)$ shown in [link].



The graph of $y = \ln\left(x^2\right)$

The graphs appear to be identical when $x > 0$. A quick check confirms this conclusion: $y = \ln\left(x^2\right) = 2\ln(x)$ for $x > 0$.

However, if $x < 0$, the graph of $y = \ln\left(x^2\right)$ includes a “extra” branch, as shown in [link]. This occurs because, while $y = 2\ln(x)$ cannot have negative values in the domain (as such values would force the argument to be negative), the function $y = \ln\left(x^2\right)$ can have negative domain values.



Note:
Exercise:

Problem: Does a linear, exponential, or logarithmic model best fit the data in [link]? Find the model.

x	1	2	3	4	5	6	7	8	9
y	3.297	5.437	8.963	14.778	24.365	40.172	66.231	109.196	180.034

Solution:

Exponential. $y = 2e^{0.5x}$.

Expressing an Exponential Model in Base e

While powers and logarithms of any base can be used in modeling, the two most common bases are 10 and e. In science and mathematics, the base e is often preferred. We can use laws of exponents and laws of logarithms to change any base to base e.

Note:
Given a model with the form $y = ab^x$, change it to the form $y = A_0e^{kx}$.

1. Rewrite $y = ab^x$ as $y = ae^{\ln(b^x)}$.
2. Use the power rule of logarithms to rewrite y as $y = ae^{x\ln(b)} = ae^{\ln(b)x}$.
3. Note that $a = A_0$ and $k = \ln(b)$ in the equation $y = A_0e^{kx}$.

Example:
Exercise:

Problem:
Changing to base *e*

Change the function $y=2.5(3.1)^{x}$ so that this same function is written in the form $y=A_0e^{kx}$.

Solution:

The formula is derived as follows
Equation:

$$\begin{aligned} y &= 2.5(3.1)^{x} & \\ &= 2.5e^{\ln(3.1^{x})} & \text{Insert exponential and its inverse.} \\ &= 2.5e^{x\ln 3.1} & \text{Laws of logs.} \\ &= 2.5e^{(\ln 3.1)x} & \text{Commutative law of multiplication} \end{aligned}$$

Note:
Exercise:

Problem: Change the function $y=3(0.5)^{x}$ to one having e as the base.

Solution:

$y=3e^{(\ln 0.5)x}$

Note:
Access these online resources for additional instruction and practice with exponential and logarithmic models.

- Logarithm Application – pH
- Exponential Model – Age Using Half-Life
- Newton's Law of Cooling
- Exponential Growth Given Doubling Time
- Exponential Growth – Find Initial Amount Given Doubling Time

Key Equations

Half-life formula	If $A=A_0e^{kt}$, $k<0$, the half-life is $t=-\frac{\ln(2)}{k}$.
Carbon-14 dating	$t=\frac{\ln\left(\frac{A}{A_0}\right)}{-0.000121}$.

	A_0 is the amount of carbon-14 when the plant or animal died A is the amount of carbon-14 remaining today t is the age of the fossil in years
Doubling time formula	If $A = A_0e^{kt}$, $k > 0$, the doubling time is $t = \frac{\ln 2}{k}$
Newton's Law of Cooling	$T(t) = Ae^{kt} + T_s$, where T_s is the ambient temperature, $A = T(0) - T_s$, and k is the continuous rate of cooling.

Key Concepts

- The basic exponential function is $f(x) = ab^x$. If $b > 1$,we have exponential growth; if $0 < b < 1$, we have exponential decay.
- We can also write this formula in terms of continuous growth as $A = A_0e^{kx}$, where A_0 is the starting value. If A_0 is positive, then we have exponential growth when $k > 0$ and exponential decay when $k < 0$. See [link].
- In general, we solve problems involving exponential growth or decay in two steps. First, we set up a model and use the model to find the parameters. Then we use the formula with these parameters to predict growth and decay. See [link].
- We can find the age, t, of an organic artifact by measuring the amount, k, of carbon-14 remaining in the artifact and using the formula $t = \frac{\ln(k)}{-0.000121}$ to solve for t. See [link].
- Given a substance's doubling time or half-time, we can find a function that represents its exponential growth or decay. See [link].
- We can use Newton's Law of Cooling to find how long it will take for a cooling object to reach a desired temperature, or to find what temperature an object will be after a given time. See [link].
- We can use logistic growth functions to model real-world situations where the rate of growth changes over time, such as population growth, spread of disease, and spread of rumors. See [link].
- We can use real-world data gathered over time to observe trends. Knowledge of linear, exponential, logarithmic, and logistic graphs help us to develop models that best fit our data. See [link].
- Any exponential function with the form $y = ab^x$ can be rewritten as an equivalent exponential function with the form $y = A_0e^{kx}$ where $k = \ln b$. See [link].

Section Exercises

Verbal

Exercise:

Problem:

With what kind of exponential model would *half-life* be associated? What role does half-life play in these models?

Solution:

Half-life is a measure of decay and is thus associated with exponential decay models. The half-life of a substance or quantity is the amount of time it takes for half of the initial amount of that substance or quantity to decay.

Exercise:

Problem:

What is carbon dating? Why does it work? Give an example in which carbon dating would be useful.

Exercise:

Problem:

With what kind of exponential model would *doubling time* be associated? What role does doubling time play in these models?

Solution:

Doubling time is a measure of growth and is thus associated with exponential growth models. The doubling time of a substance or quantity is the amount of time it takes for the initial amount of that substance or quantity to double in size.

Exercise:

Problem:

Define Newton’s Law of Cooling. Then name at least three real-world situations where Newton’s Law of Cooling would be applied.

Exercise:

Problem: What is an order of magnitude? Why are orders of magnitude useful? Give an example to explain.

Solution:

An order of magnitude is the nearest power of ten by which a quantity exponentially grows. It is also an approximate position on a logarithmic scale; Sample response: Orders of magnitude are useful when making comparisons between numbers that differ by a great amount. For example, the mass of Saturn is 95 times greater than the mass of Earth. This is the same as saying that the mass of Saturn is about 10^2 times, or *2 orders of magnitude* greater, than the mass of Earth.

Numeric

Exercise:

Problem:

The temperature of an object in degrees Fahrenheit after *t* minutes is represented by the equation $T(t)=68e^{-0.0174t}+72.$ To the nearest degree, what is the temperature of the object after one and a half hours?

For the following exercises, use the logistic growth model $f(x)=\frac{150}{1+8e^{-2x}}.$

Exercise:

Problem: Find and interpret $f(0).$ Round to the nearest tenth.

Solution:

$f(0)\approx 16.7;$ The amount initially present is about 16.7 units.

Exercise:

Problem: Find and interpret $f(4).$ Round to the nearest tenth.

Exercise:

Problem: Find the carrying capacity.

Solution:

150

Exercise:

Problem: Graph the model.

Exercise:

Problem:

Determine whether the data from the table could best be represented as a function that is linear, exponential, or logarithmic. Then write a formula for a model that represents the data.

x	$f(x)$
–2	0.694
–1	0.833
0	1
1	1.2
2	1.44
3	1.728
4	2.074
5	2.488

Solution:

exponential; $f(x)=1.2^{x}$

Exercise:

Problem: Rewrite $f(x)=1.68(0.65)^{x}$ as an exponential equation with base e to five significant digits.

Technology

For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table could represent a function that is linear, exponential, or logarithmic.

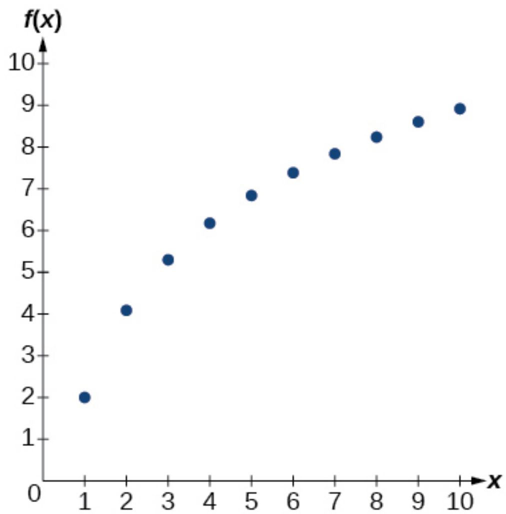
Exercise:

Problem:

x	$f(x)$
1	2
2	4.079
3	5.296
4	6.159
5	6.828
6	7.375
7	7.838
8	8.238
9	8.592
10	8.908

Solution:

logarithmic



Exercise:

Problem:

x	$f(x)$

1	2.4
2	2.88
3	3.456
4	4.147
5	4.977
6	5.972
7	7.166
8	8.6
9	10.32
10	12.383

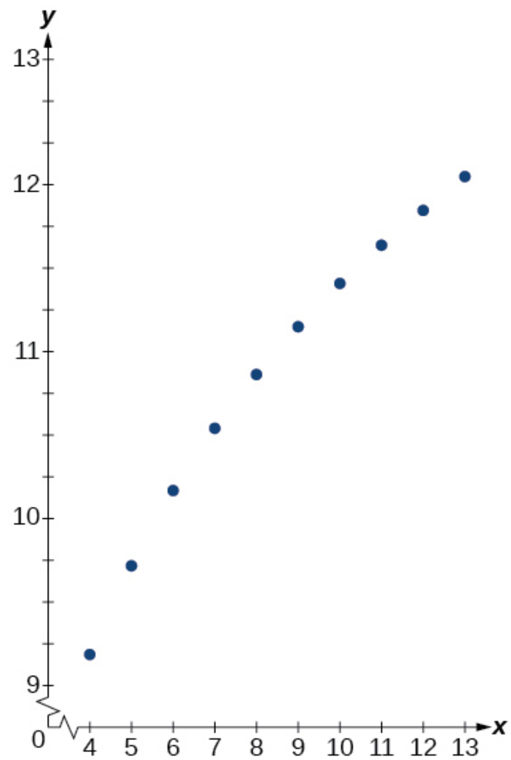
Exercise:

Problem:

x	$f(x)$
4	9.429
5	9.972
6	10.415
7	10.79
8	11.115
9	11.401
10	11.657
11	11.889
12	12.101
13	12.295

Solution:

logarithmic



Exercise:

Problem:

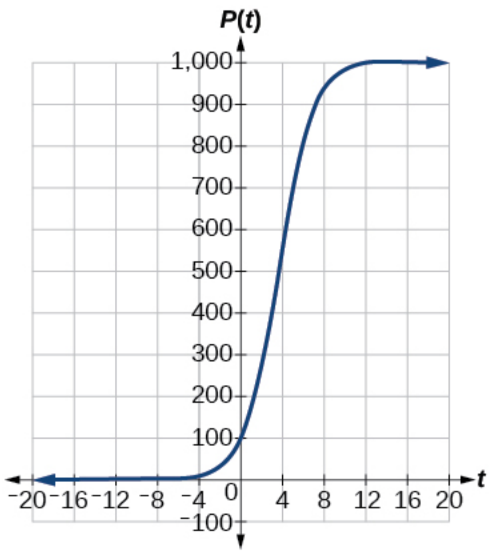
x	$f(x)$
1.25	5.75
2.25	8.75
3.56	12.68
4.2	14.6
5.65	18.95
6.75	22.25
7.25	23.75
8.6	27.8
9.25	29.75
10.5	33.5

For the following exercises, use a graphing calculator and this scenario: the population of a fish farm in t years is modeled by the equation $P(t) = \frac{1000}{1+9e^{-0.6t}}$.

Exercise:

Problem: Graph the function.

Solution:



Exercise:

Problem: What is the initial population of fish?

Exercise:

Problem: To the nearest tenth, what is the doubling time for the fish population?

Solution:

about 1.4 years

Exercise:

Problem: To the nearest whole number, what will the fish population be after 2 years?

Exercise:

Problem: To the nearest tenth, how long will it take for the population to reach 900?

Solution:

about 7.3 years

Exercise:

Problem: What is the carrying capacity for the fish population? Justify your answer using the graph of P.

Extensions

Exercise:

Problem:

A substance has a half-life of 2.045 minutes. If the initial amount of the substance was 132.8 grams, how many half-lives will have passed before the substance decays to 8.3 grams? What is the total time of decay?

Solution:

4 half-lives; 8.18 minutes

Exercise:

Problem:

The formula for an increasing population is given by $P(t) = P_0 e^{rt}$ where P_0 is the initial population and $r > 0$. Derive a general formula for the time *t* it takes for the population to increase by a factor of *M*.

Exercise:

Problem:

Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. Show each step for solving this equation algebraically for the seismic moment S.

Solution:

$$\begin{aligned} M &= \frac{2}{3}\log\left(\frac{S}{S_0}\right) \\ \log\left(\frac{S}{S_0}\right) &= \frac{3}{2}M \\ \frac{S}{S_0} &= 10^{\frac{3M}{2}} \\ S &= S_0 10^{\frac{3M}{2}} \end{aligned}$$

Exercise:

Problem:

What is the *y*-intercept of the logistic growth model $y = \frac{c}{1+ae^{-rx}}$? Show the steps for calculation. What does this point tell us about the population?

Exercise:

Problem: Prove that $b^x = e^{x\ln(b)}$ for positive $b \neq 1$.

Solution:

Let $y = b^x$ for some non-negative real number b such that $b \neq 1$. Then,

$$\begin{aligned} \ln(y) &= \ln(b^x) \\ \ln(y) &= x\ln(b) \\ e^{\ln(y)} &= e^{x\ln(b)} \\ y &= e^{x\ln(b)} \end{aligned}$$

Real-World Applications

For the following exercises, use this scenario: A doctor prescribes 125 milligrams of a therapeutic drug that decays by about 30% each hour.

Exercise:

Problem: To the nearest hour, what is the half-life of the drug?

Exercise:

Problem:

Write an exponential model representing the amount of the drug remaining in the patient's system after t hours. Then use the formula to find the amount of the drug that would remain in the patient's system after 3 hours. Round to the nearest milligram.

Solution:

$A = 125e^{(-0.3567t)}$; $A \approx 43$ mg

Exercise:

Problem:

Using the model found in the previous exercise, find $f(10)$ and interpret the result. Round to the nearest hundredth.

For the following exercises, use this scenario: A tumor is injected with 0.5 grams of Iodine-125, which has a decay rate of 1.15% per day.

Exercise:

Problem: To the nearest day, how long will it take for half of the Iodine-125 to decay?

Solution:

about 60 days

Exercise:

Problem:

Write an exponential model representing the amount of Iodine-125 remaining in the tumor after t days. Then use the formula to find the amount of Iodine-125 that would remain in the tumor after 60 days. Round to the nearest tenth of a gram.

Exercise:

Problem:

A scientist begins with 250 grams of a radioactive substance. After 250 minutes, the sample has decayed to 32 grams. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest minute, what is the half-life of this substance?

Solution:

$f(t) = 250e^{(-0.00914t)}$; half-life: about 76 minutes

Exercise:

Problem:

The half-life of Radium-226 is 1590 years. What is the annual decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

Exercise:

Problem:

The half-life of Erbium-165 is 10.4 hours. What is the hourly decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

Solution:

$r \approx -0.0667$, So the hourly decay rate is about 6.67%

Exercise:

Problem:

A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. To the nearest year, about how many years old is the artifact? (The half-life of carbon-14 is 5730 years.)

Exercise:

Problem:

A research student is working with a culture of bacteria that doubles in size every twenty minutes. The initial population count was 1350 bacteria. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest whole number, what is the population size after 3 hours?

Solution:

$f(t) = 1350e^{(0.03466t)}$; after 3 hours: $P(180) \approx 691,200$

For the following exercises, use this scenario: A biologist recorded a count of 360 bacteria present in a culture after 5 minutes and 1000 bacteria present after 20 minutes.

Exercise:

Problem: To the nearest whole number, what was the initial population in the culture?

Exercise:

Problem:

Rounding to six significant digits, write an exponential equation representing this situation. To the nearest minute, how long did it take the population to double?

Solution:

$f(t) = 256e^{(0.068110t)}$; doubling time: about 10 minutes

For the following exercises, use this scenario: A pot of boiling soup with an internal temperature of 100° Fahrenheit was taken off the stove to cool in a 69° F room. After fifteen minutes, the internal temperature of the soup was 95° F.

Exercise:

Problem: Use Newton’s Law of Cooling to write a formula that models this situation.

Exercise:

Problem: To the nearest minute, how long will it take the soup to cool to $80^\circ\ \text{F}$?

Solution:

about 88 minutes

Exercise:

Problem: To the nearest degree, what will the temperature be after 2 and a half hours?

For the following exercises, use this scenario: A turkey is taken out of the oven with an internal temperature of 165°F and is allowed to cool in a 75°F room. After half an hour, the internal temperature of the turkey is 145°F.

Exercise:

Problem: Write a formula that models this situation.

Solution:

$T(t) = 90e^{(-0.008377t)} + 75$, where t is in minutes.

Exercise:

Problem: To the nearest degree, what will the temperature be after 50 minutes?

Exercise:

Problem: To the nearest minute, how long will it take the turkey to cool to $110^\circ\ \text{F}$?

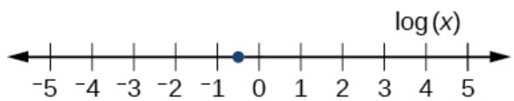
Solution:

about 113 minutes

For the following exercises, find the value of the number shown on each logarithmic scale. Round all answers to the nearest thousandth.

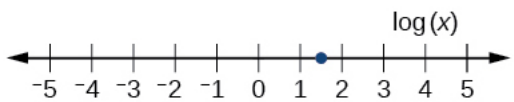
Exercise:

Problem:



Exercise:

Problem:



Solution:

$\log(x) = 1.5;\ x \approx 31.623$

Exercise:

Problem:

Plot each set of approximate values of intensity of sounds on a logarithmic scale: Whisper: $10^{-10}\ \frac{W}{m^2}$, Vacuum: $10^{-4}\frac{W}{m^2}$,Jet: $10^{2}\ \frac{W}{m^2}$

Exercise:

Problem:

Recall the formula for calculating the magnitude of an earthquake, $M=\frac{2}{3}\log\left(\frac{S}{S_0}\right).$ One earthquake has magnitude 3.9 on the MMS scale. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake. Round to the nearest hundredth.

Solution:

MMS magnitude: 5.82

For the following exercises, use this scenario: The equation $N\left(t\right)=\frac{500}{1+49e^{-0.7t}}$ models the number of people in a town who have heard a rumor after *t* days.

Exercise:

Problem: How many people started the rumor?

Exercise:

Problem: To the nearest whole number, how many people will have heard the rumor after 3 days?

Solution:

$N(3)\approx 71$

Exercise:

Problem: As t increases without bound, what value does $N\left(t\right)$ approach? Interpret your answer.

For the following exercise, choose the correct answer choice.

Exercise:

Problem:

A doctor and injects a patient with 13 milligrams of radioactive dye that decays exponentially. After 12 minutes, there are 4.75 milligrams of dye remaining in the patient's system. Which is an appropriate model for this situation?

A. $f\left(t\right)=13(0.0805)^{t}$
B. $f\left(t\right)=13e^{0.9195t}$
C. $f(t)=13e^{(-0.0839t)}$
D. $f\left(t\right)=\frac{4.75}{1+13e^{-0.83925t}}$

Solution:

C

Glossary

carrying capacity
: in a logistic model, the limiting value of the output

doubling time
: the time it takes for a quantity to double

half-life
: the length of time it takes for a substance to exponentially decay to half of its original quantity

logistic growth model
: a function of the form $f(x) = \frac{c}{1+ae^{-bx}}$ where $\frac{c}{1+a}$ is the initial value, c is the carrying capacity, or limiting value, and b is a constant determined by the rate of growth

Newton’s Law of Cooling
: the scientific formula for temperature as a function of time as an object’s temperature is equalized with the ambient temperature

order of magnitude
: the power of ten, when a number is expressed in scientific notation, with one non-zero digit to the left of the decimal

Modeling Using Variation
In this section, you will:

- Solve direct variation problems.
- Solve inverse variation problems.
- Solve problems involving joint variation.

A used-car company has just offered their best candidate, Nicole, a position in sales. The position offers 16% commission on her sales. Her earnings depend on the amount of her sales. For instance, if she sells a vehicle for $4,600, she will earn $736. She wants to evaluate the offer, but she is not sure how. In this section, we will look at relationships, such as this one, between earnings, sales, and commission rate.

Solving Direct Variation Problems

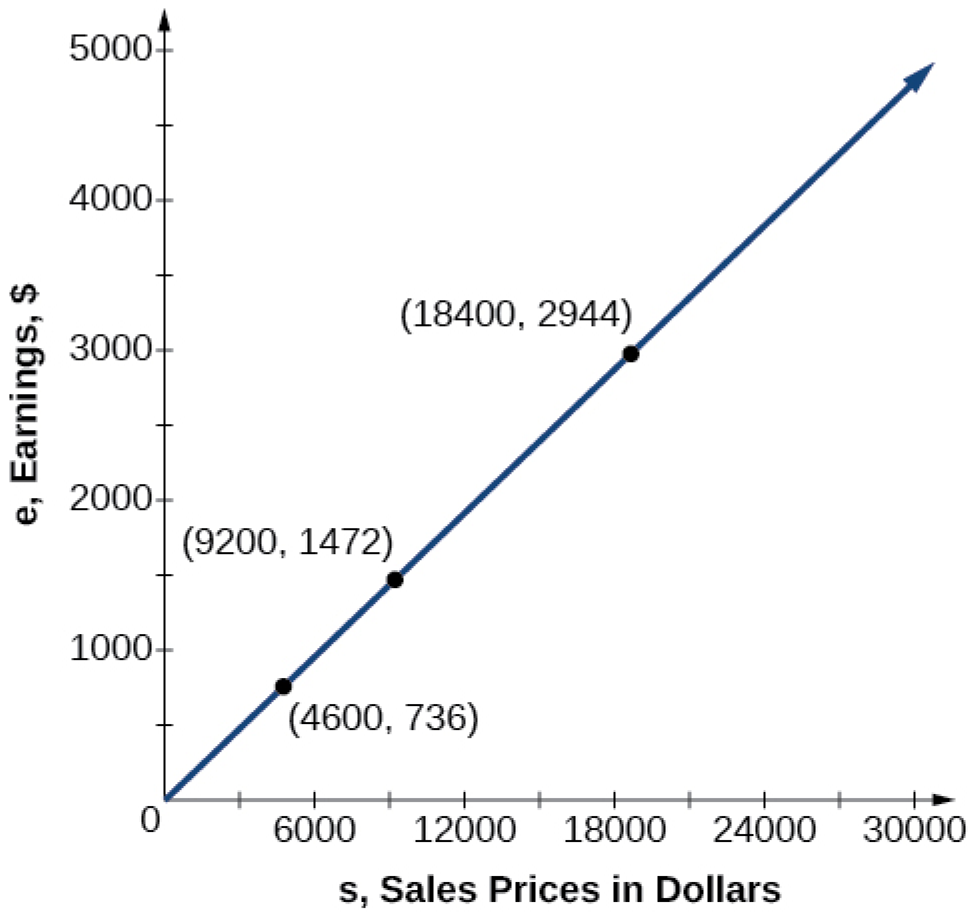
In the example above, Nicole's earnings can be found by multiplying her sales by her commission. The formula $e=0.16s$ tells us her earnings, e, come from the product of 0.16, her commission, and the sale price of the vehicle. If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive. See [link].

s **, sales prices**	$e=0.16s$	**Interpretation**
$4,600	$e=0.16\left(4{,}600\right)=736$	A sale of a $4,600 vehicle results in $736 earnings.
$9,200	$e=0.16\left(9{,}200\right)=1{,}472$	A sale of a $9,200 vehicle results in $1472 earnings.

s , **sales prices**	$e = 0.16s$	**Interpretation**
\$18,400	$e = 0.16\,(18{,}400) = 2{,}944$	A sale of a \$18,400 vehicle results in \$2944 earnings.

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from \$4,600 to \$9,200, and we double the earnings from \$736 to \$1,472. As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called **direct variation**. Each variable in this type of relationship **varies directly** with the other.

[link] represents the data for Nicole's potential earnings. We say that earnings vary directly with the sales price of the car. The formula $y = kx^n$ is used for direct variation. The value k is a nonzero constant greater than zero and is called the **constant of variation**. In this case, $k = 0.16$ and $n = 1$.



Note:
Direct Variation
If x and y are related by an equation of the form
Equation:

$$y = kx^n$$

then we say that the relationship is **direct variation** and y **varies directly** with the nth power of x. In direct variation relationships, there is a nonzero constant ratio $k = \frac{y}{x^n}$, where k is called the **constant of variation**, which help defines the relationship between the variables.

Note:
Given a description of a direct variation problem, solve for an unknown.

1. Identify the input, x,and the output, y.
2. Determine the constant of variation. You may need to divide y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Example:
Exercise:

Problem:
Solving a Direct Variation Problem

The quantity y varies directly with the cube of $x.$ If $y = 25$ when $x = 2,$ find y when x is 6.

Solution:

The general formula for direct variation with a cube is $y = kx^3.$ The constant can be found by dividing y by the cube of $x.$
Equation:

$$\begin{aligned} k &= \frac{y}{x^3} \\ &= \frac{25}{2^3} \\ &= \frac{25}{8} \end{aligned}$$

Now use the constant to write an equation that represents this relationship.
Equation:

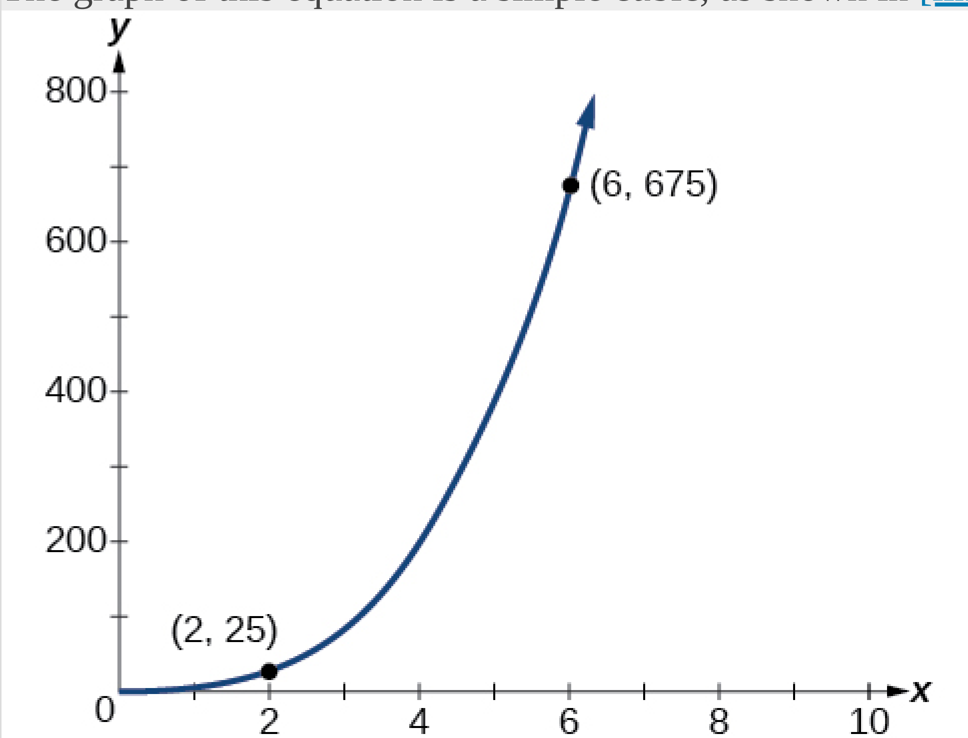
$$y = \frac{25}{8}x^3$$

Substitute $x = 6$ and solve for y.
Equation:

$$\begin{aligned} y &= \tfrac{25}{8}(6)^3 \\ &= 675 \end{aligned}$$

Analysis

The graph of this equation is a simple cubic, as shown in [link].



Note:
Do the graphs of all direct variation equations look like [link]?
No. Direct variation equations are power functions—they may be linear, quadratic, cubic, quartic, radical, etc. But all of the graphs pass through $(0,0)$.

Note:
Exercise:

Problem:

The quantity y varies directly with the square of x. If $y = 24$ when $x = 3,$ find y when x is 4.

Solution:

$\frac{128}{3}$

Solving Inverse Variation Problems

Water temperature in an ocean varies inversely to the water's depth. Between the depths of 250 feet and 500 feet, the formula $T = \frac{14{,}000}{d}$ gives us the temperature in degrees Fahrenheit at a depth in feet below Earth's surface. Consider the Atlantic Ocean, which covers 22% of Earth's surface. At a certain location, at the depth of 500 feet, the temperature may be 28°F.

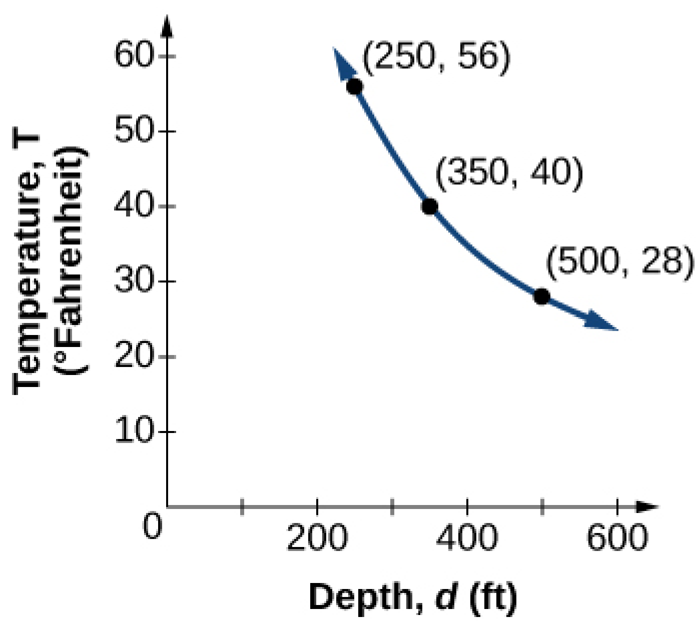
If we create [link], we observe that, as the depth increases, the water temperature decreases.

d, **depth**	$T = \frac{14{,}000}{d}$	**Interpretation**
500 ft	$\frac{14{,}000}{500} = 28$	At a depth of 500 ft, the water temperature is 28° F.

d, **depth**	$T = \frac{14,000}{d}$	**Interpretation**
350 ft	$\frac{14,000}{350} = 40$	At a depth of 350 ft, the water temperature is 40° F.
250 ft	$\frac{14,000}{250} = 56$	At a depth of 250 ft, the water temperature is 56° F.

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be **inversely proportional** and each term **varies inversely** with the other. Inversely proportional relationships are also called **inverse variations**.

For our example, [link] depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula $y = \frac{k}{x}$ for inverse variation in this case uses $k = 14,000$.



Note:
Inverse Variation
If x and y are related by an equation of the form
Equation:

$$y = \frac{k}{x^n}$$

where k is a nonzero constant, then we say that y **varies inversely** with the $n\text{th}$ power of $x.$ In **inversely proportional** relationships, or **inverse variations**, there is a constant multiple $k = x^n y$.

Example:
Exercise:

Problem:
Writing a Formula for an Inversely Proportional Relationship

A tourist plans to drive 100 miles. Find a formula for the time the trip will take as a function of the speed the tourist drives.

Solution:

Recall that multiplying speed by time gives distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which the tourist drives, then $vt = \text{distance}$. Because the distance is fixed at 100 miles, $vt = 100$. Solving this relationship for the time gives us our function.
Equation:

$$\begin{aligned} t(v) &= \tfrac{100}{v} \\ &= 100v^{-1} \end{aligned}$$

We can see that the constant of variation is 100 and, although we can write the relationship using the negative exponent, it is more common

to see it written as a fraction.

Note:
Given a description of an indirect variation problem, solve for an unknown.

1. Identify the input, $x,$ and the output, $y.$
2. Determine the constant of variation. You may need to multiply y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Example:
Exercise:

Problem:
Solving an Inverse Variation Problem

A quantity y varies inversely with the cube of $x.$ If $y=25$ when $x=2,$ find y when x is 6.

Solution:

The general formula for inverse variation with a cube is $y=\frac{k}{x^3}.$ The constant can be found by multiplying y by the cube of $x.$
Equation:

$$\begin{aligned} k &= x^3y \\ &= 2^3 \cdot 25 \\ &= 200 \end{aligned}$$

Now we use the constant to write an equation that represents this relationship.

Equation:

$$\begin{aligned} y &= \frac{k}{x^3}, \ k = 200 \\ y &= \frac{200}{x^3} \end{aligned}$$

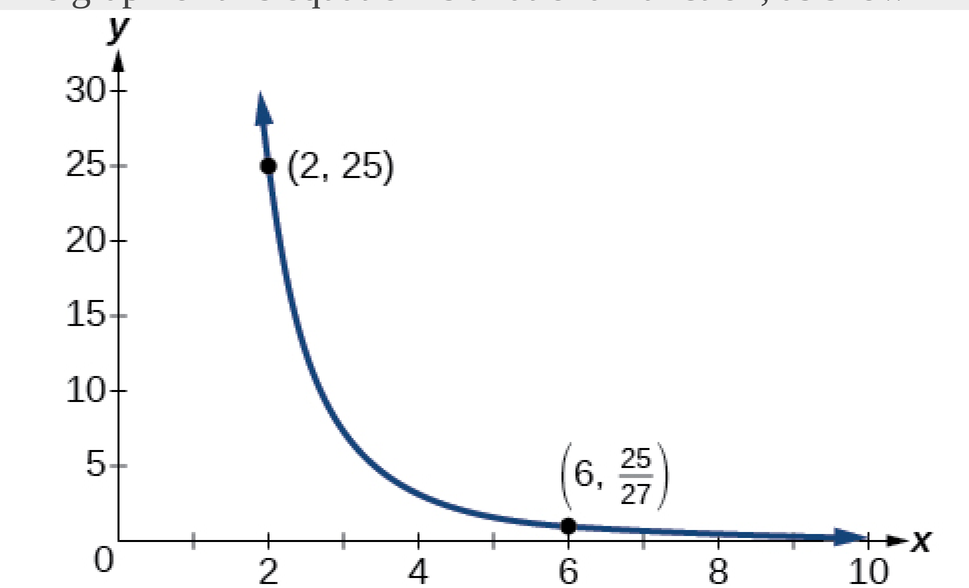
Substitute $x = 6$ and solve for y.

Equation:

$$\begin{aligned} y &= \frac{200}{6^3} \\ &= \frac{25}{27} \end{aligned}$$

Analysis

The graph of this equation is a rational function, as shown in [link].



Note:

Exercise:

Problem:

A quantity y varies inversely with the square of $x.$ If $y = 8$ when $x = 3,$ find y when x is 4.

Solution:

$\frac{9}{2}$

Solving Problems Involving Joint Variation

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called **joint variation**. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable c, cost, varies jointly with the number of students, n, and the distance, d.

Note:
Joint Variation
Joint variation occurs when a variable varies directly or inversely with multiple variables.
For instance, if x varies directly with both y and $z,$ we have $x = kyz.$ If x varies directly with y and inversely with z, we have $x = \frac{ky}{z}.$ Notice that we only use one constant in a joint variation equation.

Example:
Exercise:

Problem:

Solving Problems Involving Joint Variation

A quantity x varies directly with the square of y and inversely with the cube root of z. If $x = 6$ when $y = 2$ and $z = 8,$ find x when $y = 1$ and $z = 27.$

Solution:

Begin by writing an equation to show the relationship between the variables.
Equation:

$$x = \frac{ky^2}{\sqrt[3]{z}}$$

Substitute $x = 6,$ $y = 2,$ and $z = 8$ to find the value of the constant $k.$
Equation:

$$\begin{gathered} 6 = \frac{k2^2}{\sqrt[3]{8}} \\ 6 = \frac{4k}{2} \\ 3 = k \end{gathered}$$

Now we can substitute the value of the constant into the equation for the relationship.
Equation:

$$x = \frac{3y^2}{\sqrt[3]{z}}$$

To find x when $y = 1$ and $z = 27,$ we will substitute values for y and z into our equation.

Equation:

$$\begin{aligned} x &= \frac{3(1)^2}{\sqrt[3]{27}} \\ &= 1 \end{aligned}$$

Note:

Exercise:

Problem:

x varies directly with the square of y and inversely with z. If $x = 40$ when $y = 4$ and $z = 2$, find x when $y = 10$ and $z = 25$.

Solution:

$x = 20$

Note:

Access these online resources for additional instruction and practice with direct and inverse variation.

- Direct Variation
- Inverse Variation
- Direct and Inverse Variation

Visit this website for additional practice questions from Learningpod.

Key Equations

Direct variation	$y = kx^n, k$ is a nonzero constant.
Inverse variation	$y = \frac{k}{x^n}, k$ is a nonzero constant.

Key Concepts

- A relationship where one quantity is a constant multiplied by another quantity is called direct variation. See [link].
- Two variables that are directly proportional to one another will have a constant ratio.
- A relationship where one quantity is a constant divided by another quantity is called inverse variation. See [link].
- Two variables that are inversely proportional to one another will have a constant multiple. See [link].
- In many problems, a variable varies directly or inversely with multiple variables. We call this type of relationship joint variation. See [link].

Section Exercises

Verbal

Exercise:

Problem:

What is true of the appearance of graphs that reflect a direct variation between two variables?

Solution:

The graph will have the appearance of a power function.

Exercise:

Problem:

If two variables vary inversely, what will an equation representing their relationship look like?

Exercise:

Problem:

Is there a limit to the number of variables that can jointly vary? Explain.

Solution:

No. Multiple variables may jointly vary.

Algebraic

For the following exercises, write an equation describing the relationship of the given variables.

Exercise:

Problem: y varies directly as x and when $x = 6,\ y = 12.$

Exercise:

Problem:

y varies directly as the square of x and when $x = 4,\ \ y = 80.$

Solution:

$y = 5x^2$

Exercise:

Problem:

y varies directly as the square root of x and when $x = 36,\ \ y = 24.$

Exercise:

Problem: y varies directly as the cube of x and when $x = 36,\ y = 24.$

Solution:

$y = 10x^3$

Exercise:

Problem:

y varies directly as the cube root of x and when $x = 27,\ \ y = 15.$

Exercise:

Problem:

y varies directly as the fourth power of x and when $x = 1,\ \ y = 6.$

Solution:

$y = 6x^4$

Exercise:

Problem: y varies inversely as x and when $x = 4,\ \ y = 2.$

Exercise:

Problem:

y varies inversely as the square of x and when $x = 3,\ \ y = 2.$

Solution:

$y = \frac{18}{x^2}$

Exercise:

Problem: y varies inversely as the cube of x and when $x = 2,\ \ y = 5.$

Exercise:

Problem:

y varies inversely as the fourth power of x and when $x = 3,\ \ y = 1.$

Solution:

$y = \frac{81}{x^4}$

Exercise:

Problem:

y varies inversely as the square root of x and when $x = 25,\ \ y = 3.$

Exercise:

Problem:

y varies inversely as the cube root of x and when $x = 64,\ \ y = 5.$

Solution:

$y = \frac{20}{\sqrt[3]{x}}$

Exercise:

Problem:

y varies jointly with x and z and when $x = 2$ and $z = 3,\ \ y = 36.$

Exercise:

Problem:

y varies jointly as $x,\ z,$ and w and when
$x = 1,\ \ z = 2,\ \ w = 5,$ then $y = 100.$

Solution:

$y = 10xzw$

Exercise:

Problem:

y varies jointly as the square of x and the square of z and when
$x = 3$ and $z = 4,$ then $y = 72.$

Exercise:

Problem:

y varies jointly as x and the square root of z and when
$x = 2$ and $z = 25,$ then $y = 100.$

Solution:

$y = 10x\sqrt{z}$

Exercise:

Problem:

y varies jointly as the square of x the cube of z and the square root of
w. When $x = 1, z = 2,$ and $w = 36,$ then $y = 48.$

Exercise:

Problem:

y varies jointly as x and z and inversely as w. When $x = 3,\ z = 5,$ and $w = 6,$ then $y = 10.$

Solution:

$y = 4\frac{xz}{w}$

Exercise:

Problem:

y varies jointly as the square of x and the square root of z and inversely as the cube of w. When $x = 3, z = 4,$ and $w = 3,$ then $y = 6.$

Exercise:

Problem:

y varies jointly as x and z and inversely as the square root of w and the square of t .When $x = 3, z = 1, w = 25,$ and $t = 2,$ then $y = 6.$

Solution:

$y = 40\frac{xz}{\sqrt{w}t^2}$

Numeric

For the following exercises, use the given information to find the unknown value.

Exercise:

Problem:

y varies directly as x. When $x = 3,$ then $y = 12.$ Find y when $x = 20.$

Exercise:

Problem:

y varies directly as the square of x. When $x = 2,$ then $y = 16$. Find y when $x = 8$.

Solution:

$y = 256$

Exercise:

Problem:

y varies directly as the cube of x. When $x = 3,$ then $y = 5.$ Find y when $x = 4$.

Exercise:

Problem:

y varies directly as the square root of x. When $x = 16,$ then $y = 4$. Find y when $x = 36$.

Solution:

$y = 6$

Exercise:

Problem:

y varies directly as the cube root of x. When $x = 125,$ then $y = 15$. Find y when $x = 1,000$.

Exercise:

Problem:

y varies inversely with x. When $x = 3,$ then $y = 2$. Find y when $x = 1$.

Solution:

$y = 6$

Exercise:

Problem:

y varies inversely with the square of x. When $x = 4,$ then $y = 3$. Find y when $x = 2$.

Exercise:

Problem:

y varies inversely with the cube of x. When $x = 3,$ then $y = 1$. Find y when $x = 1$.

Solution:

$y = 27$

Exercise:

Problem:

y varies inversely with the square root of x. When $x = 64,$ then $y = 12$. Find y when $x = 36$.

Exercise:

Problem:

y varies inversely with the cube root of x. When $x = 27,$ then $y = 5$. Find y when $x = 125$.

Solution:

$y = 3$

Exercise:

Problem:

y varies jointly as x and z.When $x = 4$ and $z = 2,$ then $y = 16.$ Find y when $x = 3$ and $z = 3.$

Exercise:

Problem:

y varies jointly as $x,\ z,$ and w.When $x = 2,\ z = 1,$ and $w = 12,$ then $y = 72.$ Find y when $x = 1,\ z = 2,$ and $w = 3.$

Solution:

$y = 18$

Exercise:

Problem:

y varies jointly as x and the square of z. When $x = 2$ and $z = 4,$ then $y = 144.$ Find y when $x = 4$ and $z = 5.$

Exercise:

Problem:

y varies jointly as the square of x and the square root of $z.$ When $x = 2$ and $z = 9,$ then $y = 24.$ Find y when $x = 3$ and $z = 25.$

Solution:

$y = 90$

Exercise:

Problem:

y varies jointly as x and z and inversely as $w.$ When $x = 5,\ z = 2,$ and $w = 20,$ then $y = 4.$ Find y when $x = 3$ and $z = 8,$ and $w = 48.$

Exercise:

Problem:

y varies jointly as the square of x and the cube of z and inversely as the square root of w. When $x = 2,\ z = 2,$ and $w = 64,$ then $y = 12$. Find y when $x = 1,\ z = 3,$ and $w = 4$.

Solution:

$y = \frac{81}{2}$

Exercise:

Problem:

y varies jointly as the square of x and of z and inversely as the square root of w and of t .When $x = 2,\ z = 3,\ w = 16,$ and $t = 3,$ then $y = 1$. Find y when $x = 3,\ z = 2,\ w = 36,$ and $t = 5$.

Technology

For the following exercises, use a calculator to graph the equation implied by the given variation.

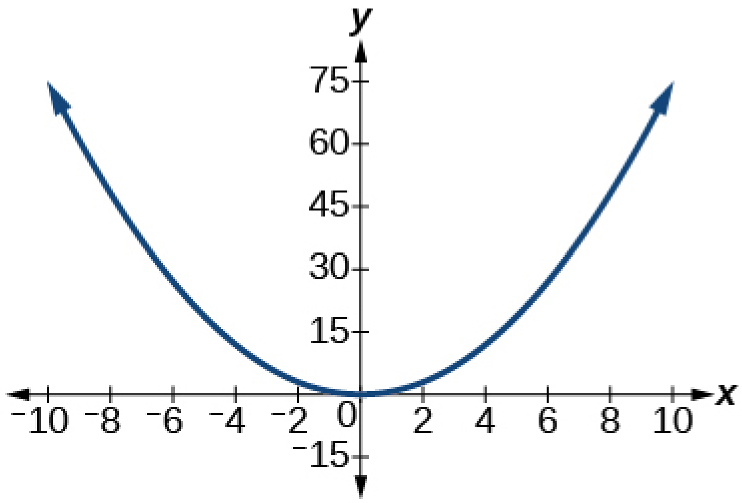
Exercise:

Problem:

y varies directly with the square of x and when $x = 2,\ y = 3$.

Solution:

$y = \frac{3}{4}x^2$



Exercise:

Problem: y varies directly as the cube of x and when $x = 2,\ y = 4.$

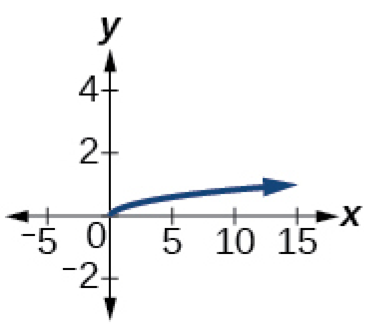
Exercise:

Problem:

y varies directly as the square root of x and when $x = 36,\ y = 2.$

Solution:

$y = \frac{1}{3}\sqrt{x}$



Exercise:

Problem: y varies inversely with x and when $x = 6,\ y = 2.$

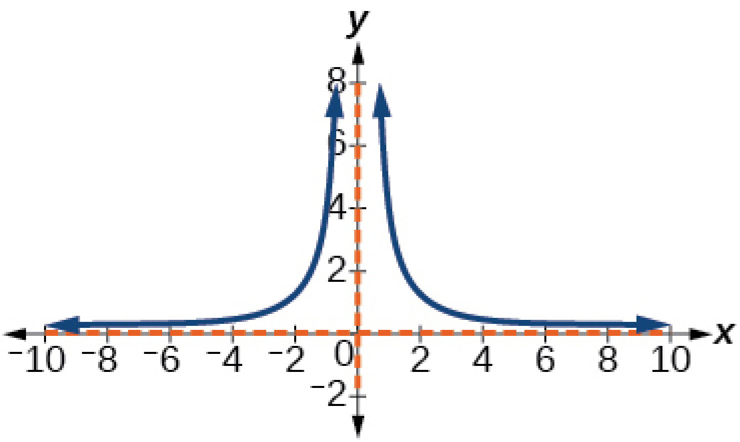
Exercise:

Problem:

y varies inversely as the square of x and when $x = 1,\ y = 4.$

Solution:

$y = \frac{4}{x^2}$



Extensions

For the following exercises, use Kepler’s Law, which states that the square of the time, $T,$ required for a planet to orbit the Sun varies directly with the cube of the mean distance, $a,$ that the planet is from the Sun.

Exercise:

Problem:

Using the Earth’s time of 1 year and mean distance of 93 million miles, find the equation relating T and $a.$

Exercise:

Problem:

Use the result from the previous exercise to determine the time required for Mars to orbit the Sun if its mean distance is 142 million miles.

Solution:

1.89 years

Exercise:

Problem:

Using Earth’s distance of 150 million kilometers, find the equation relating T and a.

Exercise:

Problem:

Use the result from the previous exercise to determine the time required for Venus to orbit the Sun if its mean distance is 108 million kilometers.

Solution:

0.61 years

Exercise:

Problem:

Using Earth’s distance of 1 astronomical unit (A.U.), determine the time for Saturn to orbit the Sun if its mean distance is 9.54 A.U.

Real-World Applications

For the following exercises, use the given information to answer the questions.

Exercise:

Problem:

The distance s that an object falls varies directly with the square of the time, $t,$ of the fall. If an object falls 16 feet in one second, how long for it to fall 144 feet?

Solution:

3 seconds

Exercise:

Problem:

The velocity v of a falling object varies directly to the time, $t,$ of the fall. If after 2 seconds, the velocity of the object is 64 feet per second, what is the velocity after 5 seconds?

Exercise:

Problem:

The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 24 inches long and vibrates 128 times per second, what is the length of a string that vibrates 64 times per second?

Solution:

48 inches

Exercise:

Problem:

The volume of a gas held at constant temperature varies indirectly as the pressure of the gas. If the volume of a gas is 1200 cubic centimeters when the pressure is 200 millimeters of mercury, what is the volume when the pressure is 300 millimeters of mercury?

Exercise:

Problem:

The weight of an object above the surface of the Earth varies inversely with the square of the distance from the center of the Earth. If a body weighs 50 pounds when it is 3960 miles from Earth's center, what would it weigh it were 3970 miles from Earth's center?

Solution:

49.75 pounds

Exercise:

Problem:

The intensity of light measured in foot-candles varies inversely with the square of the distance from the light source. Suppose the intensity of a light bulb is 0.08 foot-candles at a distance of 3 meters. Find the intensity level at 8 meters.

Exercise:

Problem:

The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is 40 amperes, the resistance is 10 ohms. Find the current if the resistance is 12 ohms.

Solution:

33.33 amperes

Exercise:

Problem:

The force exerted by the wind on a plane surface varies jointly with the square of the velocity of the wind and with the area of the plane surface. If the area of the surface is 40 square feet surface and the wind velocity is 20 miles per hour, the resulting force is 15 pounds. Find the force on a surface of 65 square feet with a velocity of 30 miles per hour.

Exercise:

Problem:

The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm)) and the cube of the diameter. If the shaft of a certain material 3 inches in diameter can transmit 45 hp at 100 rpm, what must the diameter be in order to transmit 60 hp at 150 rpm?

Solution:

2.88 inches

Exercise:

Problem:

The kinetic energy K of a moving object varies jointly with its mass m and the square of its velocity $v.$ If an object weighing 40 kilograms with a velocity of 15 meters per second has a kinetic energy of 1000 joules, find the kinetic energy if the velocity is increased to 20 meters per second.

Chapter Review Exercises

You have reached the end of Chapter 3: Polynomial and Rational Functions. Let’s review some of the Key Terms, Concepts and Equations you have learned.

Complex Numbers

Perform the indicated operation with complex numbers.

Exercise:

Problem: $(4+3i)+(-2-5i)$

Solution:

$2-2i$

Exercise:

Problem: $(6-5i)-(10+3i)$

Exercise:

Problem: $(2-3i)\,(3+6i)$

Solution:

$24+3i$

Exercise:

Problem: $\frac{2-i}{2+i}$

Solve the following equations over the complex number system.

Exercise:

Problem: $x^2-4x+5=0$

Solution:

$\{2+i,\ 2-i\}$

Exercise:

Problem: $x^2 + 2x + 10 = 0$

Quadratic Functions

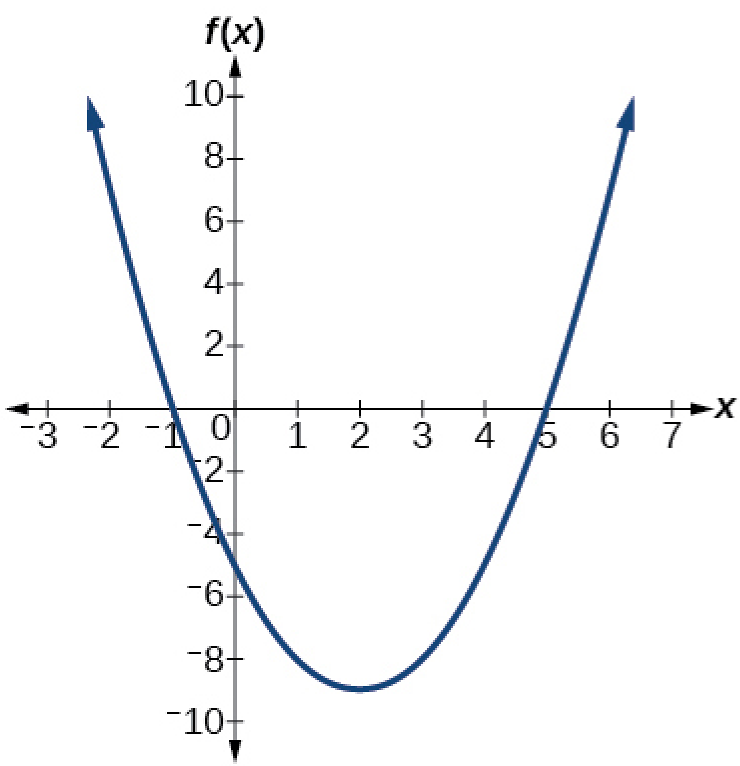
For the following exercises, write the quadratic function in standard form. Then, give the vertex and axes intercepts. Finally, graph the function.

Exercise:

Problem: $f(x) = x^2 - 4x - 5$

Solution:

$f(x) = (x - 2)^2 - 9$ vertex $(2, -9)$, intercepts $(5, 0)$; $(-1, 0)$; $(0, -5)$



Exercise:

Problem: $f(x) = -2x^2 - 4x$

For the following problems, find the equation of the quadratic function using the given information.

Exercise:

Problem: The vertex is $(-2, 3)$ and a point on the graph is $(3, 6)$.

Solution:

$f(x) = \frac{3}{25}(x + 2)^2 + 3$

Exercise:

Problem: The vertex is $(-3, 6.5)$ and a point on the graph is $(2, 6)$.

Answer the following questions.

Exercise:

Problem:

A rectangular plot of land is to be enclosed by fencing. One side is along a river and so needs no fence. If the total fencing available is 600 meters, find the dimensions of the plot to have maximum area.

Solution:

300 meters by 150 meters, the longer side parallel to river.

Exercise:

Problem:

An object projected from the ground at a 45 degree angle with initial velocity of 120 feet per second has height, $h,$ in terms of horizontal distance traveled, $x,$ given by $h(x)=\frac{-32}{(120)^2}x^2+x.$ Find the maximum height the object attains.

Power Functions and Polynomial Functions

For the following exercises, determine if the function is a polynomial function and, if so, give the degree and leading coefficient.

Exercise:

Problem: $f(x)=4x^5-3x^3+2x-1$

Solution:

Yes, degree = 5, leading coefficient = 4

Exercise:

Problem: $f(x)=5^{x+1}-x^2$

Exercise:

Problem: $f(x)=x^2\left(3-6x+x^2\right)$

Solution:

Yes, degree = 4, leading coefficient = 1

For the following exercises, determine end behavior of the polynomial function.

Exercise:

Problem: $f(x) = 2x^4 + 3x^3 - 5x^2 + 7$

Exercise:

Problem: $f(x) = 4x^3 - 6x^2 + 2$

Solution:

As $x \to -\infty,\ f(x) \to -\infty,$ as $x \to \infty,\ f(x) \to \infty$

Exercise:

Problem: $f(x) = 2x^2(1 + 3x - x^2)$

Graphs of Polynomial Functions

For the following exercises, find all zeros of the polynomial function, noting multiplicities.

Exercise:

Problem: $f(x) = (x + 3)^2(2x - 1)(x + 1)^3$

Solution:

–3 with multiplicity 2, $-\frac{1}{2}$ with multiplicity 1, –1 with multiplicity 3

Exercise:

Problem: $f(x) = x^5 + 4x^4 + 4x^3$

Exercise:

Problem: $f(x) = x^3 - 4x^2 + x - 4$

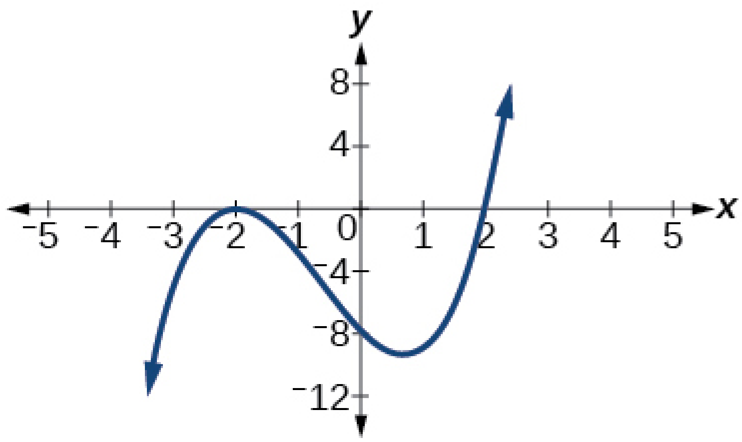
Solution:

4 with multiplicity 1

For the following exercises, based on the given graph, determine the zeros of the function and note multiplicity.

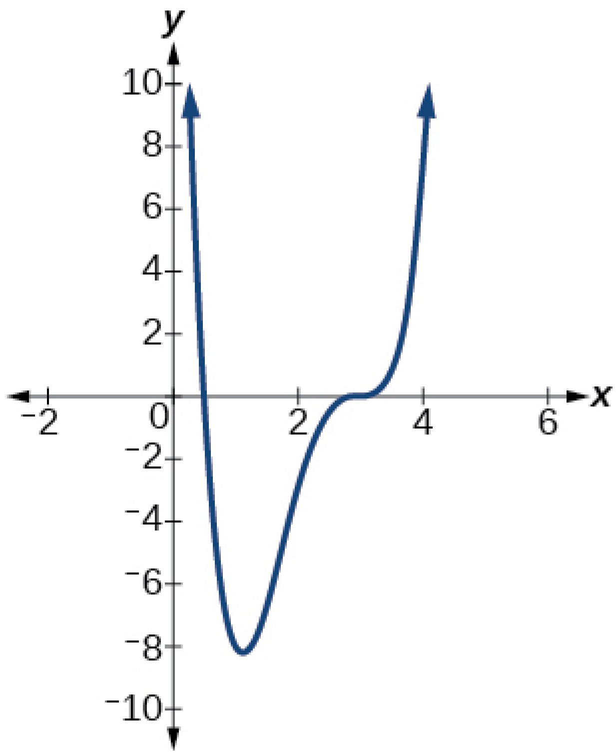
Exercise:

Problem:



Exercise:

Problem:



Solution:

$\frac{1}{2}$ with multiplicity 1, 3 with multiplicity 3

Exercise:

Problem:

Use the Intermediate Value Theorem to show that at least one zero lies between 2 and 3 for the function $f(x) = x^3 - 5x + 1$

Dividing Polynomials

For the following exercises, use long division to find the quotient and remainder.

Exercise:

Problem: $\frac{x^3-2x^2+4x+4}{x-2}$

Solution:

x^2+4 with remainder 12

Exercise:

Problem: $\frac{3x^4-4x^2+4x+8}{x+1}$

For the following exercises, use synthetic division to find the quotient. If the divisor is a factor, then write the factored form.

Exercise:

Problem: $\frac{x^3-2x^2+5x-1}{x+3}$

Solution:

$x^2-5x+20-\frac{61}{x+3}$

Exercise:

Problem: $\frac{x^3+4x+10}{x-3}$

Exercise:

Problem: $\frac{2x^3+6x^2-11x-12}{x+4}$

Solution:

$2x^2-2x-3$, so factored form is $(x+4)(2x^2-2x-3)$

Exercise:

Problem: $\frac{3x^4+3x^3+2x+2}{x+1}$

Zeros of Polynomial Functions

For the following exercises, use the Rational Zero Theorem to help you solve the polynomial equation.

Exercise:

Problem: $2x^3 - 3x^2 - 18x - 8 = 0$

Solution:

$\{-2,\ 4,\ -\frac{1}{2}\}$

Exercise:

Problem: $3x^3 + 11x^2 + 8x - 4 = 0$

Exercise:

Problem: $2x^4 - 17x^3 + 46x^2 - 43x + 12 = 0$

Solution:

$\{1,\ 3,\ 4,\ \frac{1}{2}\}$

Exercise:

Problem: $4x^4 + 8x^3 + 19x^2 + 32x + 12 = 0$

For the following exercises, use Descartes' Rule of Signs to find the possible number of positive and negative solutions.

Exercise:

Problem: $x^3 - 3x^2 - 2x + 4 = 0$

Solution:

0 or 2 positive, 1 negative

Exercise:

Problem: $2x^4 - x^3 + 4x^2 - 5x + 1 = 0$

Rational Functions

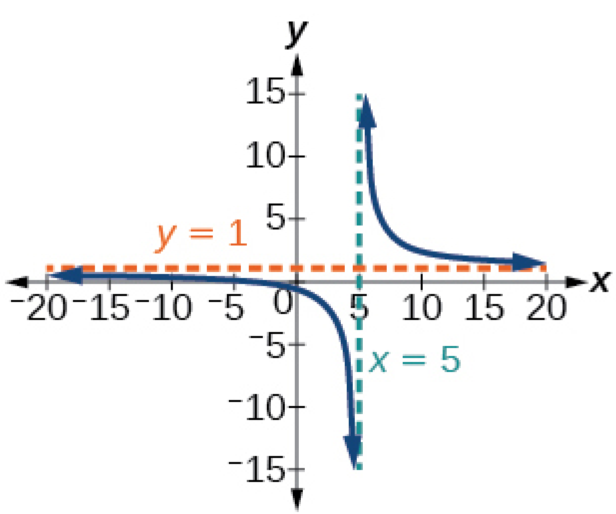
For the following rational functions, find the intercepts and the vertical and horizontal asymptotes, and then use them to sketch a graph.

Exercise:

Problem: $f(x) = \frac{x+2}{x-5}$

Solution:

Intercepts $(-2, 0)$ and $\left(0, -\frac{2}{5}\right)$, Asymptotes $x = 5$ and $y = 1$.



Exercise:

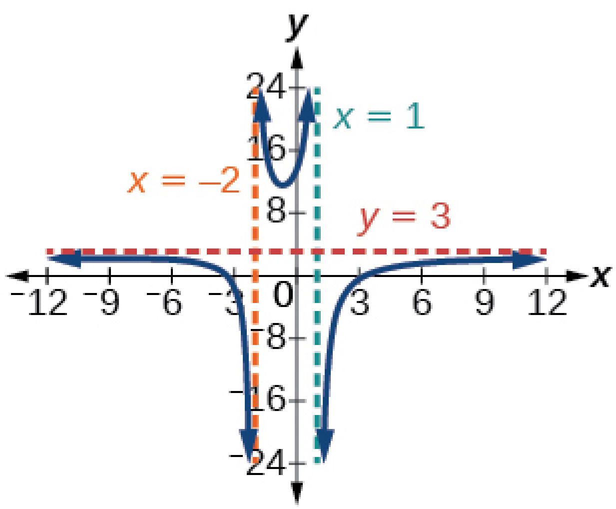
Problem: $f(x) = \frac{x^2+1}{x^2-4}$

Exercise:

Problem: $f(x) = \frac{3x^2-27}{x^2+x-2}$

Solution:

Intercepts (3, 0), (-3, 0), and $\left(0, \frac{27}{2}\right)$, Asymptotes $x = 1,\ x = -2,\ y = 3.$



Exercise:

Problem: $f(x) = \frac{x+2}{x^2-9}$

For the following exercises, find the slant asymptote.

Exercise:

Problem: $f(x) = \frac{x^2-1}{x+2}$

Solution:

$y = \ x - 2$

Exercise:

Problem: $f(x) = \frac{2x^3 - x^2 + 4}{x^2 + 1}$

Inverses and Radical Functions

For the following exercises, find the inverse of the function with the domain given.

Exercise:

Problem: $f(x) = (x - 2)^2, \ x \geq 2$

Solution:

$f^{-1}(x) = \sqrt{x} + 2$

Exercise:

Problem: $f(x) = (x + 4)^2 - 3, \ x \geq -4$

Exercise:

Problem: $f(x) = x^2 + 6x - 2, \ x \geq -3$

Solution:

$f^{-1}(x) = \sqrt{x + 11} - 3$

Exercise:

Problem: $f(x) = 2x^3 - 3$

Exercise:

Problem: $f(x) = \sqrt{4x+5} - 3$

Solution:

$f^{-1}(x) = \frac{(x+3)^2 - 5}{4}$, $x \geq -3$

Exercise:

Problem: $f(x) = \frac{x-3}{2x+1}$

Modeling Using Variation

For the following exercises, find the unknown value.

Exercise:

Problem:

y varies directly as the square of x. If when $x = 3,\ y = 36,$ find y if $x = 4$.

Solution:

$y = 64$

Exercise:

Problem:

y varies inversely as the square root of x If when $x = 25,\ y = 2,$ find y if $x = 4$.

Exercise:

Problem:

y varies jointly as the cube of x and as $z.$ If when $x = 1$ and $z = 2,$ $y = 6,$ find y if $x = 2$ and $z = 3.$

Solution:

$y\ =\ 72$

Exercise:

Problem:

y varies jointly as x and the square of z and inversely as the cube of $w.$ If when $x = 3,$ $z = 4,$ and $w = 2,$ $y = 48,$ find y if $x = 4,$ $z = 5,$ and $w = 3.$

For the following exercises, solve the application problem.

Exercise:

Problem:

The weight of an object above the surface of the earth varies inversely with the distance from the center of the earth. If a person weighs 150 pounds when he is on the surface of the earth (3,960 miles from center), find the weight of the person if he is 20 miles above the surface.

Solution:

148.5 pounds

Exercise:

Problem:

The volume V of an ideal gas varies directly with the temperature T and inversely with the pressure P. A cylinder contains oxygen at a temperature of 310 degrees K and a pressure of 18 atmospheres in a volume of 120 liters. Find the pressure if the volume is decreased to 100 liters and the temperature is increased to 320 degrees K.

Chapter Test

Perform the indicated operation or solve the equation.
Exercise:

Problem: $(3-4i)(4+2i)$

Solution:

$20-10i$

Exercise:

Problem: $\frac{1-4i}{3+4i}$

Exercise:

Problem: $x^2-4x+13=0$

Solution:

$\{2+3i,\ 2-3i\}$

Give the degree and leading coefficient of the following polynomial function.
Exercise:

Problem: $f(x) = x^3 \left(3 - 6x^2 - 2x^2\right)$

Determine the end behavior of the polynomial function.

Exercise:

Problem: $f(x) = 8x^3 - 3x^2 + 2x - 4$

Solution:

$As\, x \to -\infty,\ f(x) \to -\infty,\ as\, x \to \infty,\ f(x) \to \infty$

Exercise:

Problem: $f(x) = -2x^2(4 - 3x - 5x^2)$

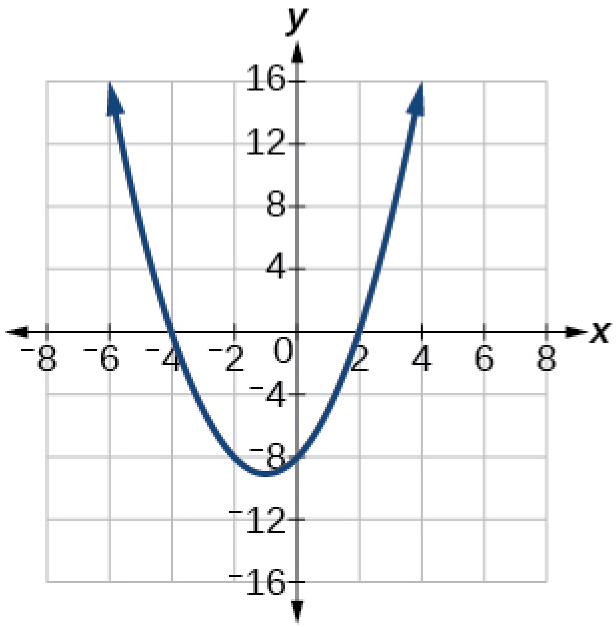
Write the quadratic function in standard form. Determine the vertex and axes intercepts and graph the function.

Exercise:

Problem: $f(x) = x^2 + 2x - 8$

Solution:

$f(x) = (x + 1)^2 - 9$, vertex $(-1, -9)$, intercepts $(2, 0); (-4, 0);\ (0, -8)$



Given information about the graph of a quadratic function, find its equation.
Exercise:

Problem: Vertex $(2, 0)$ and point on graph $(4, 12)$.

Solve the following application problem.
Exercise:

Problem:

A rectangular field is to be enclosed by fencing. In addition to the enclosing fence, another fence is to divide the field into two parts, running parallel to two sides. If 1,200 feet of fencing is available, find the maximum area that can be enclosed.

Solution:

60,000 square feet

Find all zeros of the following polynomial functions, noting multiplicities.
Exercise:

Problem: $f(x)=(x-3)^3(3x-1)(x-1)^2$

Exercise:

Problem: $f(x)=2x^6-12x^5+18x^4$

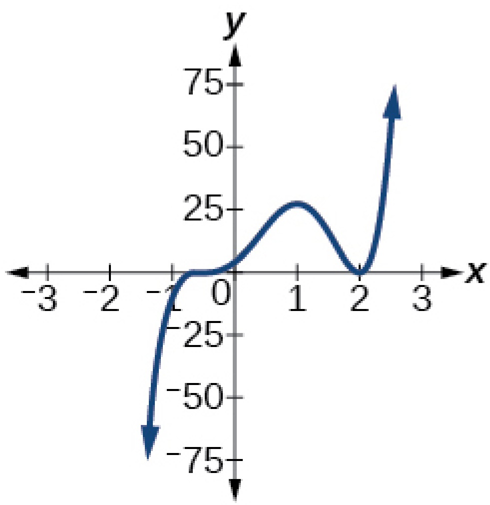
Solution:

0 with multiplicity 4, 3 with multiplicity 2

Based on the graph, determine the zeros of the function and multiplicities.

Exercise:

Problem:



Use long division to find the quotient.

Exercise:

Problem: $\frac{2x^3+3x-4}{x+2}$

Solution:

$2x^2 - 4x + 11 - \frac{26}{x+2}$

Use synthetic division to find the quotient. If the divisor is a factor, write the factored form.

Exercise:

Problem: $\frac{x^4+3x^2-4}{x-2}$

Exercise:

Problem: $\frac{2x^3+5x^2-7x-12}{x+3}$

Solution:

$2x^2 - x - 4$. So factored form is $(x+3)(2x^2 - x - 4)$

Use the Rational Zero Theorem to help you find the zeros of the polynomial functions.

Exercise:

Problem: $f(x) = 2x^3 + 5x^2 - 6x - 9$

Exercise:

Problem: $f(x) = 4x^4 + 8x^3 + 21x^2 + 17x + 4$

Solution:

$-\frac{1}{2}$ (has multiplicity 2), $\frac{-1 \pm i\sqrt{15}}{2}$

Exercise:

Problem: $f(x) = 4x^4 + 16x^3 + 13x^2 - 15x - 18$

Exercise:

Problem: $f(x) = x^5 + 6x^4 + 13x^3 + 14x^2 + 12x + 8$

Solution:

-2 (has multiplicity 3), $\pm i$

Given the following information about a polynomial function, find the function.

Exercise:

Problem:

It has a double zero at $x = 3$ and zeroes at $x = 1$ and $x = -2$. Its *y*-intercept is $(0, 12)$.

Exercise:

Problem:

It has a zero of multiplicity 3 at $x = \frac{1}{2}$ and another zero at $x = -3$. It contains the point $(1, 8)$.

Solution:

$f(x) = 2(2x - 1)^3 (x + 3)$

Use Descartes' Rule of Signs to determine the possible number of positive and negative solutions.

Exercise:

Problem: $8x^3 - 21x^2 + 6 = 0$

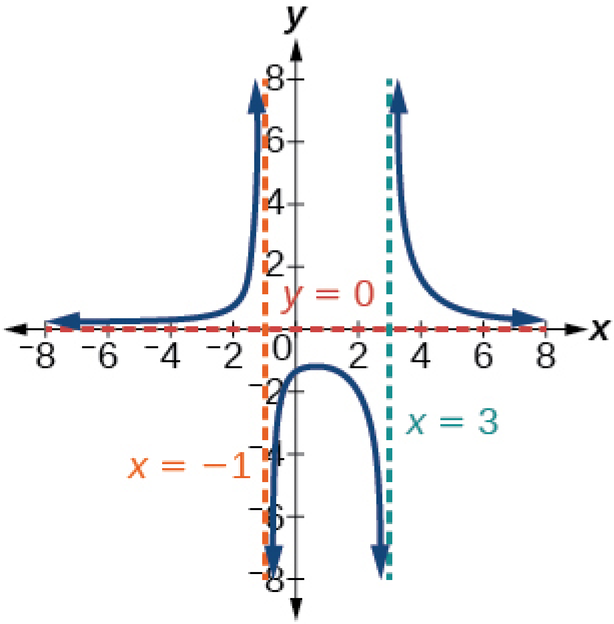
For the following rational functions, find the intercepts and horizontal and vertical asymptotes, and sketch a graph.

Exercise:

Problem: $f(x) = \frac{x+4}{x^2-2x-3}$

Solution:

Intercepts $(-4, 0)$, $\left(0, -\frac{4}{3}\right)$, Asymptotes $x = 3$, $x = -1$, $y = 0$.



Exercise:

Problem: $f(x) = \frac{x^2+2x-3}{x^2-4}$

Find the slant asymptote of the rational function.

Exercise:

Problem: $f(x) = \frac{x^2+3x-3}{x-1}$

Solution:

$y = x + 4$

Find the inverse of the function.
Exercise:

Problem: $f(x) = \sqrt{x-2} + 4$

Exercise:

Problem: $f(x) = 3x^3 - 4$

Solution:

$f^{-1}(x) = \sqrt[3]{\frac{x+4}{3}}$

Exercise:

Problem: $f(x) = \frac{2x+3}{3x-1}$

Find the unknown value.
Exercise:

Problem:

y varies inversely as the square of x and when $x = 3,\ y = 2.$ Find y if $x = 1.$

Solution:

$y = 18$

Exercise:

Problem:

y varies jointly with x and the cube root of $z.$ If when $x = 2$ and $z = 27,\ y = 12,$ find y if $x = 5$ and $z = 8.$

Solve the following application problem.
Exercise:

Problem:

The distance a body falls varies directly as the square of the time it falls. If an object falls 64 feet in 2 seconds, how long will it take to fall 256 feet?

Solution:

4 seconds

Glossary

constant of variation
: the non-zero value k that helps define the relationship between variables in direct or inverse variation

direct variation
: the relationship between two variables that are a constant multiple of each other; as one quantity increases, so does the other

inverse variation
: the relationship between two variables in which the product of the variables is a constant

inversely proportional
: a relationship where one quantity is a constant divided by the other quantity; as one quantity increases, the other decreases

joint variation
: a relationship where a variable varies directly or inversely with multiple variables

varies directly
: a relationship where one quantity is a constant multiplied by the other quantity

varies inversely

a relationship where one quantity is a constant divided by the other quantity

Quadratic Functions
In this section, you will:

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function’s minimum or maximum value.
- Solve problems involving a quadratic function’s minimum or maximum value.

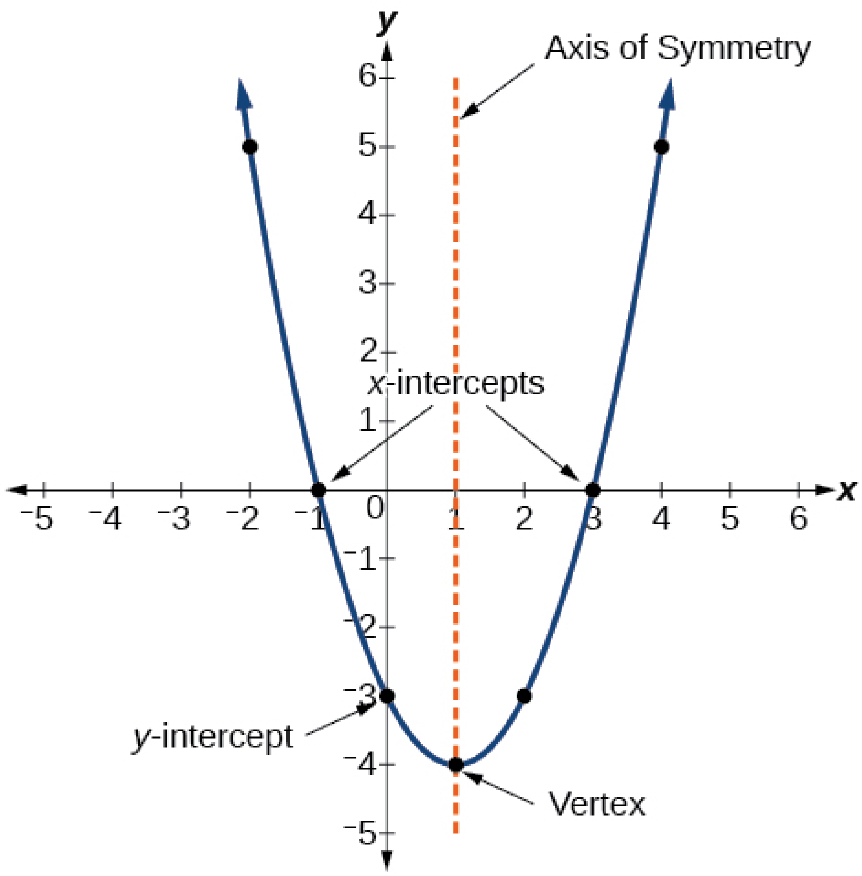
An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)

Curved antennas, such as the ones shown in [link], are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in [link].



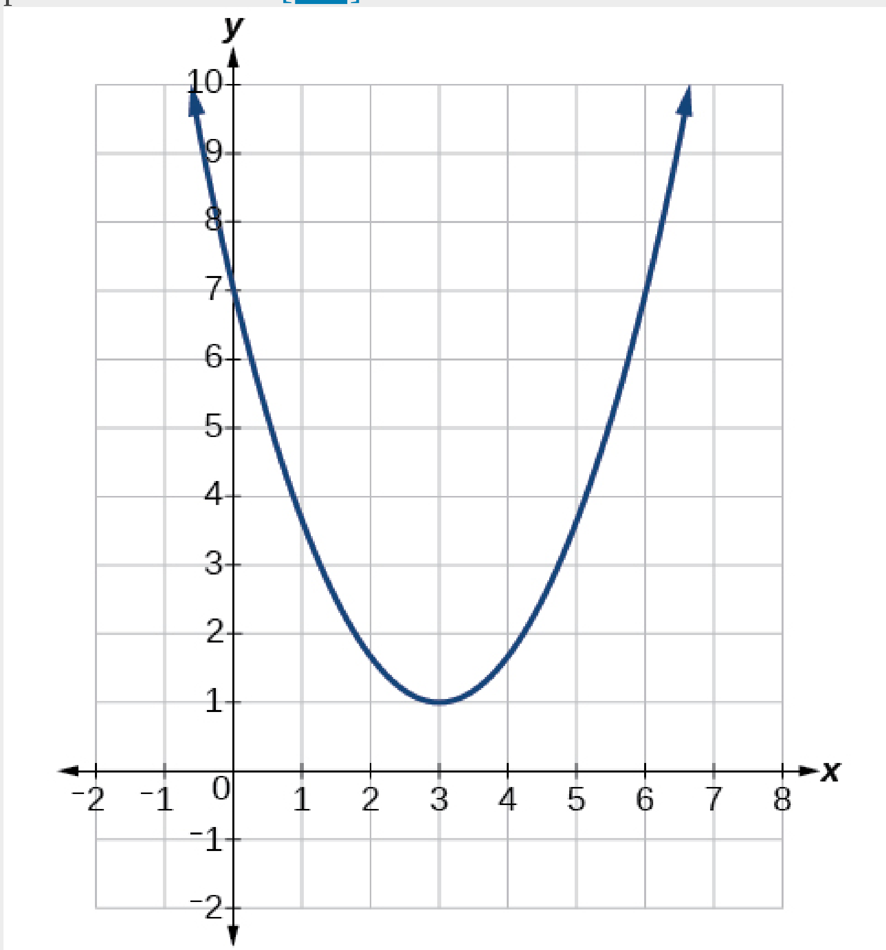
The *y*-intercept is the point at which the parabola crosses the *y*-axis. The *x*-intercepts are the points at which the parabola crosses the *x*-axis. If they exist, the *x*-intercepts represent the **zeros,** or **roots**, of the quadratic function, the values of x at which $y = 0$.

Example:

Exercise:

Problem:
Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in [link].



Solution:

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$ so this is the y-intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

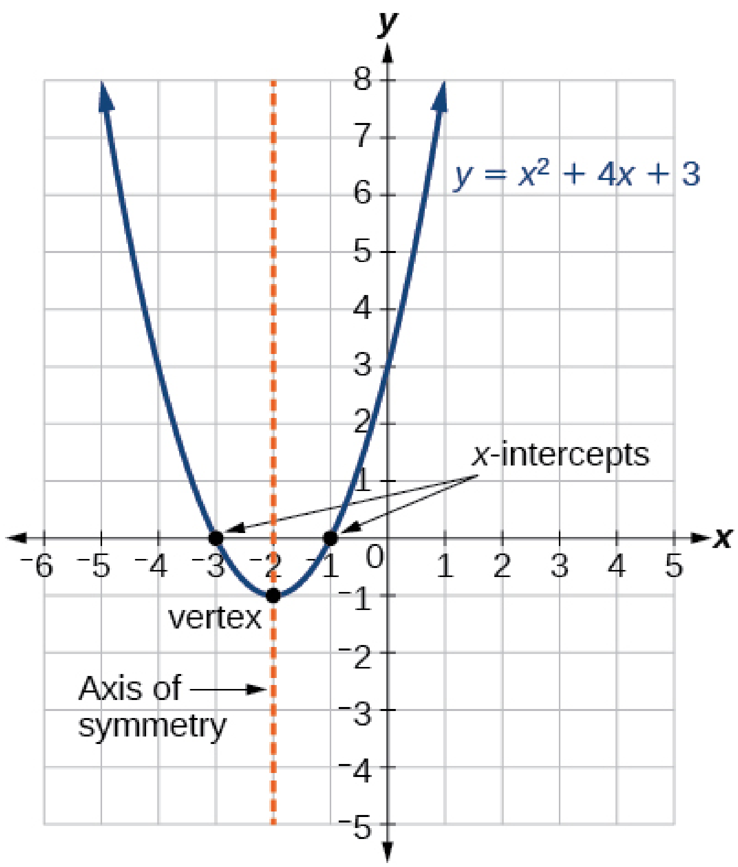
The **general form of a quadratic function** presents the function in the form
Equation:

$$f(x) = ax^2 + bx + c$$

where $a, b,$ and c are real numbers and $a \neq 0$. If $a > 0,$ the parabola opens upward. If $a < 0,$ the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of x halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

[link] represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1, b = 4,$ and $c = 3$. Because $a > 0,$ the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

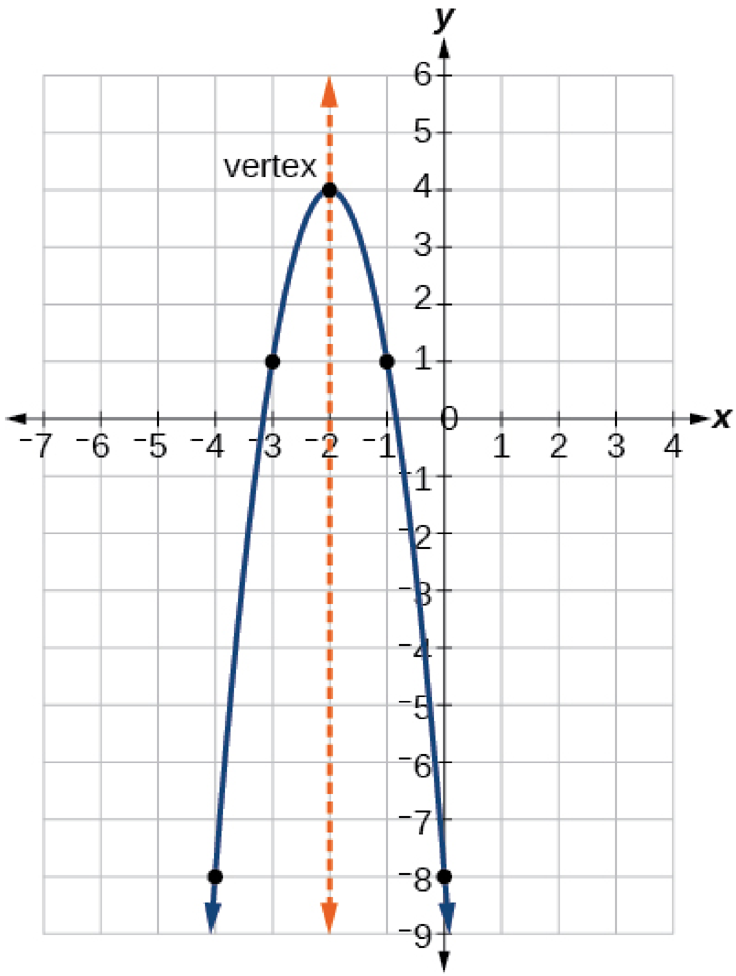


The **standard form of a quadratic function** presents the function in the form
Equation:

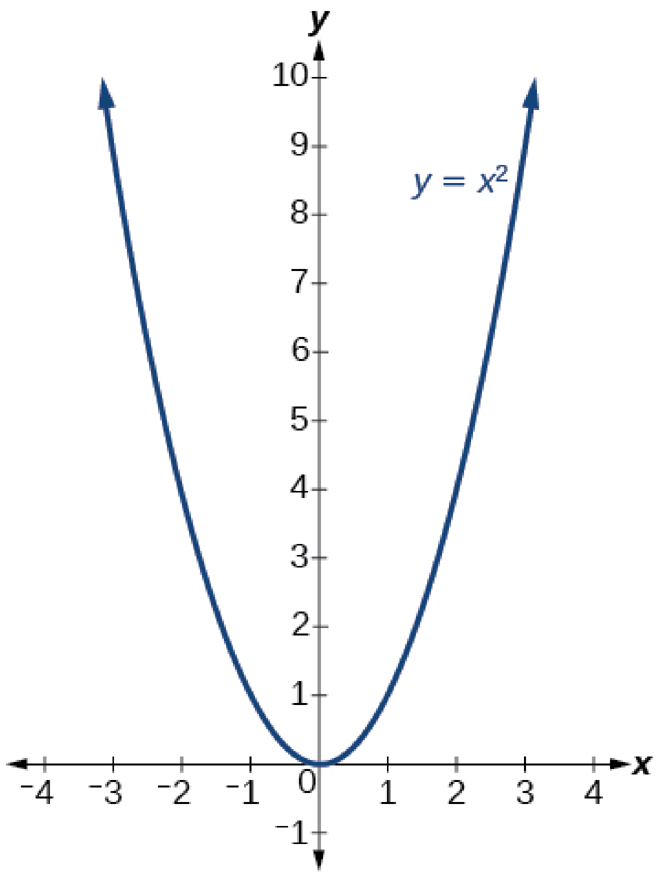
$$f(x) = a(x - h)^2 + k$$

where $(h,\ k)$ is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0,$ the parabola opens upward and the vertex is a minimum. If $a < 0,$ the parabola opens downward, and the vertex is a maximum. [link] represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3, h = -2,$ and $k = 4$. Because $a < 0,$ the parabola opens downward. The vertex is at $(-2,\ 4)$.



The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. [link] is the graph of this basic function.



If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In [link], $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In [link], $h < 0$, so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In [link], $|a| > 1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

Equation:

$$\begin{aligned} a(x-h)^2 + k &= ax^2 + bx + c \\ ax^2 - 2ahx + (ah^2 + k) &= ax^2 + bx + c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.
Equation:

$$-2ah = b, \text{ so } h = -\frac{b}{2a}.$$

This is the axis of symmetry we defined earlier. Setting the constant terms equal:
Equation:

$$\begin{aligned} ah^2 + k &= c \\ k &= c - ah^2 \\ &= c - a\left(-\tfrac{b}{2a}\right)^2 \\ &= c - \tfrac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h) = k$.

Note:
Forms of Quadratic Functions
A quadratic function is a function of degree two. The graph of a quadratic function is a parabola. The **general form of a quadratic function** is $f(x) = ax^2 + bx + c$ where $a, b,$ and c are real numbers and $a \neq 0$.
The **standard form of a quadratic function** is $f(x) = a(x-h)^2 + k$.
The vertex (h, k) is located at
Equation:

$$h = -\frac{b}{2a},\ k = f(h) = f\left(\frac{-b}{2a}\right).$$

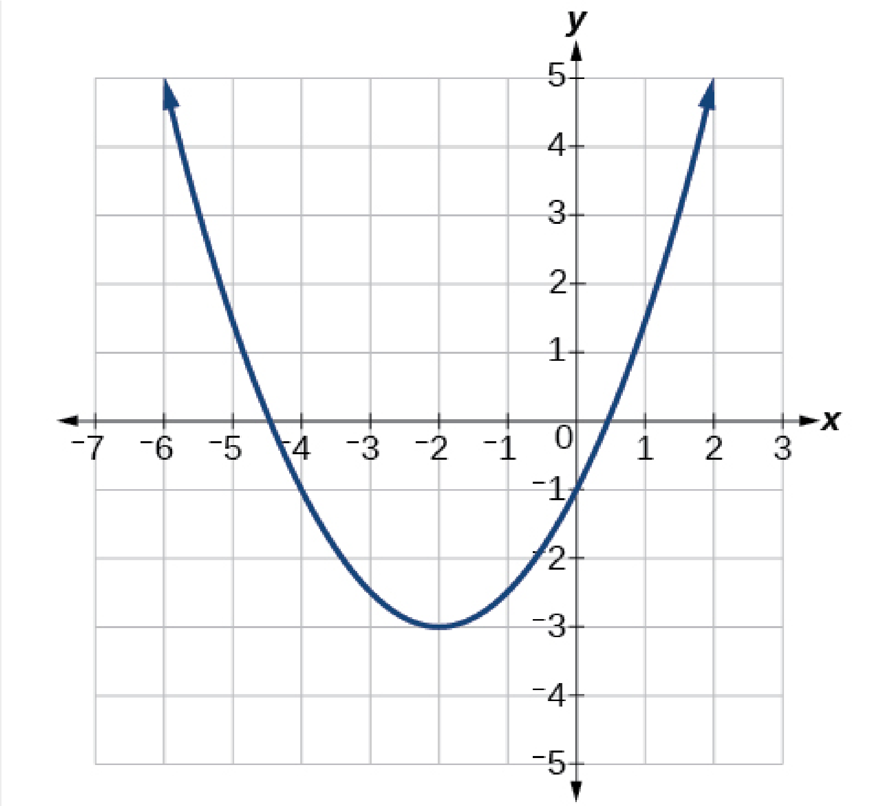
Note:
Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is $h.$ Identify the vertical shift of the parabola; this value is $k.$
2. Substitute the values of the horizontal and vertical shift for h and $k.$ in the function $f(x)=a(x–h)^2+k.$
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x).$
4. Solve for the stretch factor, $|a|.$
5. If the parabola opens up, $a>0.$ If the parabola opens down, $a<0$ since this means the graph was reflected about the x-axis.
6. Expand and simplify to write in general form.

Example:
Exercise:

Problem:
Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in [link] as a transformation of $f(x)=x^2,$ and then expand the formula, and simplify terms to write the equation in general form.



Solution:

We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

Equation:

$$\begin{aligned} -1 &= a(0+2)^2 - 3 \\ 2 &= 4a \\ a &= \tfrac{1}{2} \end{aligned}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x+2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

Equation:

$$\begin{aligned} g(x) &= \frac{1}{2}(x+2)^2 - 3 \\ &= \frac{1}{2}(x+2)(x+2) - 3 \\ &= \frac{1}{2}(x^2+4x+4) - 3 \\ &= \frac{1}{2}x^2 + 2x + 2 - 3 \\ &= \frac{1}{2}x^2 + 2x - 1 \end{aligned}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Analysis

We can check our work using the table feature on a graphing utility. First enter $\text{Y1} = \frac{1}{2}(x+2)^2 - 3$. Next, select TBLSET, then use $\text{TblStart} = -6$ and $\Delta\text{Tbl} = 2$, and select TABLE. See [link].

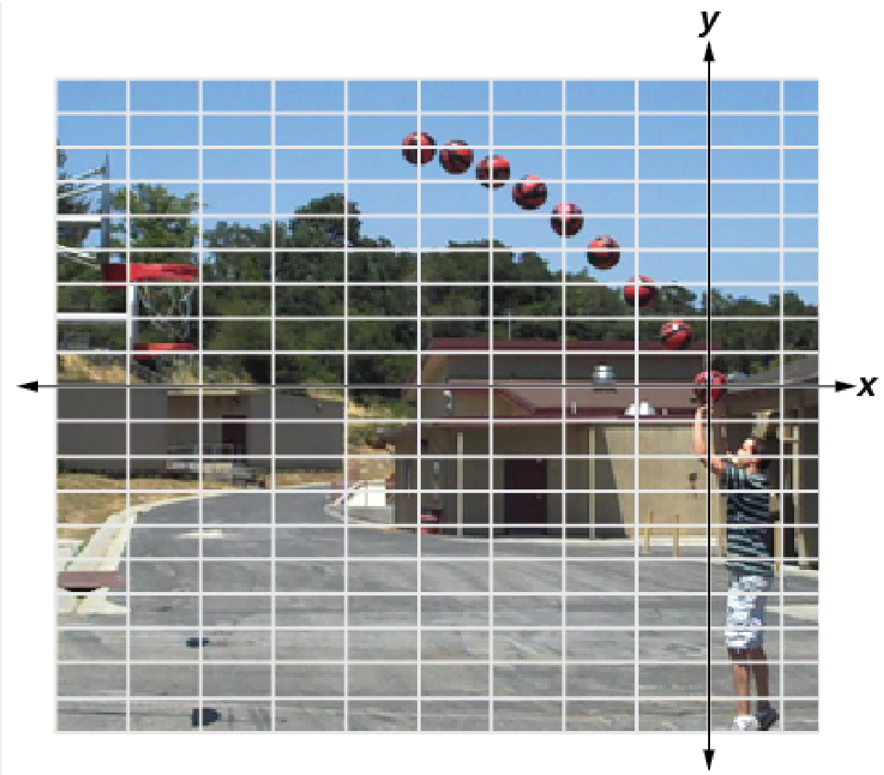
x	–6	–4	–2	0	2
y	5	–1	–3	–1	5

The ordered pairs in the table correspond to points on the graph.

Note:
Exercise:

Problem:

A coordinate grid has been superimposed over the quadratic path of a basketball in [link]. Find an equation for the path of the ball. Does the shooter make the basket?



(credit: modification of work by Dan Meyer)

Solution:

The path passes through the origin and has vertex at $(-4,\ 7)$, so $(h)x =- \frac{7}{16}(x+4)^2+7$. To make the shot, $h\left(-7.5\right)$ would need to be about 4 but $h(-7.5)\approx 1.64;$ he doesn't make it.

Note:
Given a quadratic function in general form, find the vertex of the parabola.

1. Identify $a,\ b,\ $ and $c.$
2. Find $h,$ the *x*-coordinate of the vertex, by substituting a and b into $h =- \frac{b}{2a}.$
3. Find $k,$ the *y*-coordinate of the vertex, by evaluating $k=f\left(h\right)=f\left(-\frac{b}{2a}\right).$

Example:
Exercise:

Problem:
Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 – 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution:

The horizontal coordinate of the vertex will be at
Equation:

$$\begin{aligned} h &= -\tfrac{b}{2a} \\ &= -\tfrac{-6}{2(2)} \\ &= \tfrac{6}{4} \\ &= \tfrac{3}{2} \end{aligned}$$

The vertical coordinate of the vertex will be at
Equation:

$$\begin{aligned} k &= f(h) \\ &= f\left(\tfrac{3}{2}\right) \\ &= 2\left(\tfrac{3}{2}\right)^2 - 6\left(\tfrac{3}{2}\right) + 7 \\ &= \tfrac{5}{2} \end{aligned}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.
Equation:

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ f(x) &= 2x^2 - 6x + 7 \end{aligned}$$

Using the vertex to determine the shifts,
Equation:

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$$

Analysis

One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs, (k), and where it occurs, (x).

Note:
Exercise:

Problem:

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Solution:

$g(x) = x^2 - 6x + 13$ in general form; $g(x) = (x-3)^2 + 4$ in standard form

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all *y*-values greater than or equal to the *y*-coordinate at the turning point or less than or equal to the *y*-coordinate at the turning point, depending on whether the parabola opens up or down.

Note:
Domain and Range of a Quadratic Function
The domain of any quadratic function is all real numbers.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $\left[f\left(-\frac{b}{2a}\right), \infty\right)$; the range of a quadratic function written in general form with a negative a value is $f(x) \leq f\left(-\frac{b}{2a}\right)$, or $\left(-\infty, f\left(-\frac{b}{2a}\right)\right]$.

The range of a quadratic function written in standard form $f(x) = a(x-h)^2 + k$ with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

Note:

Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

Example:

Exercise:

Problem:

Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution:

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

Equation:

$$
\begin{aligned}
h &= -\frac{b}{2a} \\
&= -\frac{9}{2(-5)} \\
&= \frac{9}{10}
\end{aligned}
$$

The maximum value is given by $f(h)$.

Equation:

$$
\begin{aligned}
f\left(\frac{9}{10}\right) &= -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 \\
&= \frac{61}{20}
\end{aligned}
$$

The range is $f(x) \le \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$.

Note:
Exercise:

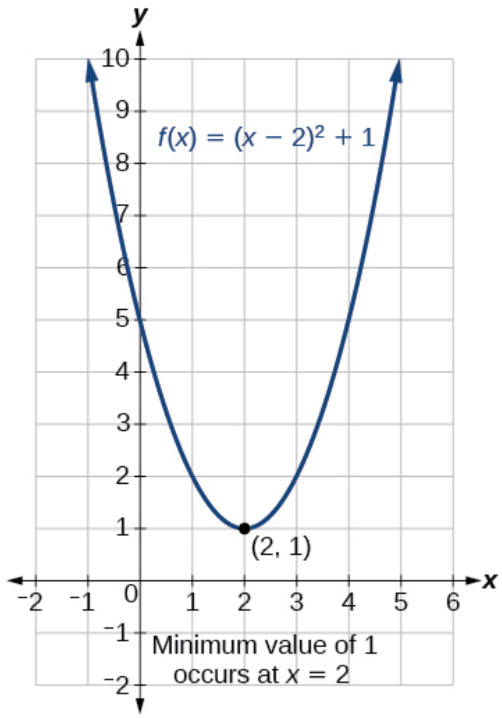
Problem: Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Solution:

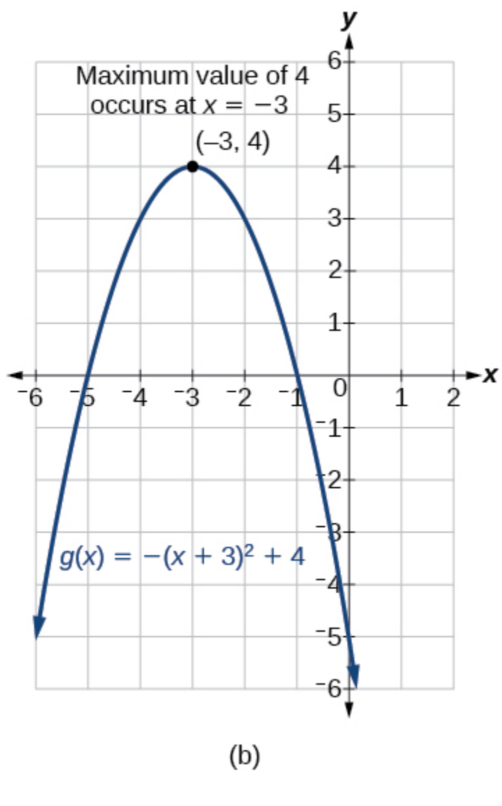
The domain is all real numbers. The range is $f(x) \ge \frac{8}{11}$, or $\left[\frac{8}{11}, \infty\right)$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the parabola. We can see the maximum and minimum values in [link].



(a)



(b)

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and

revenue.

Example:
Exercise:

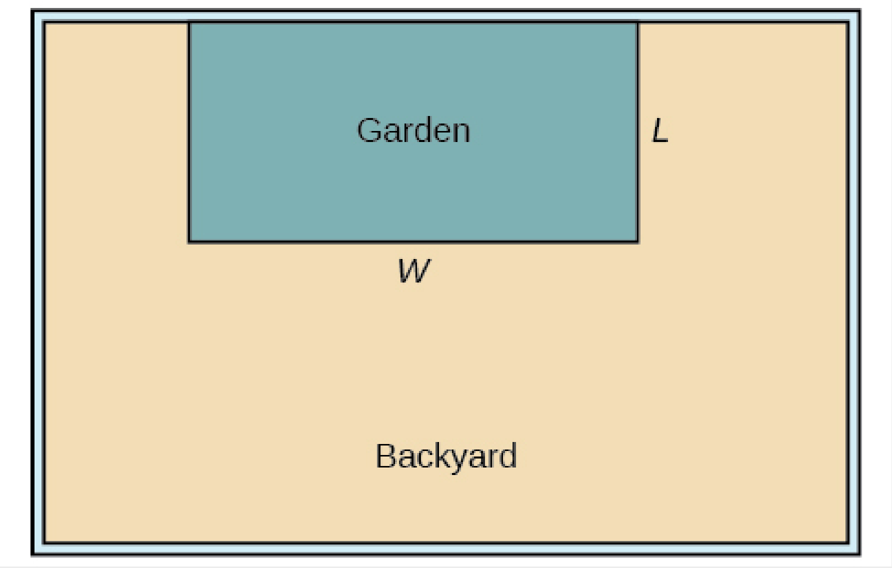
Problem:
Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
b. What dimensions should she make her garden to maximize the enclosed area?

Solution:

Let’s use a diagram such as [link] to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.



a. We know we have only 80 feet of fence available, and $L+W+L=80,$ or more simply, $2L+W=80.$ This allows us to represent the width, $W,$ in terms of $L.$
Equation:

$$W=80-2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so
Equation:

$$\begin{aligned}A&=LW=L(80-2L)\\A(L)&=80L-2L^2\end{aligned}$$

This formula represents the area of the fence in terms of the variable length $L.$ The function, written in general form, is
Equation:

$$A(L)=-2L^2+80L.$$

b. The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a=-2, b=80,$ and $c=0.$

To find the vertex:
Equation:

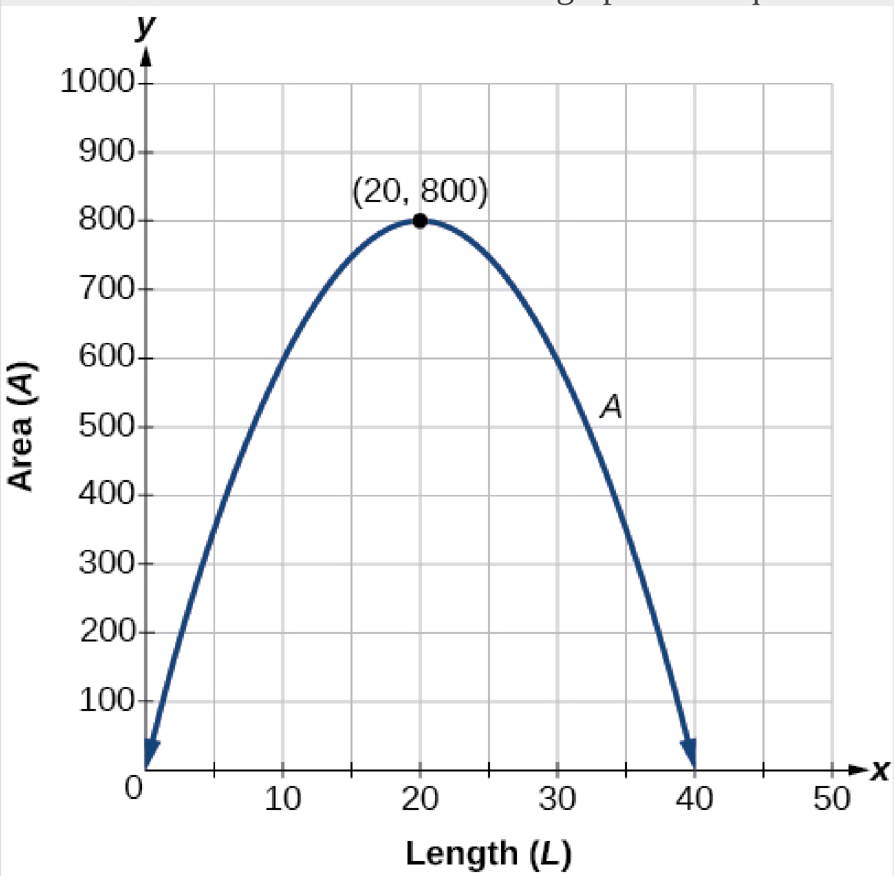
$$\begin{aligned}h&=-\tfrac{80}{2(-2)} & \quad & k=A(20)\\&=20 & \text{and} & =80(20)-2(20)^2\\& & & =800\end{aligned}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L=20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose

the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in [link].



Note:
Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

Example:
Exercise:

Problem:
Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of \$30. Market research has suggested that if the owners raise the price to \$32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution:

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to \$32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be
Equation:

$$\begin{aligned} m &= \frac{79{,}000-84{,}000}{32-30} \\ &= \frac{-5{,}000}{2} \\ &= -2,500 \end{aligned}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the *y*-intercept.
Equation:

$$\begin{array}{rl} Q = -2500p + b & \text{Substitute in the point } Q = 84,000 \text{ and } p = 30 \\ 84,000 = -2500(30) + b & \text{Solve for } b \\ b = 159,000 & \end{array}$$

This gives us the linear equation $Q = -2{,}500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

Equation:

$$\begin{aligned} \text{Revenue} &= pQ \\ \text{Revenue} &= p(-2,500p + 159,000) \\ \text{Revenue} &= -2,500p^2 + 159,000p \end{aligned}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

Equation:

$$\begin{aligned} h &= -\frac{159{,}000}{2(-2{,}500)} \\ &= 31.8 \end{aligned}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

Equation:

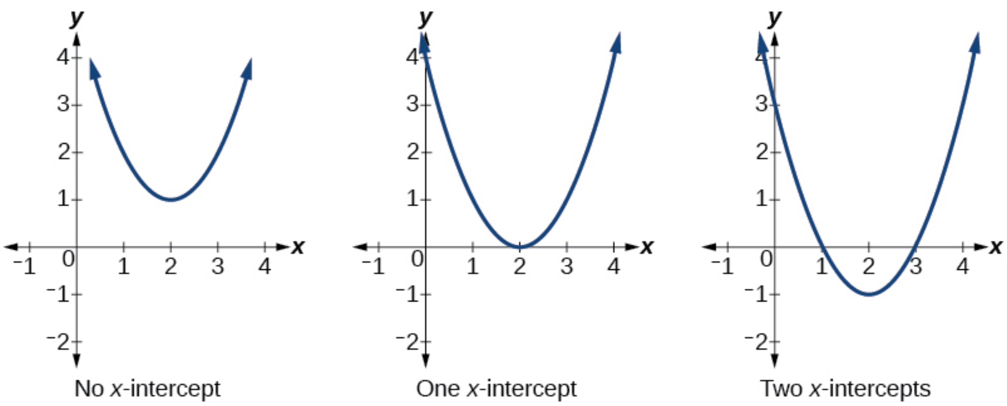
$$\begin{aligned} \text{maximum revenue} &= -2{,}500(31.8)^2 + 159{,}000(31.8) \\ &= 2{,}528{,}100 \end{aligned}$$

Analysis

This could also be solved by graphing the quadratic as in [link]. We can see the maximum revenue on a graph of the quadratic function.

Finding the *x*- and *y*-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in [link] that the number of x-intercepts can vary depending upon the location of the graph.



Number of *x*-intercepts of a parabola

Note:
Given a quadratic function $f(x)$, find the y- and *x*-intercepts.

1. Evaluate $f(0)$ to find the y-intercept.
2. Solve the quadratic equation $f(x) = 0$ to find the *x*-intercepts.

Example:
Exercise:

Problem:
Finding the *y*- and *x*-Intercepts of a Parabola

Find the *y*- and *x*-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution:

We find the *y*-intercept by evaluating $f(0)$.
Equation:

$$\begin{aligned} f(0) &= 3(0)^2 + 5(0) - 2 \\ &= -2 \end{aligned}$$

So the *y*-intercept is at $(0, -2)$.

For the *x*-intercepts, we find all solutions of $f(x) = 0$.
Equation:

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.
Equation:

$$0 = (3x - 1)(x + 2)$$

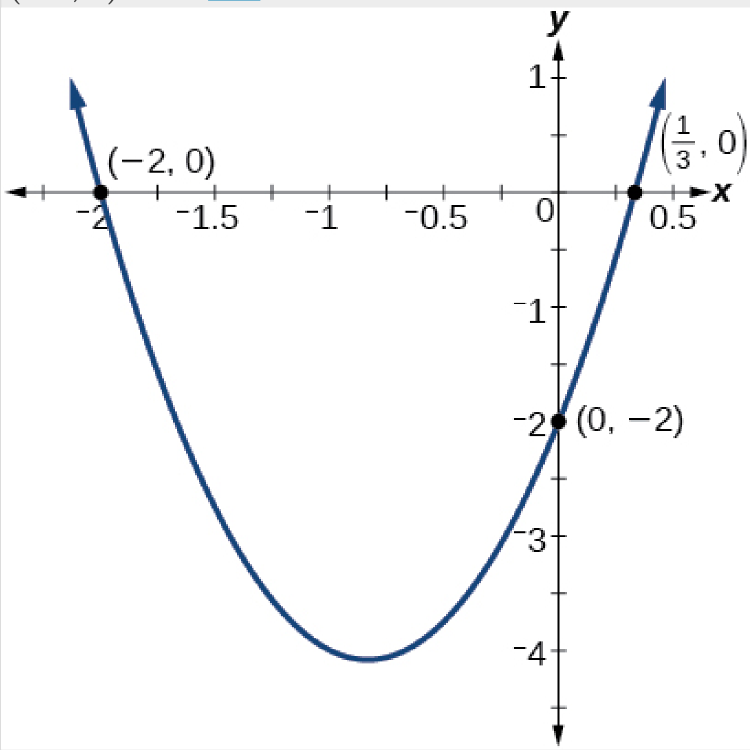
Equation:

$$\begin{array}{lcl} 0 = 3x - 1 & & 0 = x + 2 \\ x = \frac{1}{3} & \text{or} & x = -2 \end{array}$$

So the *x*-intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.

Analysis

By graphing the function, we can confirm that the graph crosses the *y*-axis at $(0, -2)$. We can also confirm that the graph crosses the *x*-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.See [link]



Rewriting Quadratics in Standard Form

In [link], the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

Note:

Given a quadratic function, find the x-intercepts by rewriting in standard form.

1. Substitute a and b into $h=-\frac{b}{2a}$.
2. Substitute $x=h$ into the general form of the quadratic function to find k.
3. Rewrite the quadratic in standard form using h and k.
4. Solve for when the output of the function will be zero to find the x-intercepts.

Example:
Exercise:

Problem:
Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x)=2x^2+4x-4$.

Solution:

We begin by solving for when the output will be zero.
Equation:

$$0=2x^2+4x-4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.
Equation:

$$f\left(x\right)=a(x-h)^2+k$$

We know that $a=2$. Then we solve for h and k.
Equation:

$$\begin{aligned} h &= -\tfrac{b}{2a} & \qquad k &= f(-1) \\ &= -\tfrac{4}{2(2)} & &= 2(-1)^2+4(-1)-4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

Equation:

$$f(x) = 2(x+1)^2 - 6$$

We can now solve for when the output will be zero.

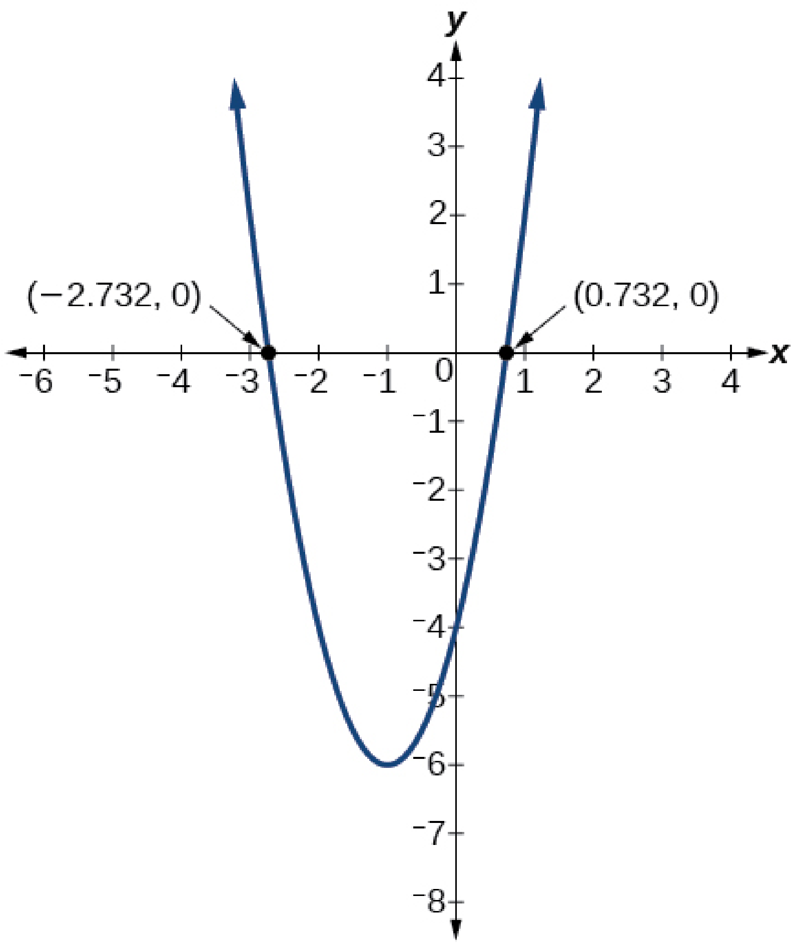
Equation:

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$.

Analysis

We can check our work by graphing the given function on a graphing utility and observing the x-intercepts. See [link].



Note:
Exercise:

Problem:

In a separate Try It, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x.$ Now find the y- and x-intercepts (if any).

Solution:

y-intercept at (0, 13), No x-intercepts

Example:

Exercise:

Problem:
Solving a Quadratic Equation with the Quadratic Formula

Solve $x^2+x+2=0$.

Solution:

Let's begin by writing the quadratic formula: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

When applying the quadratic formula, we identify the coefficients $a,\ b$ and $c.$ For the equation $x^2+x+2=0,$ we have $a=1,\ b=1,$ and $c=2.$ Substituting these values into the formula we have:

Equation:

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-1\pm\sqrt{1^2-4\cdot1\cdot(2)}}{2\cdot1} \\ &= \frac{-1\pm\sqrt{1-8}}{2} \\ &= \frac{-1\pm\sqrt{-7}}{2} \\ &= \frac{-1\pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $x=\frac{-1+i\sqrt{7}}{2}$ and $x=\frac{-1-i\sqrt{7}}{2}$ or $x=\frac{-1}{2}+\frac{i\sqrt{7}}{2}$ and $x=\frac{-1}{2}-\frac{i\sqrt{7}}{2}$.

Example:
Exercise:

Problem:
Applying the Vertex and *x*-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

a. When does the ball reach the maximum height?
b. What is the maximum height of the ball?
c. When does the ball hit the ground?

Solution:

a. The ball reaches the maximum height at the vertex of the parabola.
   Equation:

$$\begin{aligned} h &= -\tfrac{80}{2(-16)} \\ &= \tfrac{80}{32} \\ &= \tfrac{5}{2} \\ &= 2.5 \end{aligned}$$

   The ball reaches a maximum height after 2.5 seconds.
b. To find the maximum height, find the y-coordinate of the vertex of the parabola.
   Equation:

$$\begin{aligned} k &= H\left(-\tfrac{b}{2a}\right) \\ &= H\left(2.5\right) \\ &= -16(2.5)^2 + 80\left(2.5\right) + 40 \\ &= 140 \end{aligned}$$

   The ball reaches a maximum height of 140 feet.
c. To find when the ball hits the ground, we need to determine when the height is zero, $H\left(t\right) = 0$.

   We use the quadratic formula.
   Equation:

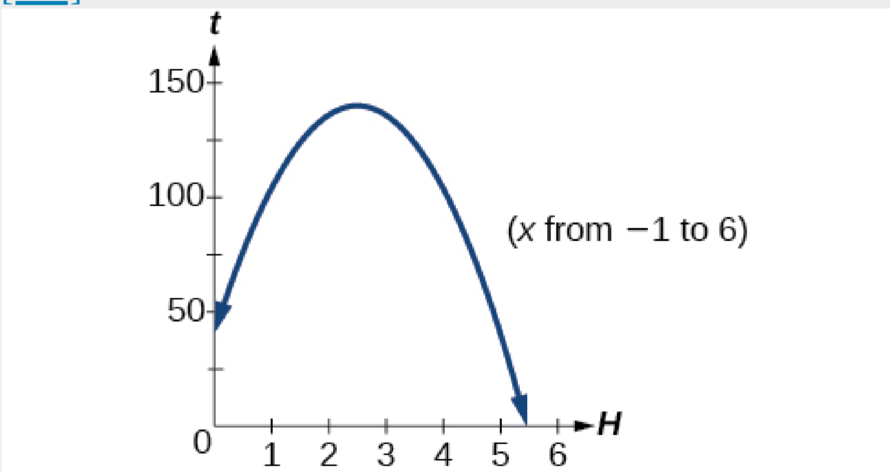
$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)}$$
$$= \frac{-80 \pm \sqrt{8960}}{-32}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

Equation:

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \quad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See [link]



Note:
Exercise:

Problem:

A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Solution:

3 seconds 256 feet 7 seconds

Note:
Access these online resources for additional instruction and practice with quadratic equations.

- Graphing Quadratic Functions in General Form
- Graphing Quadratic Functions in Standard Form
- Quadratic Function Review
- Characteristics of a Quadratic Function

Key Equations

general form of a quadratic function	$f(x) = ax^2 + bx + c$
the quadratic formula	

	$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
standard form of a quadratic function	$f(x)=a(x-h)^2+k$

Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the y-axis. See [link], [link], and [link].
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph. See [link].
- The vertex can be found from an equation representing a quadratic function. See [link].
- The domain of a quadratic function is all real numbers. The range varies with the function. See [link].
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue. See [link] and [link].
- Some quadratic equations must be solved by using the quadratic formula. See [link].
- The vertex and the intercepts can be identified and interpreted to solve real-world problems. See [link].

Section Exercises

Verbal

Exercise:

Problem:

Explain the advantage of writing a quadratic function in standard form.

Solution:

When written in that form, the vertex can be easily identified.

Exercise:

Problem:

How can the vertex of a parabola be used in solving real world problems?

Exercise:

Problem:

Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

Solution:

If $a = 0$ then the function becomes a linear function.

Exercise:

Problem: What is another name for the standard form of a quadratic function?

Exercise:

Problem:

What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

Solution:

If possible, we can use factoring. Otherwise, we can use the quadratic formula.

Algebraic

For the following exercises, rewrite the quadratic functions in standard form and give the vertex.

Exercise:

Problem: $f\left(x\right)={x}^{2}-12x+32$

Exercise:

Problem:$g\left(x\right)={x}^{2}+2x-3$

Solution:

$f(x)={\left(x+1\right)}^{2}-2,$ Vertex $\left(-1,-4\right)$

Exercise:

Problem:$f(x)={x}^{2}-x$

Exercise:

Problem:$f(x)={x}^{2}+5x-2$

Solution:

$f(x)={\left(x+\frac{5}{2}\right)}^{2}-\frac{33}{4},$ Vertex $\left(-\frac{5}{2},-\frac{33}{4}\right)$

Exercise:

Problem:$h\left(x\right)=2{x}^{2}+8x-10$

Exercise:

Problem:$k\left(x\right)=3{x}^{2}-6x-9$

Solution:

$f(x)=3{\left(x-1\right)}^{2}-12,$ Vertex $\left(1,-12\right)$

Exercise:

Problem:$f(x)=2{x}^{2}-6x$

Exercise:

Problem: $f(x) = 3x^2 - 5x - 1$

Solution:

$f(x) = 3\left(x - \frac{5}{6}\right)^2 - \frac{37}{12}$, Vertex $\left(\frac{5}{6}, -\frac{37}{12}\right)$

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

Exercise:

Problem: $y\,(x) = 2x^2 + 10x + 12$

Exercise:

Problem: $f\,(x) = 2x^2 - 10x + 4$

Solution:

Minimum is $-\frac{17}{2}$ and occurs at $\frac{5}{2}$. Axis of symmetry is $x = \frac{5}{2}$.

Exercise:

Problem: $f(x) = -x^2 + 4x + 3$

Exercise:

Problem: $f(x) = 4x^2 + x - 1$

Solution:

Minimum is $-\frac{17}{16}$ and occurs at $-\frac{1}{8}$. Axis of symmetry is $x = -\frac{1}{8}$.

Exercise:

Problem: $h\,(t) = -4t^2 + 6t - 1$

Exercise:

Problem: $f(x) = \frac{1}{2}x^2 + 3x + 1$

Solution:

Minimum is $-\frac{7}{2}$ and occurs at $-3.$ Axis of symmetry is $x = -3.$

Exercise:

Problem: $f(x) = -\frac{1}{3}x^2 - 2x + 3$

For the following exercises, determine the domain and range of the quadratic function.

Exercise:

Problem: $f(x) = (x - 3)^2 + 2$

Solution:

Domain is $(-\infty, \infty).$ Range is $[2, \infty).$

Exercise:

Problem: $f(x) = -2(x + 3)^2 - 6$

Exercise:

Problem: $f(x) = x^2 + 6x + 4$

Solution:

Domain is $(-\infty, \infty).$ Range is $[-5, \infty).$

Exercise:

Problem: $f(x) = 2x^2 - 4x + 2$

Exercise:

Problem: $k(x) = 3x^2 - 6x - 9$

Solution:

Domain is $(-\infty, \infty)$. Range is $[-12, \infty)$.

For the following exercises, solve the equations over the complex numbers.

Exercise:

Problem: $x^2 = -25$

Exercise:

Problem: $x^2 = -8$

Solution:

$\left\{2i\sqrt{2}, -2i\sqrt{2}\right\}$

Exercise:

Problem: $x^2 + 36 = 0$

Exercise:

Problem: $x^2 + 27 = 0$

Solution:

$\left\{3i\sqrt{3}, -3i\sqrt{3}\right\}$

Exercise:

Problem: $x^2 + 2x + 5 = 0$

Exercise:

Problem: $x^2 - 4x + 5 = 0$

Solution:

$\{2 + i, 2 - i\}$

Exercise:

Problem:$x^2 + 8x + 25 = 0$

Exercise:

Problem:$x^2 - 4x + 13 = 0$

Solution:

$\{2 + 3i, 2 - 3i\}$

Exercise:

Problem:$x^2 + 6x + 25 = 0$

Exercise:

Problem:$x^2 - 10x + 26 = 0$

Solution:

$\{5 + i, 5 - i\}$

Exercise:

Problem:$x^2 - 6x + 10 = 0$

Exercise:

Problem:$x(x - 4) = 20$

Solution:

$\left\{2 + 2\sqrt{6},\ 2 - 2\sqrt{6}\right\}$

Exercise:

Problem:$x(x - 2) = 10$

Exercise:

Problem: $2x^2 + 2x + 5 = 0$

Solution:

$\{-\frac{1}{2} + \frac{3}{2}i, \; -\frac{1}{2} - \frac{3}{2}i\}$

Exercise:

Problem: $5x^2 - 8x + 5 = 0$

Exercise:

Problem: $5x^2 + 6x + 2 = 0$

Solution:

$\{-\frac{3}{5} + \frac{1}{5}i, \; -\frac{3}{5} - \frac{1}{5}i\}$

Exercise:

Problem: $2x^2 - 6x + 5 = 0$

Exercise:

Problem: $x^2 + x + 2 = 0$

Solution:

$\left\{-\frac{1}{2} + \frac{1}{2}i\sqrt{7}, \; -\frac{1}{2} - \frac{1}{2}i\sqrt{7}\right\}$

Exercise:

Problem: $x^2 - 2x + 4 = 0$

For the following exercises, use the vertex (h, k) and a point on the graph (x, y) to find the general form of the equation of the quadratic function.

Exercise:

Problem: $(h, k) = (2, 0), (x, y) = (4, 4)$

Solution:

$f(x) = x^2 - 4x + 4$

Exercise:

Problem: $(h, k) = (-2, -1), (x, y) = (-4, 3)$

Exercise:

Problem: $(h, k) = (0, 1), (x, y) = (2, 5)$

Solution:

$f(x) = x^2 + 1$

Exercise:

Problem: $(h, k) = (2, 3), (x, y) = (5, 12)$

Exercise:

Problem: $(h, k) = (-5, 3), (x, y) = (2, 9)$

Solution:

$f(x) = \frac{6}{49}x^2 + \frac{60}{49}x + \frac{297}{49}$

Exercise:

Problem: $(h, k) = (3, 2), (x, y) = (10, 1)$

Exercise:

Problem: $(h, k) = (0, 1), (x, y) = (1, 0)$

Solution:

$f(x) = -x^2 + 1$

Exercise:

Problem: $(h, k) = (1, 0), (x, y) = (0, 1)$

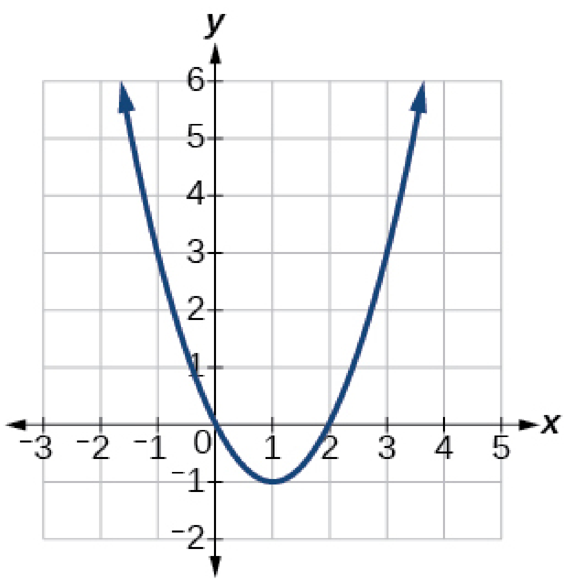
Graphical

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

Exercise:

Problem: $f(x) = x^2 - 2x$

Solution:



Vertex$(1,\ -1)$, Axis of symmetry is $x = 1$. Intercepts are $(0, 0),\ (2, 0)$.

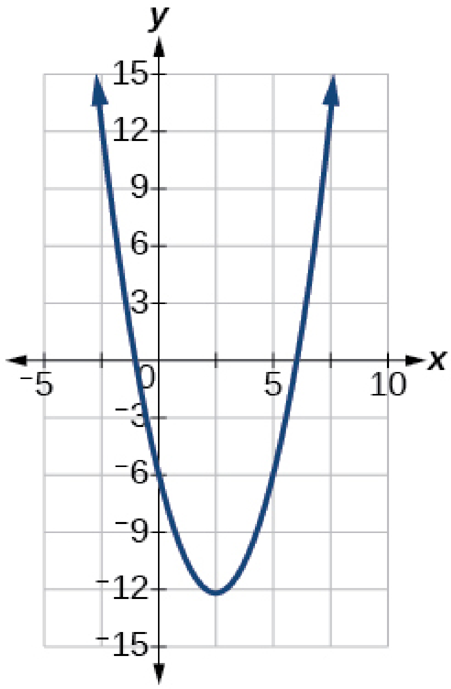
Exercise:

Problem: $f(x) = x^2 - 6x - 1$

Exercise:

Problem: $f(x) = x^2 - 5x - 6$

Solution:



Vertex $\left(\frac{5}{2}, \frac{-49}{4}\right)$, Axis of symmetry is $(0, -6), (-1, 0), (6, 0)$.

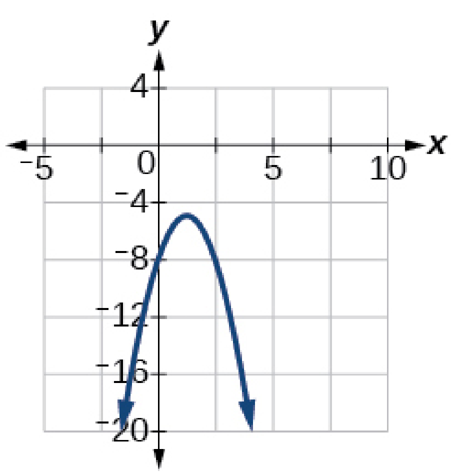
Exercise:

Problem: $f(x) = x^2 - 7x + 3$

Exercise:

Problem: $f(x) = -2x^2 + 5x - 8$

Solution:

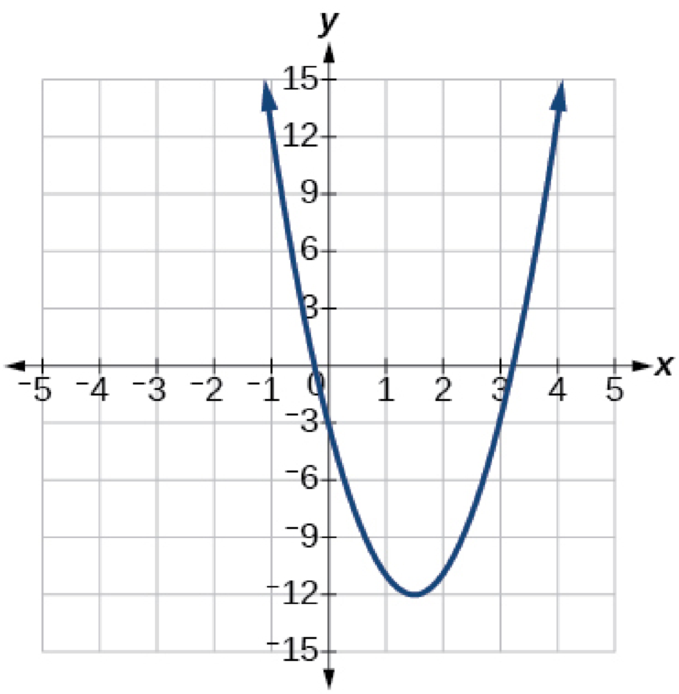


Vertex $\left(\frac{5}{4}, \ -\frac{39}{8}\right)$, Axis of symmetry is $x = \frac{5}{4}$. Intercepts are $(0, \ -8)$.

Exercise:

Problem: $f(x) = 4x^2 - 12x - 3$

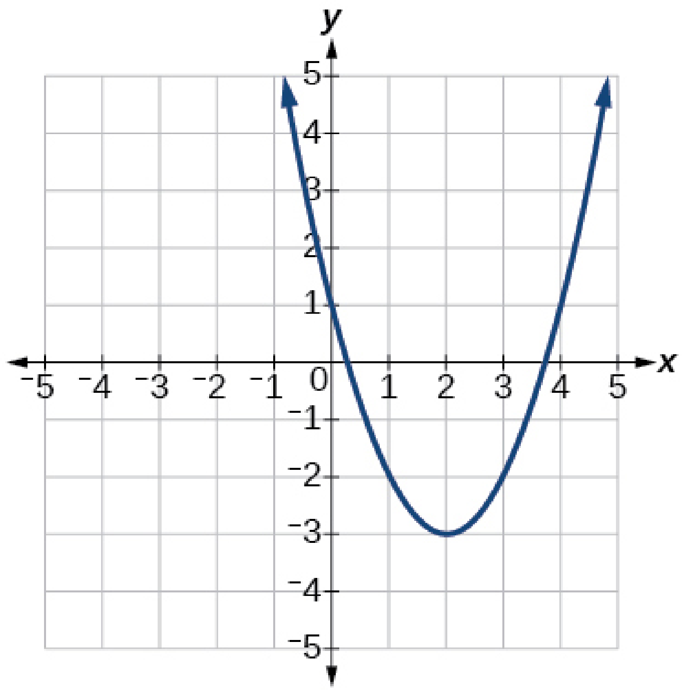
Solution:



For the following exercises, write the equation for the graphed function.

Exercise:

Problem:

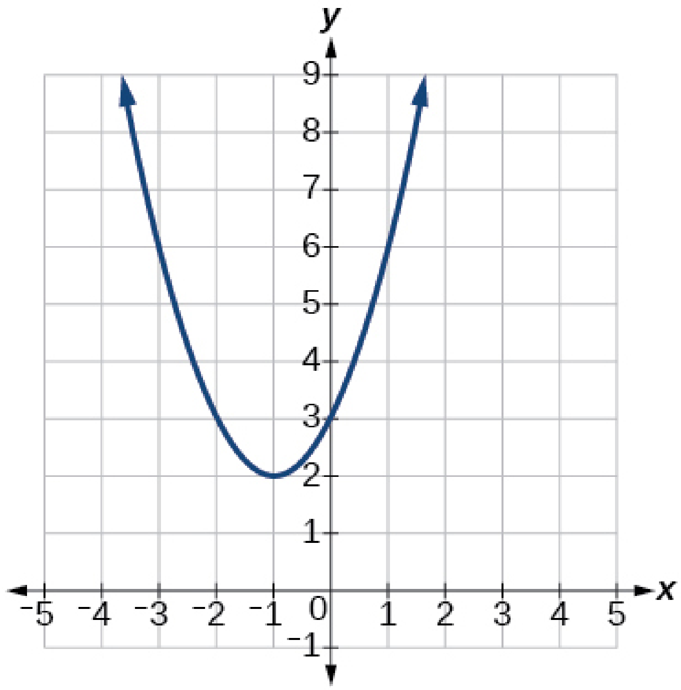


Solution:

$f(x) = x^2 - 4x + 1$

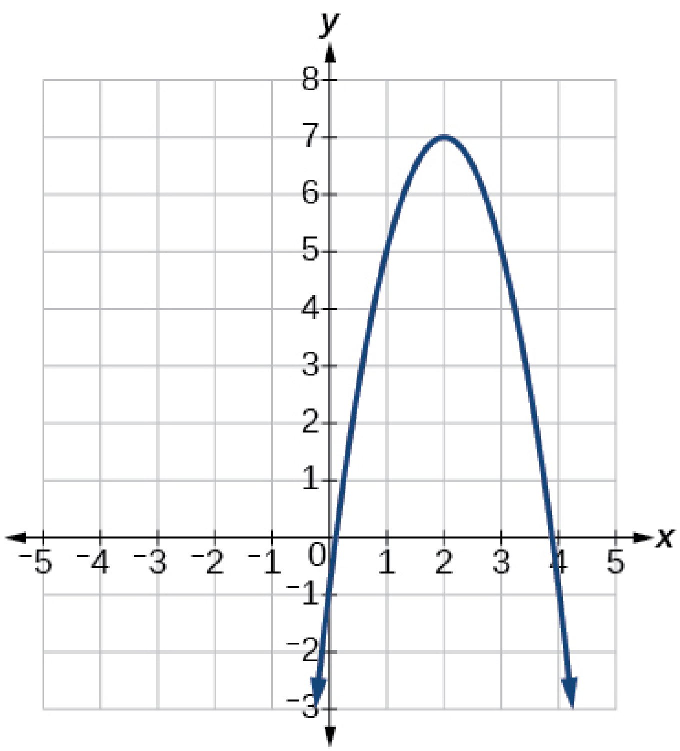
Exercise:

Problem:



Exercise:

Problem:

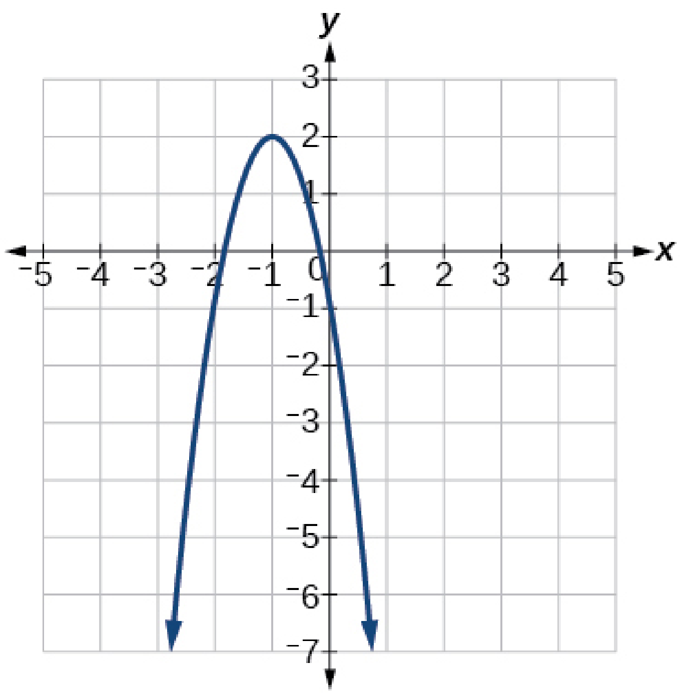


Solution:

$f(x) = -2x^2 + 8x - 1$

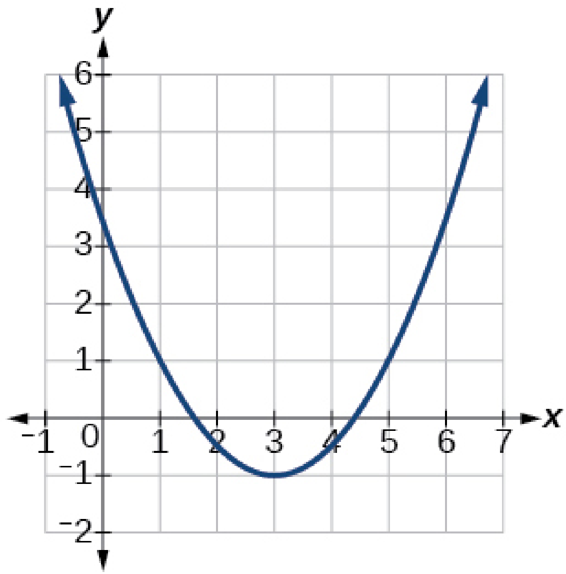
Exercise:

Problem:



Exercise:

Problem:

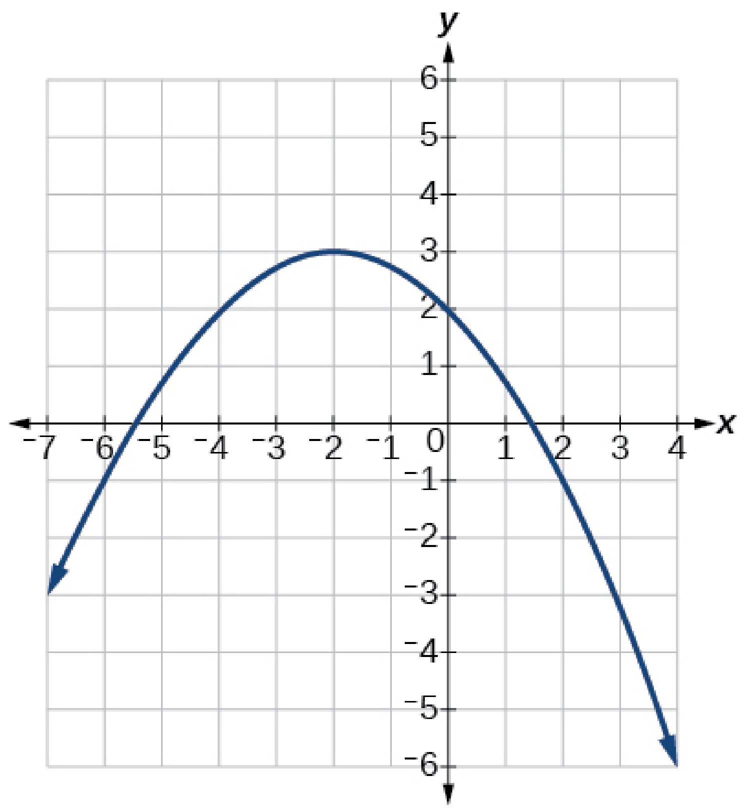


Solution:

$f(x) = \frac{1}{2}x^2 - 3x + \frac{7}{2}$

Exercise:

Problem:



Numeric

For the following exercises, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

Exercise:

Problem:

x	–2	–1	0	1	2
y	5	2	1	2	5

Solution:

$f(x)=x^2+1$

Exercise:

Problem:

x	–2	–1	0	1	2
y	1	0	1	4	9

Exercise:

Problem:

x	–2	–1	0	1	2
y	–2	1	2	1	–2

Solution:

$f(x)=2-x^2$

Exercise:

Problem:

x	–2	–1	0	1	2
y	–8	–3	0	1	0

Exercise:

Problem:

x	–2	–1	0	1	2
y	8	2	0	2	8

Solution:

$f(x) = 2x^2$

Technology

For the following exercises, use a calculator to find the answer.

Exercise:

Problem:

Graph on the same set of axes the functions $f(x) = x^2, f(x) = 2x^2,$ and $f(x) = \frac{1}{3}x^2.$

What appears to be the effect of changing the coefficient?

Exercise:

Problem:

Graph on the same set of axes $f(x) = x^2, f(x) = x^2 + 2$ and $f(x) = x^2, f(x) = x^2 + 5$ and $f(x) = x^2 - 3.$ What appears to be the effect of adding a constant?

Solution:

The graph is shifted up or down (a vertical shift).

Exercise:

Problem:

Graph on the same set of axes $f(x) = x^2, f(x) = (x-2)^2, f(x-3)^2,$ and $f(x) = (x+4)^2.$

What appears to be the effect of adding or subtracting those numbers?

Exercise:

Problem:

The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x) = \frac{-32}{(80)^2}x^2 + x$ where x is the horizontal distance traveled and $h\left(x\right)$ is the height in feet. Use the TRACE feature of your calculator to determine the height of the object when it has traveled 100 feet away horizontally.

Solution:

50 feet

Exercise:

Problem:

A suspension bridge can be modeled by the quadratic function $h(x) = .0001x^2$ with $-2000 \leq x \leq 2000$ where $|x|$ is the number of feet from the center and $h\left(x\right)$ is height in feet. Use the TRACE feature of your calculator to estimate how far from the center does the bridge have a height of 100 feet.

Extensions

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

Exercise:

Problem: Vertex $(1,-2)$, opens up.

Solution:

Domain is $(-\infty,\infty)$. Range is $[-2,\infty)$.

Exercise:

Problem: Vertex $(-1,2)$ opens down.

Exercise:

Problem: Vertex $(-5,11)$, opens down.

Solution:

Domain is $(-\infty,\infty)$ Range is $(-\infty,11]$.

Exercise:

Problem: Vertex $(-100,100)$, opens up.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

Exercise:

Problem:

Contains $(1,1)$ and has shape of $f(x)=2x^2$. Vertex is on the y-axis.

Solution:

$f(x)=2x^2-1$

Exercise:

Problem:

Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the y-axis.

Exercise:

Problem:

Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$. Vertex is on the y-axis.

Solution:

$f(x) = 3x^2 - 9$

Exercise:

Problem:

Contains $(1, -3)$ and has the shape of $f(x) = -x^2$. Vertex is on the y-axis.

Exercise:

Problem:

Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$. Vertex is on the y-axis.

Solution:

$f(x) = 5x^2 - 77$

Exercise:

Problem:

Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has x-coordinate of -1.

Real-World Applications

Exercise:

Problem:

Find the dimensions of the rectangular corral producing the greatest enclosed area given 200 feet of fencing.

Solution:

50 feet by 50 feet. Maximize $f(x) = -x^2 + 100x$.

Exercise:

Problem:

Find the dimensions of the rectangular corral split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.

Exercise:

Problem:

Find the dimensions of the rectangular corral producing the greatest enclosed area split into 3 pens of the same size given 500 feet of fencing.

Solution:

125 feet by 62.5 feet. Maximize $f(x) = -2x^2 + 250x$.

Exercise:

Problem:

Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?

Exercise:

Problem:

Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?

Solution:

6 and $-6;$ product is –36; maximize $f(x) = x^2 + 12x$.

Exercise:

Problem:

Suppose that the price per unit in dollars of a cell phone production is modeled by $p = \$45 - 0.0125x,$ where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p.$ Find the production level that will maximize revenue.

Exercise:

Problem:

A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h\left(t\right) = -4.9t^2 + 229t + 234.$ Find the maximum height the rocket attains.

Solution:

2909.56 meters

Exercise:

Problem:

A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h\left(t\right) = -4.9t^2 + 24t + 8.$ How long does it take to reach maximum height?

Exercise:

Problem:

A soccer stadium holds 62,000 spectators. With a ticket price of \$11, the average attendance has been 26,000. When the price dropped to \$9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

Solution:

\$10.70

Exercise:

Problem:

A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

Glossary

axis of symmetry
: a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

general form of a quadratic function
: the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where $a, b,$ and c are real numbers and $a \neq 0$.

standard form of a quadratic function
: the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where $(h,\ k)$ is the vertex.

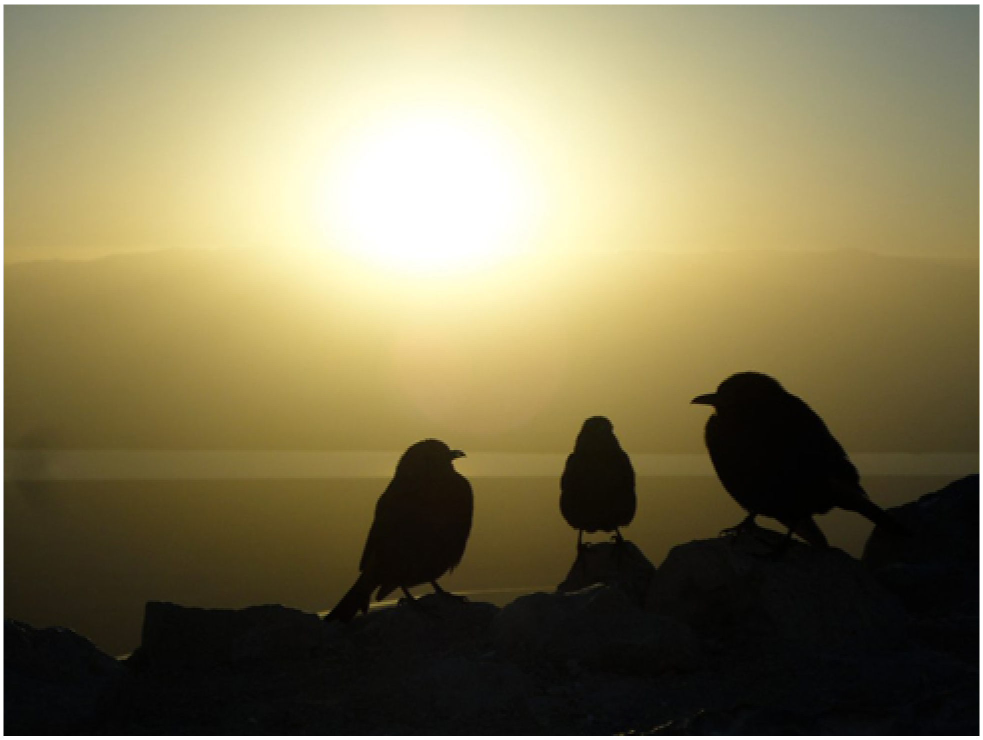
vertex
: the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

vertex form of a quadratic function
: another name for the standard form of a quadratic function

zeros
: in a given function, the values of x at which $y = 0$, also called roots

Power Functions and Polynomial Functions
In this section, you will:

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.



(credit: Jason Bay, Flickr)

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in [link].

Year					

	2009	2010	2011	2012	2013
Bird Population	800	897	992	$1,083$	$1,169$

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800,$ where $P(t)$ represents the bird population on the island t years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A **power function** is a function with a single term that is the product of a real number, a **coefficient,** and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius r is
Equation:

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius r is
Equation:

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi,$ multiplied by a variable r raised to a power.

Note:
Power Function
A **power function** is a function that can be represented in the form
Equation:

$$f(x) = kx^p$$

where k and p are real numbers, and k is known as the **coefficient**.

Note:
Is $f(x) = 2^x$ a power function?
No. A power function contains a variable base raised to a fixed power. This function has a constant base raised to a variable power. This is called an exponential function, not a power function.

Example:
Exercise:

Problem:
Identifying Power Functions

Which of the following functions are power functions?

$$
\begin{array}{ll}
f(x) = 1 & \text{Constant function} \\
f(x) = x & \text{Identify function} \\
f(x) = x^2 & \text{Quadratic function} \\
f(x) = x^3 & \text{Cubic function} \\
f(x) = \frac{1}{x} & \text{Reciprocal function} \\
f(x) = \frac{1}{x^2} & \text{Reciprocal squared function} \\
f(x) = \sqrt{x} & \text{Square root function} \\
f(x) = \sqrt[3]{x} & \text{Cube root function}
\end{array}
$$

Solution:

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The reciprocal and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and cube root functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Note:
Exercise:

Problem: Which functions are power functions?

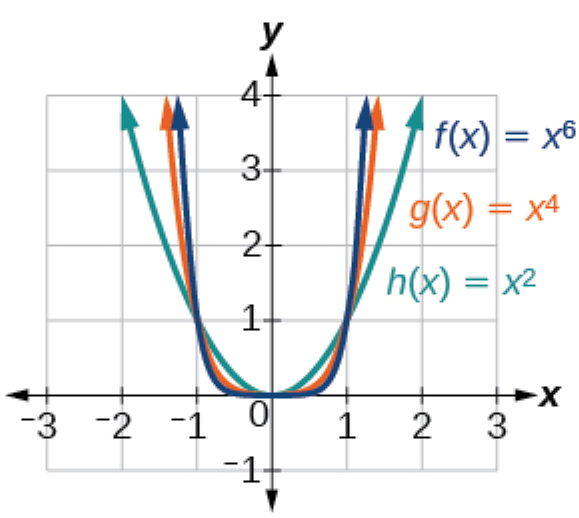
$f(x) = 2x^2 \cdot 4x^3$
$g(x) = -x^5 + 5x^3 - 4x$
$h(x) = \frac{2x^5 - 1}{3x^2 + 4}$

Solution:

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.

Identifying End Behavior of Power Functions

[link] shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.



Even-power functions

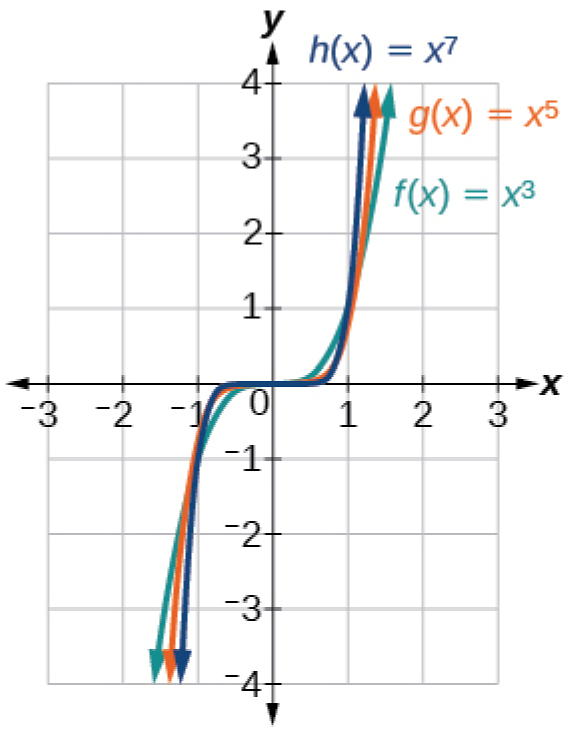
To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f\left(x\right)$ values increase without bound. In symbolic form, we could write

Equation:

$$\text{as } x \to \pm\infty, \ f(x) \to \infty$$

[link] shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.



Odd-power function

These examples illustrate that functions of the form $f\left(x\right)=x^{n}$ reveal symmetry of one kind or another. First, in [link] we see that even functions of the form $f\left(x\right)=x^{n}$, n even, are symmetric about the y-axis. In [link] we see that odd functions of the form $f\left(x\right)=x^{n}$, n odd, are symmetric about the origin.

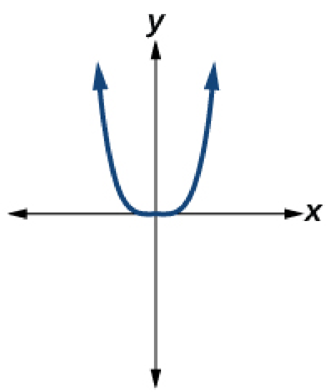
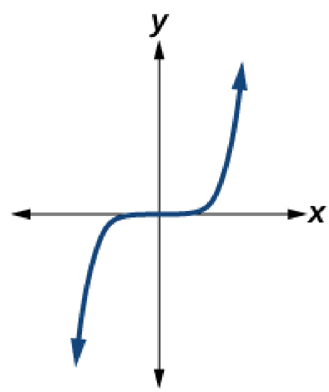
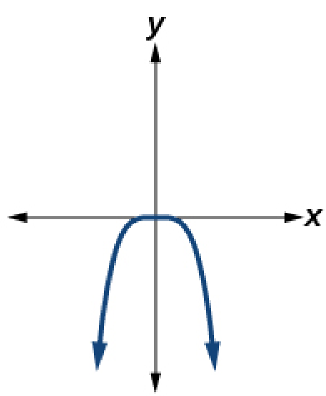
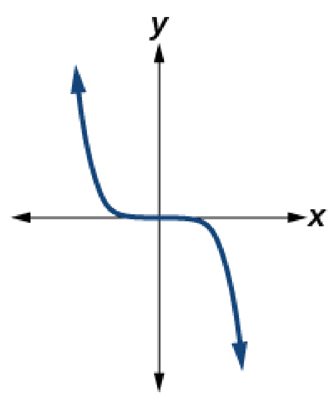
For these odd power functions, as x approaches negative infinity, $f\left(x\right)$ decreases without bound. As x approaches positive infinity, $f\left(x\right)$ increases without bound. In symbolic form we write

Equation:

$$\begin{array}{c}\text{as } x\to -\infty ,\text{ }f(x)\to -\infty \\ \text{as } x\to \infty ,\text{ }f(x)\to \infty \end{array}$$

The behavior of the graph of a function as the input values get very small ($x \to -\infty$) and get very large ($x \to \infty$) is referred to as the **end behavior** of the function. We can use words or symbols to describe end behavior.

[link] shows the end behavior of power functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power and the constant.

	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to \infty$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to -\infty$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to -\infty$

Note:
Given a power function $f(x) = kx^n$ where n is a non-negative integer, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use [link] to identify the end behavior.

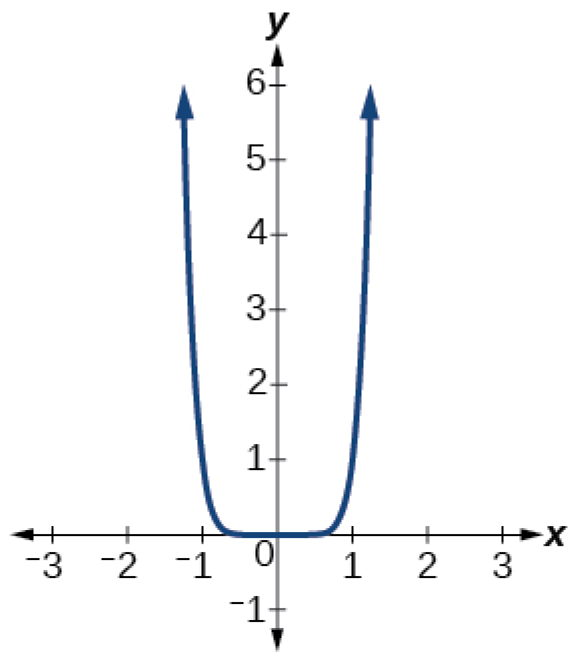
Example:
Exercise:

Problem:
Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution:

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. We write as $x \to \infty,\ f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty,\ f(x) \to \infty$. We can graphically represent the function as shown in [link].



Example:
Exercise:

Problem:
Identifying the End Behavior of a Power Function.

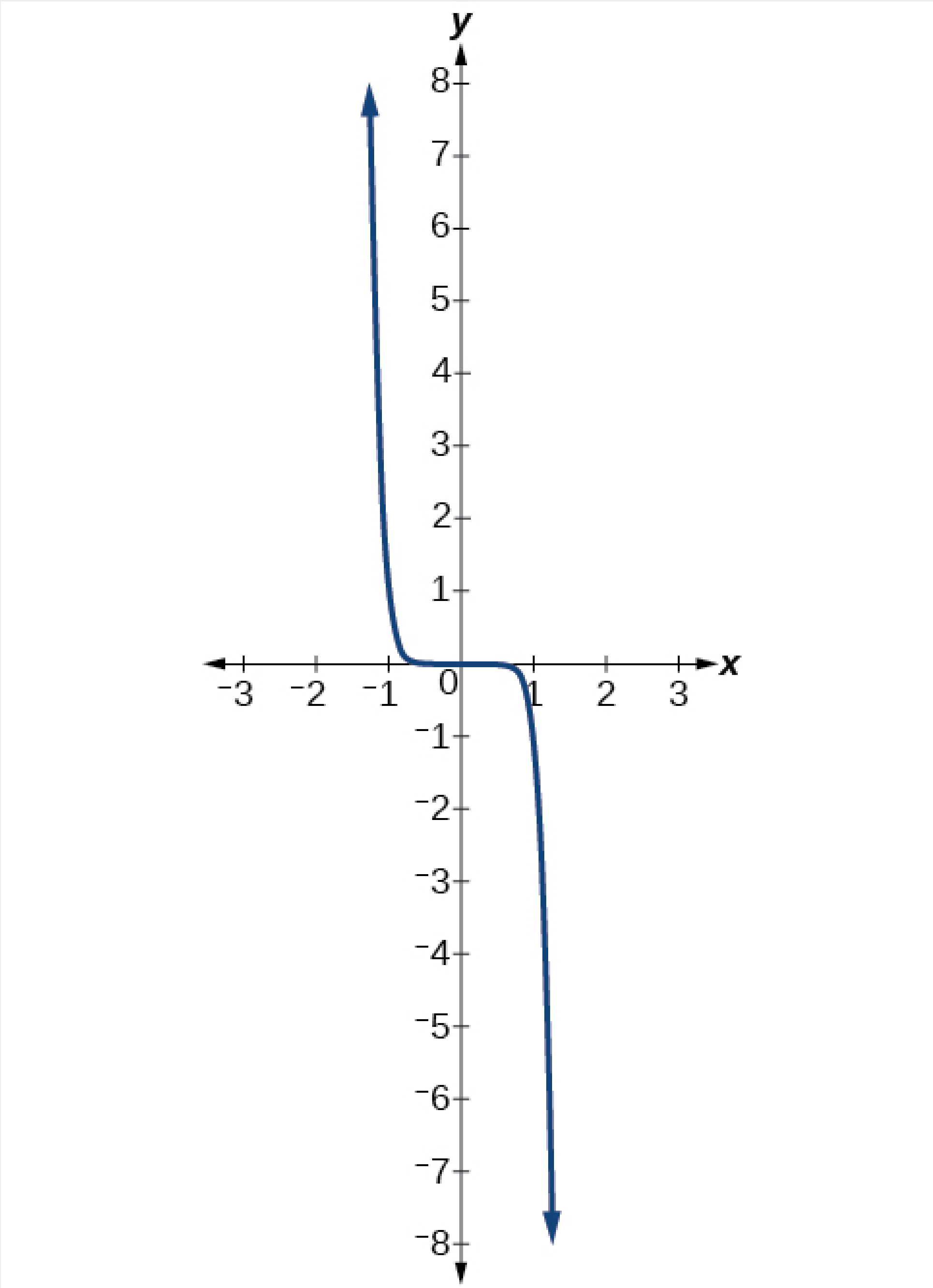
Describe the end behavior of the graph of $f(x) = -x^9$.

Solution:

The exponent of the power function is 9 (an odd number). Because the coefficient is -1 (negative), the graph is the reflection about the x-axis of the graph of $f(x) = x^9$. [link] shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

Equation:

$$\text{as } x \to -\infty, \ f(x) \to \infty$$
$$\text{as } x \to \infty, \ f(x) \to -\infty$$



Analysis

We can check our work by using the table feature on a graphing utility.

x	$f\left(x\right)$
–10	1,000,000,000
–5	1,953,125
0	0
5	–1,953,125
10	–1,000,000,000

We can see from [link] that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is very small (meaning that it is a very large negative value).

Note:
Exercise:

Problem:

Describe in words and symbols the end behavior of $f(x)=-5x^4$.

Solution:

As x approaches positive or negative infinity, $f\left(x\right)$ decreases without bound: as $x\to\pm\infty,\ f(x)\to-\infty$ because of the negative coefficient.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that

radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius r of the spill depends on the number of weeks w that have passed. This relationship is linear.

Equation:

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

Equation:

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

Equation:

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24 + 8w) \\ &= \pi(24 + 8w)^2 \end{aligned}$$

Multiplying gives the formula.

Equation:

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a **polynomial function**. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Note:
Polynomial Functions

Let n be a non-negative integer. A **polynomial function** is a function that can be written in the form

Equation:

$$f(x) = a_n x^n + a_{n\text{-}1} x^{n\text{-}1} + ... + a_2 x^2 + a_1 x + a_1$$

This is called the general form of a polynomial function. Each a_i is a coefficient and can be any real number, but a_n cannot $=$ 0. Each product $a_i x^i$ is a **term of a polynomial function**.

Example:

Exercise:

Problem:

Identifying Polynomial Functions

Which of the following are polynomial functions?

Equation:

$$f(x) = 2x^3 \cdot 3x + 4$$
$$g(x) = -x(x^2 - 4)$$
$$h(x) = 5\sqrt{x} + 2$$

Solution:

The first two functions are examples of polynomial functions because they can be written in the form $f(x) = a_n x^n + a_{n\text{-}1} x^{n\text{-}1} + ... + a_2 x^2 + a_1 x + a_1$ where the powers are non-negative integers and the coefficients are real numbers.

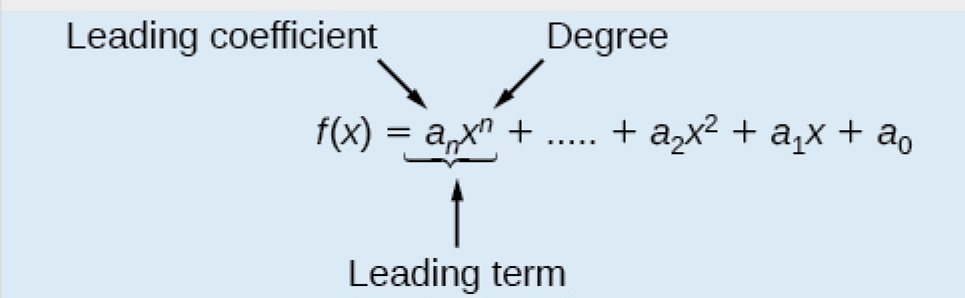
- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.

- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The **leading term** is the term containing the highest power of the variable, or the term with the highest degree. The **leading coefficient** is the coefficient of the leading term.

Note:
Terminology of Polynomial Functions
We often rearrange polynomials so that the powers are descending.



When a polynomial is written in this way, we say that it is in general form.

Note:

Given a polynomial function, identify the degree and leading coefficient.

1. Find the highest power of x to determine the degree function.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Example:
Exercise:

Problem:
Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.
Equation:

$$f(x) = 3 + 2x^2 - 4x^3$$
$$g(t) = 5t^5 - 2t^3 + 7t$$
$$h(p) = 6p - p^3 - 2$$

Solution:

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, -1.

Note:
Exercise:

Problem:

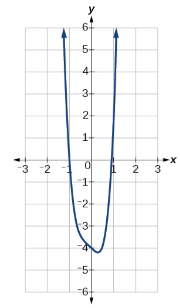
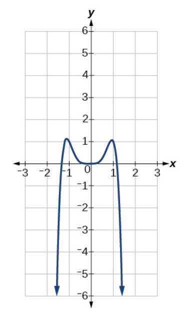
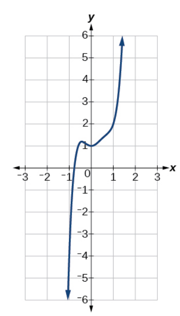
Identify the degree, leading term, and leading coefficient of the polynomial $f(x)=4x^2-x^6+2x-6$.

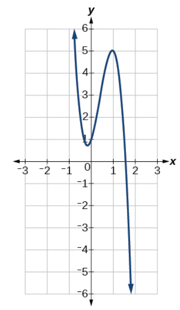
Solution:

The degree is 6. The leading term is $-x^6$. The leading coefficient is -1.

Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree. See [link].

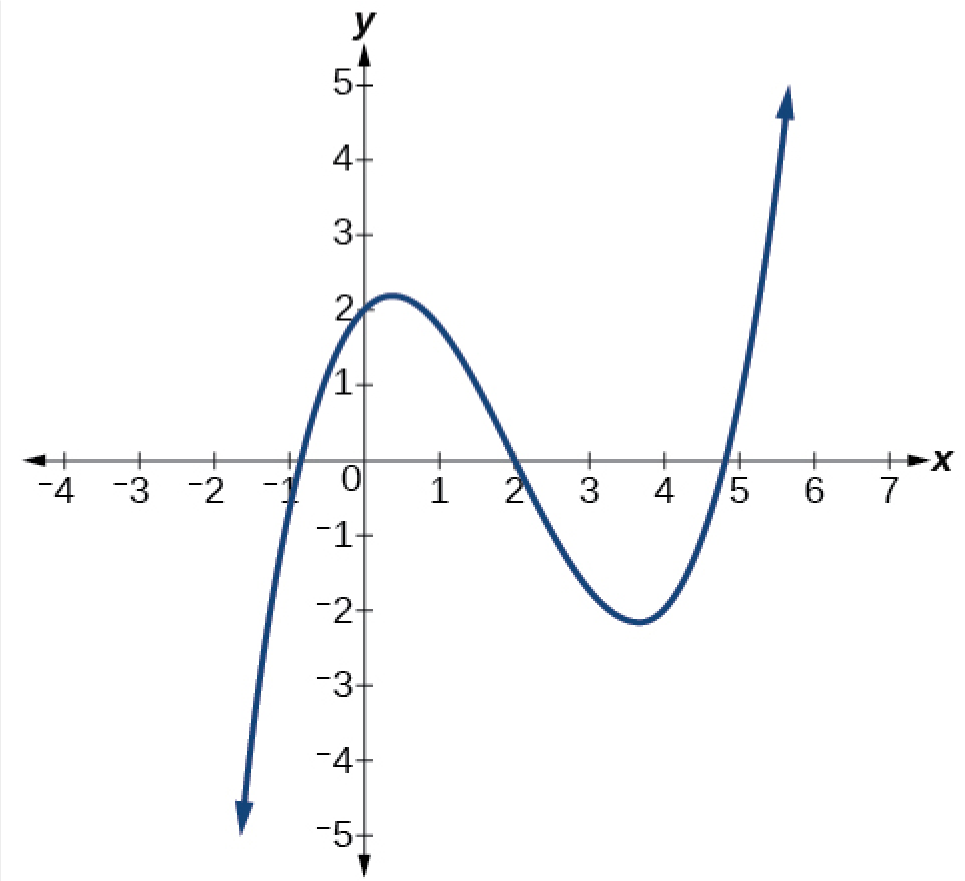
Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$	

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$	

Example:
Exercise:

Problem:
Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in [link].



Solution:

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

Equation:

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$
$$\text{as } x \to \infty,\ f(x) \to \infty$$

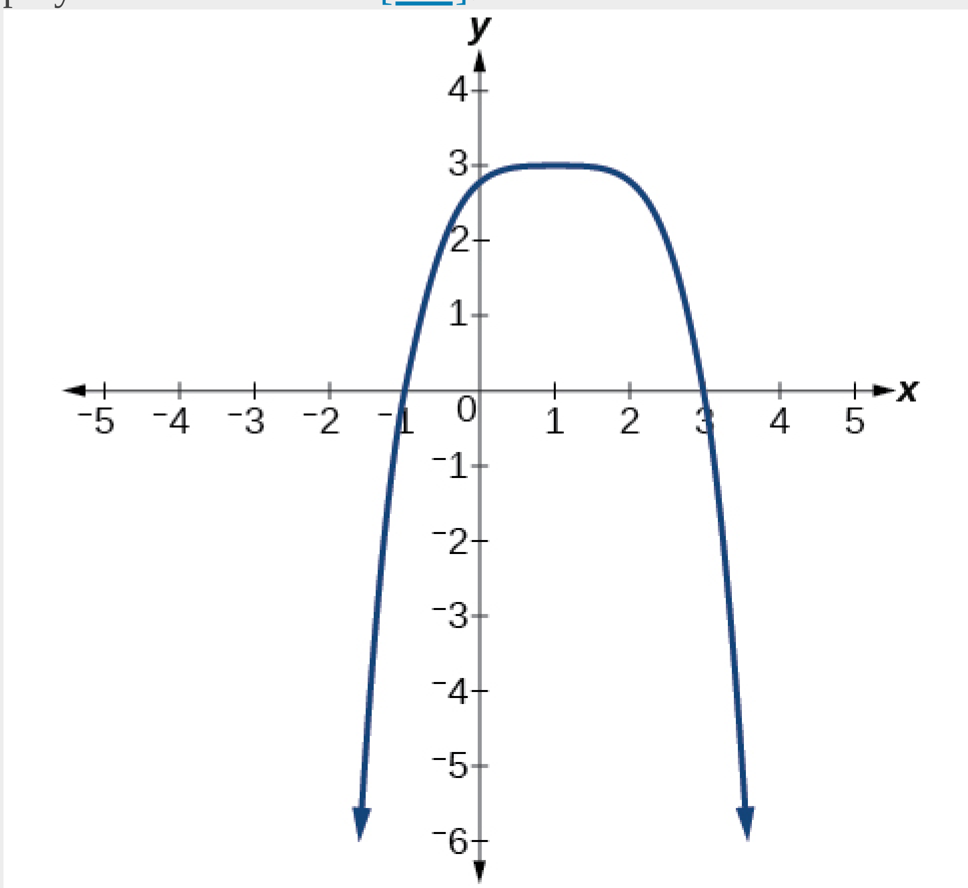
In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

Note:
Exercise:

Problem:

Describe the end behavior, and determine a possible degree of the polynomial function in [link].



Solution:

As $x \to \infty,\ f(x) \to -\infty;\ as\ x \to -\infty,\ f(x) \to -\infty.$ It has the shape of an even degree power function with a negative coefficient.

Example:
Exercise:

Problem:
Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4),$ express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution:

Obtain the general form by expanding the given expression for $f\,(x).$
Equation:

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4-9x^3+12x^2 \end{aligned}$$

The general form is $f\,(x) = -3x^4 - 9x^3 + 12x^2.$ The leading term is $-\,3x^4;$ therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is
Equation:

$$\begin{aligned} &\text{as } x \to -\infty,\ f(x) \to -\infty \\ &\text{as } x \to \infty,\ f(x) \to -\infty \end{aligned}$$

Note:
Exercise:

Problem:

Given the function $f(x) = 0.2(x-2)(x+1)(x-5),$ express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

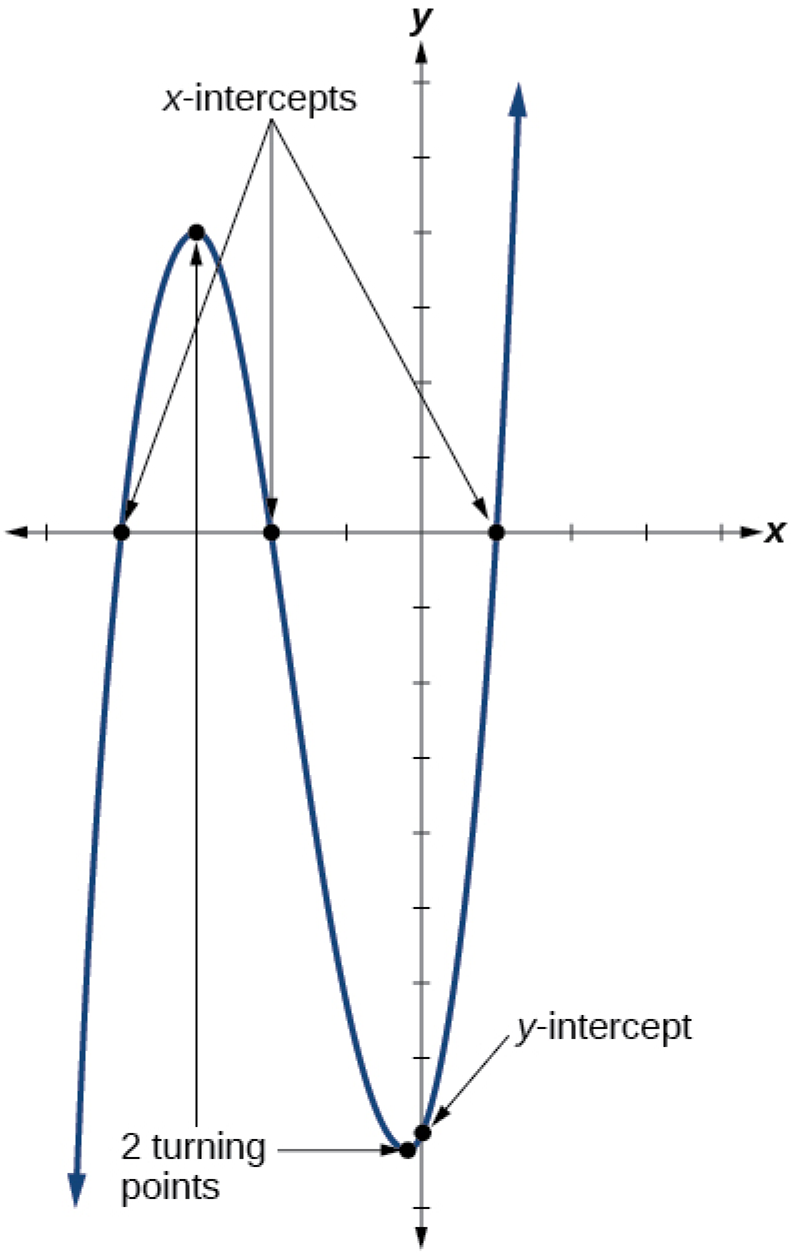
Solution:

The leading term is $0.2x^3,$ so it is a degree 3 polynomial. As x approaches positive infinity, $f\left(x\right)$ increases without bound; as x approaches negative infinity, $f\left(x\right)$ decreases without bound.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the *y*-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one *y*-intercept $\left(0,a_0\right).$ The *x*-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one *x*-intercept. See [link]**.**



Note:
Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The *y*-intercept is the point at which the function has an input value of zero. The x-intercepts are the points at which the output value is zero.

Note:
Given a polynomial function, determine the intercepts.

1. Determine the *y*-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the x-intercepts by solving for the input values that yield an output value of zero.

Example:
Exercise:

Problem:
Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x - 2)(x + 1)(x - 4)$, written in factored form for your convenience, determine the y-and x-intercepts.

Solution:

The *y*-intercept occurs when the input is zero so substitute 0 for x.
Equation:

$$\begin{aligned} f(0) &= (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

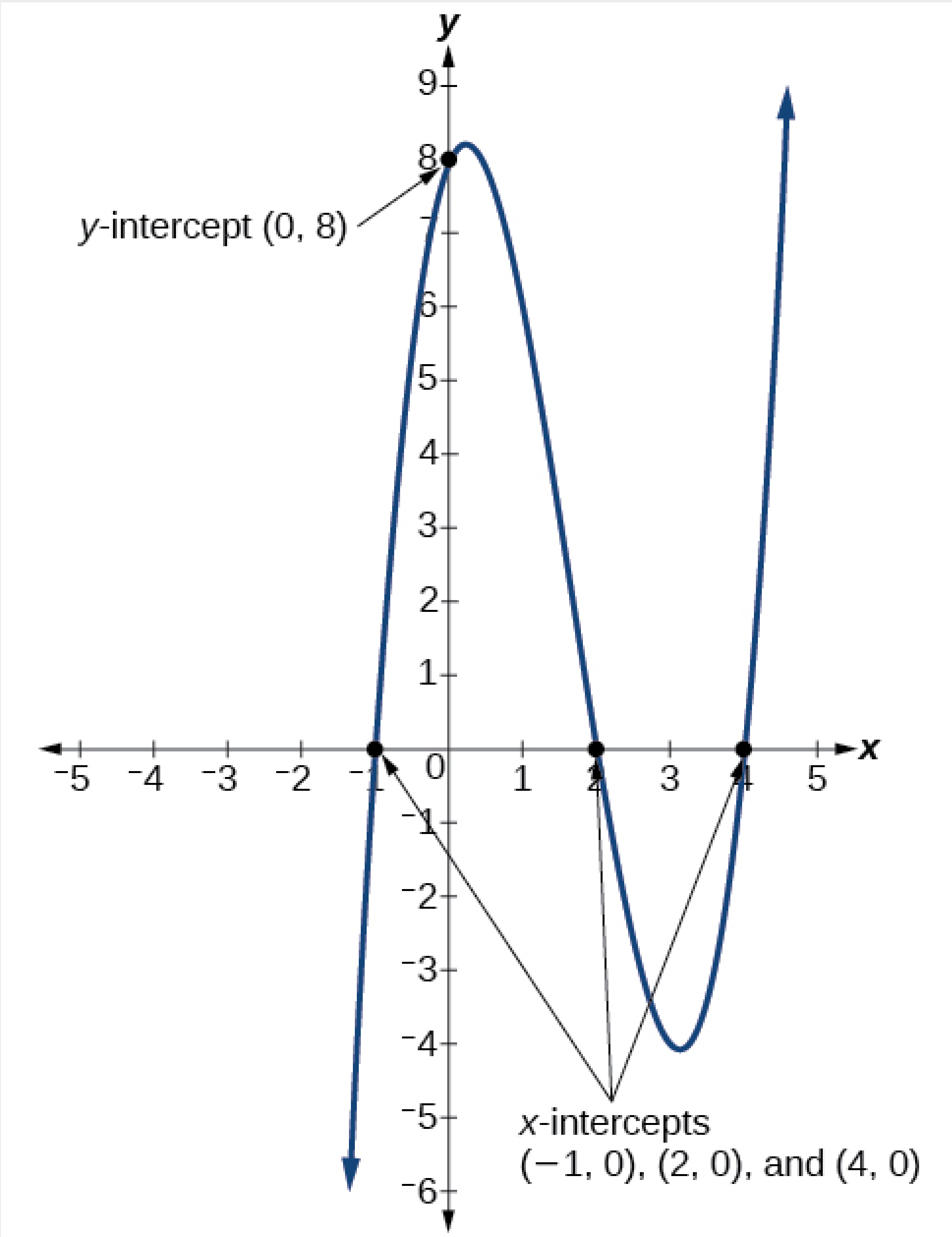
The *y*-intercept is (0, 8).

The *x*-intercepts occur when the output is zero.
Equation:

$$
\begin{array}{rcccc}
0 = (x-2)(x+1)(x-4) & & & & \\
x-2=0 & \text{or} & x+1=0 & \text{or} & x-4=0 \\
x=2 & \text{or} & x=-1 & \text{or} & x=4
\end{array}
$$

The x-intercepts are $(2, 0), (-1, 0),$ and $(4, 0)$.

We can see these intercepts on the graph of the function shown in [link].



Example:
Exercise:

Problem:
Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45,$ determine the y-and x-intercepts.

Solution:

The *y*-intercept occurs when the input is zero.
Equation:

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^2 - 45 \\ &= -45 \end{aligned}$$

The *y*-intercept is $(0, -45)$.

The *x*-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.
Equation:

$$\begin{aligned} f(x) &= x^4 - 4x^2 - 45 \\ &= (x^2 - 9)(x^2 + 5) \\ &= (x - 3)(x + 3)(x^2 + 5) \end{aligned}$$

Equation:

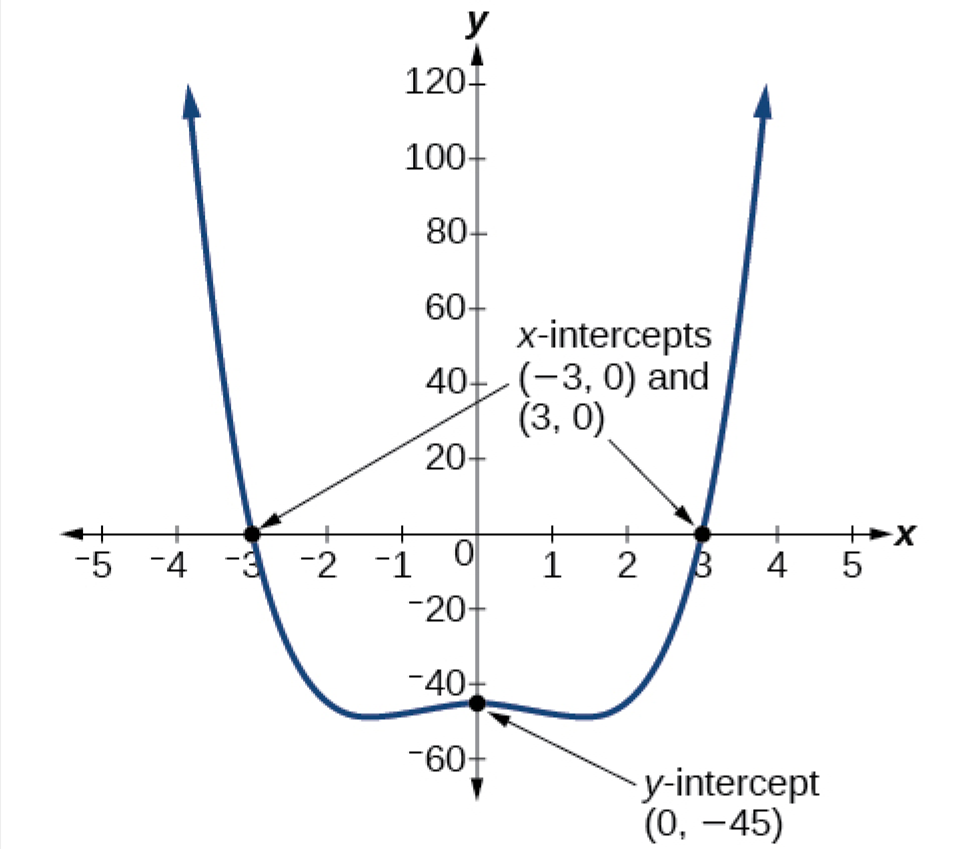
$$0 = (x - 3)(x + 3)(x^2 + 5)$$

Equation:

$$\begin{array}{lllll} x-3=0 & \text{or} & x+3=0 & \text{or} & x^2+5=0 \\ x=3 & \text{or} & x=-3 & \text{or} & \text{(no real solution)} \end{array}$$

The x-intercepts are $(3,0)$ and $(-3,0)$.

We can see these intercepts on the graph of the function shown in [link]. We can see that the function is even because $f(x)=f(-x)$.



Note:
Exercise:

Problem:

Given the polynomial function $f(x)=2x^3-6x^2-20x,$ determine the y-and x-intercepts.

Solution:

y-intercept $(0, 0)$; x-intercepts $(0, 0), (-2, 0),$ and$(5, 0)$

Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of x-intercepts and the number of turning points. A polynomial function of $n\text{th}$ degree is the product of n factors, so it will have at most n roots or zeros, or x-intercepts. The graph of the polynomial function of degree n must have at most $n-1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A **smooth curve** is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Note:
Intercepts and Turning Points of Polynomials
A polynomial of degree n will have, at most, n x-intercepts and $n-1$ turning points.

Example:
Exercise:

Problem:
Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of x-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

Solution:

The polynomial has a degree of 10, so there are at most 10 x-intercepts and at most $10 - 1 = 9$ turning points.

Note:
Exercise:

Problem:

Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$

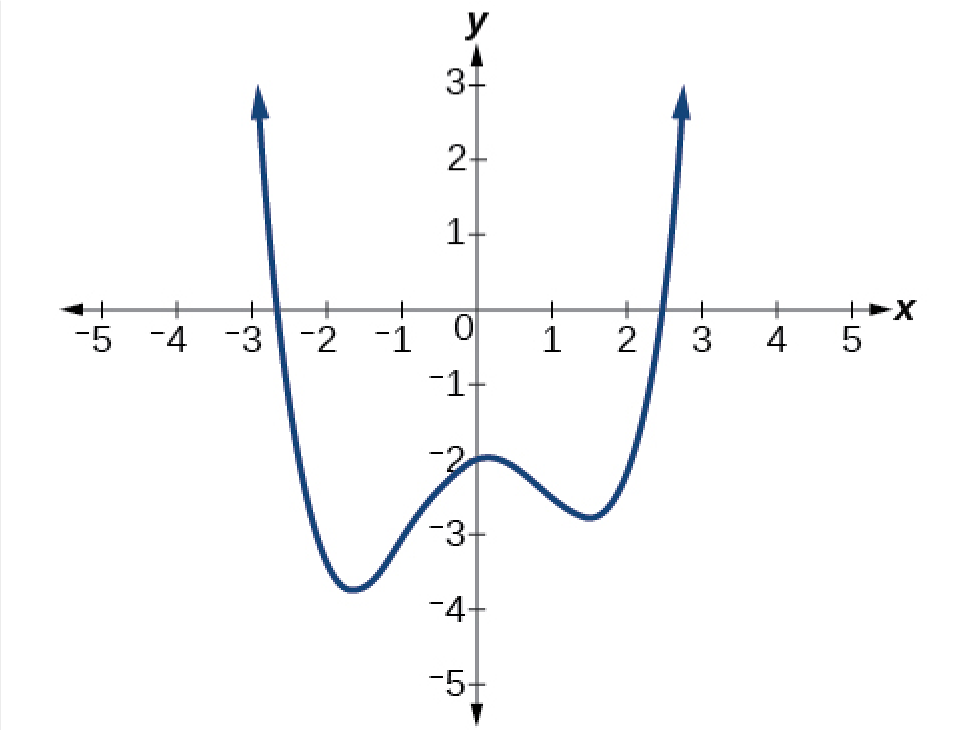
Solution:

There are at most 12 x-intercepts and at most 11 turning points.

Example:
Exercise:

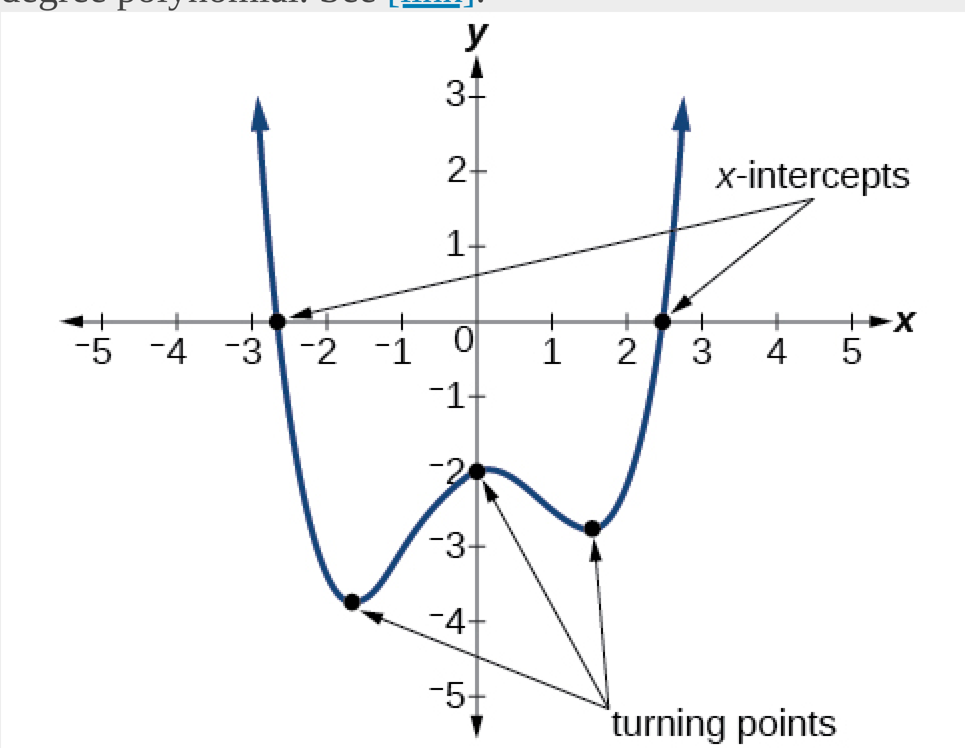
Problem:
Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in [link] based on its intercepts and turning points?



Solution:

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See [link].

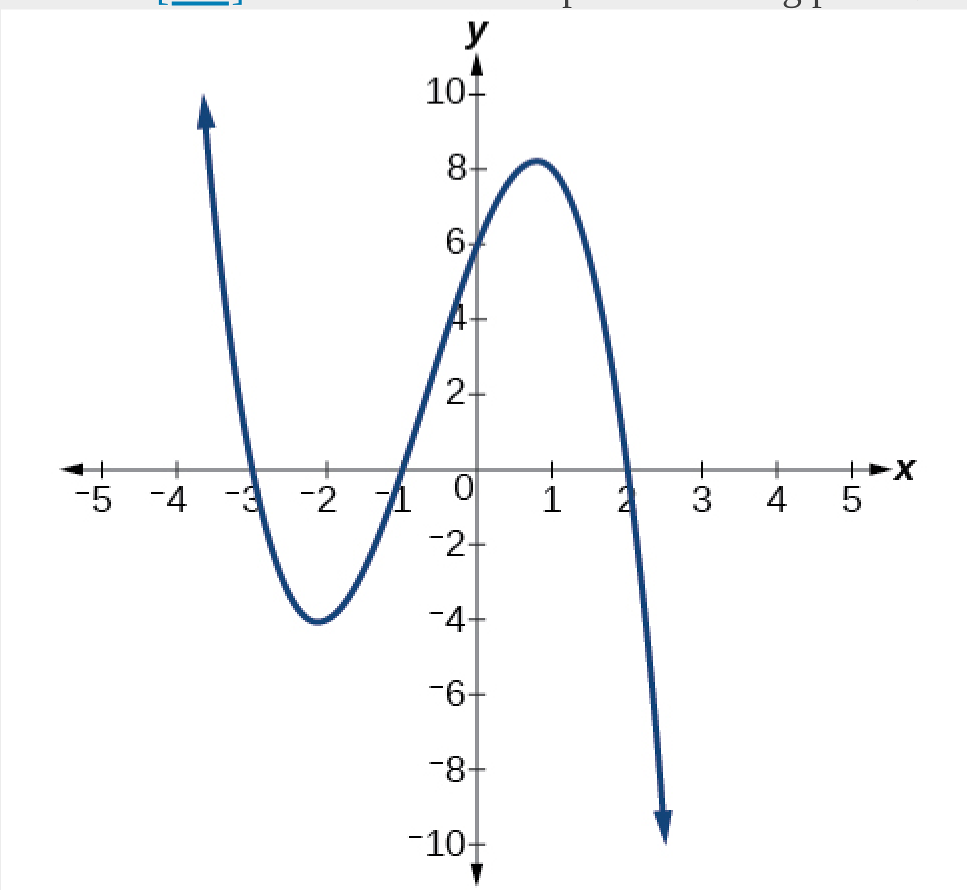


The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

Note:
Exercise:

Problem:

What can we conclude about the polynomial represented by the graph shown in [link] based on its intercepts and turning points?



Solution:

The end behavior indicates an odd-degree polynomial function; there are 3 x-intercepts and 2 turning points, so the degree is odd and at

least 3. Because of the end behavior, we know that the lead coefficient must be negative.

Example:
Exercise:

Problem:
Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x\,(x+3)\,(x-4),$ determine the local behavior.

Solution:

The y-intercept is found by evaluating $f(0)$.
Equation:

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The y-intercept is $(0,0)$.

The x-intercepts are found by determining the zeros of the function.
Equation:

$$\begin{array}{lllll} 0 = -4x(x+3)(x-4) & & & & \\ x = 0 & \text{or} & x+3=0 & \text{or} & x-4=0 \\ x = 0 & \text{or} & x=-3 & \text{or} & x=4 \end{array}$$

The x-intercepts are $(0,0), (-3,0),$ and $(4,0)$.

The degree is 3 so the graph has at most 2 turning points.

Note:
Exercise:

Problem:

Given the function $f(x) = 0.2(x-2)(x+1)(x-5),$ determine the local behavior.

Solution:

The x-intercepts are $(2, 0)$, $(-1, 0)$,and $(5, 0),$ the y-intercept is $(0, 2),$ and the graph has at most 2 turning points.

Note:
Access these online resources for additional instruction and practice with power and polynomial functions.

- Find Key Information about a Given Polynomial Function
- End Behavior of a Polynomial Function
- Turning Points and x-intercepts of Polynomial Functions
- Least Possible Degree of a Polynomial Function

Key Equations

general form of a polynomial function	$f(x) = a_n x^n + a_{n\text{-}1} x^{n\text{-}1} + \ldots + a_2 x^2 + a_1 x + a_1$

Key Concepts

- A power function is a variable base raised to a number power. See [link].
- The behavior of a graph as the input decreases beyond bound and increases beyond bound is called the end behavior.
- The end behavior depends on whether the power is even or odd. See [link] and [link].
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power. See [link].
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient. See [link].
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function. See [link] and [link].
- A polynomial of degree n will have at most n x-intercepts and at most $n-1$ turning points. See [link], [link], [link], [link], and [link].

Section Exercises

Verbal

Exercise:

Problem:

Explain the difference between the coefficient of a power function and its degree.

Solution:

The coefficient of the power function is the real number that is multiplied by the variable raised to a power. The degree is the highest

power appearing in the function.

Exercise:

Problem:

If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?

Exercise:

Problem:

In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.

Solution:

As x decreases without bound, so does $f\left(x\right).$ As x increases without bound, so does $f\left(x\right).$

Exercise:

Problem:

What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?

Exercise:

Problem:

What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x\to -\infty ,\ f(x)\to -\infty$ and as $x\to \infty ,\ f(x)\to -\infty .$

Solution:

The polynomial function is of even degree and leading coefficient is negative.

Algebraic

For the following exercises, identify the function as a power function, a polynomial function, or neither.

Exercise:

Problem: $f(x) = x^5$

Exercise:

Problem: $f(x) = \left(x^2\right)^3$

Solution:

Power function

Exercise:

Problem: $f(x) = x - x^4$

Exercise:

Problem: $f(x) = \frac{x^2}{x^2 - 1}$

Solution:

Neither

Exercise:

Problem: $f(x) = 2x\,(x+2)(x-1)^2$

Exercise:

Problem: $f(x) = 3^{x+1}$

Solution:

Neither

For the following exercises, find the degree and leading coefficient for the given polynomial.

Exercise:

Problem:$-3x^4$

Exercise:

Problem:$7-2x^2$

Solution:

Degree = 2, Coefficient = –2

Exercise:

Problem:$-2x^2-\ 3x^5+\ x-6$

Exercise:

Problem:$x\left(4-x^2\right)(2x+1)$

Solution:

Degree =4, Coefficient = –2

Exercise:

Problem:$x^2(2x-3)^2$

For the following exercises, determine the end behavior of the functions.

Exercise:

Problem:$f\left(x\right)=x^4$

Solution:

As $x \to \infty,\ f(x) \to \infty$, as $x \to -\infty,\ f(x) \to \infty$

Exercise:

Problem: $f\left(x\right) = x^3$

Exercise:

Problem: $f\left(x\right) = -x^4$

Solution:

As $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to -\infty$

Exercise:

Problem: $f\left(x\right) = -x^9$

Exercise:

Problem: $f(x) = -2x^4 -\ 3x^2 +\ x - 1$

Solution:

As $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to -\infty$

Exercise:

Problem: $f(x) = 3x^2 +\ x - 2$

Exercise:

Problem: $f(x) = x^2(2x^3 - x + 1)$

Solution:

As $x \to \infty,\ f(x) \to \infty$, as $x \to -\infty,\ f(x) \to -\infty$

Exercise:

Problem: $f(x) = (2 - x)^7$

For the following exercises, find the intercepts of the functions.

Exercise:

Problem: $f(t) = 2(t - 1)(t + 2)(t - 3)$

Solution:

y-intercept is $(0, 12)$, *t*-intercepts are $(1, 0)$; $(-2, 0)$; and $(3, 0)$.

Exercise:

Problem: $g(n) = -2(3n - 1)(2n + 1)$

Exercise:

Problem: $f(x) = x^4 - 16$

Solution:

y-intercept is $(0, -16)$. *x*-intercepts are $(2, 0)$ and $(-2, 0)$.

Exercise:

Problem: $f(x) = x^3 + 27$

Exercise:

Problem: $f(x) = x\left(x^2 - 2x - 8\right)$

Solution:

y-intercept is $(0, 0)$. *x*-intercepts are $(0, 0)$, $(4, 0)$, and $(-2,\ 0)$.

Exercise:

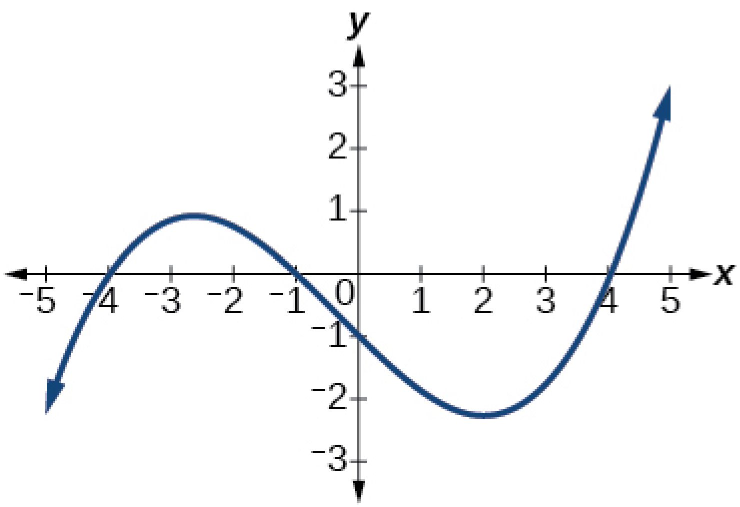
Problem: $f(x) = (x + 3)(4x^2 - 1)$

Graphical

For the following exercises, determine the least possible degree of the polynomial function shown.

Exercise:

Problem:

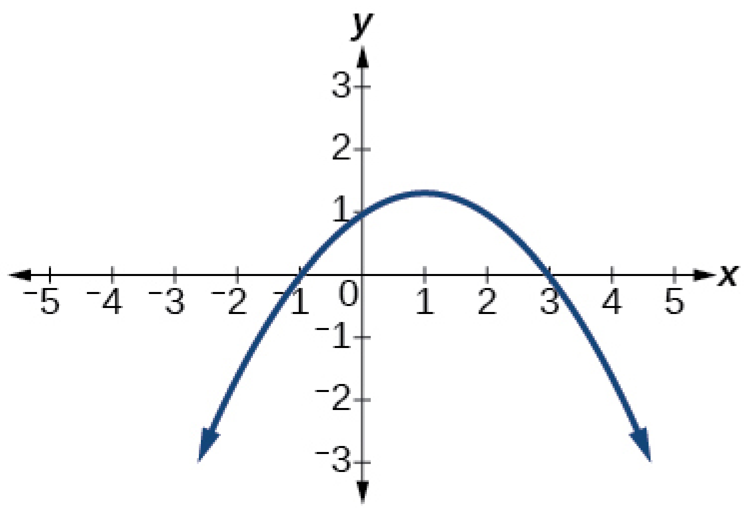


Solution:

3

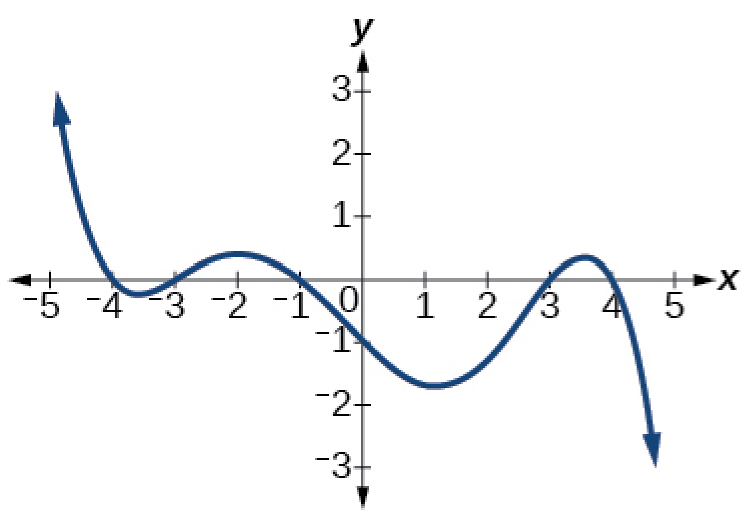
Exercise:

Problem:



Exercise:

Problem:

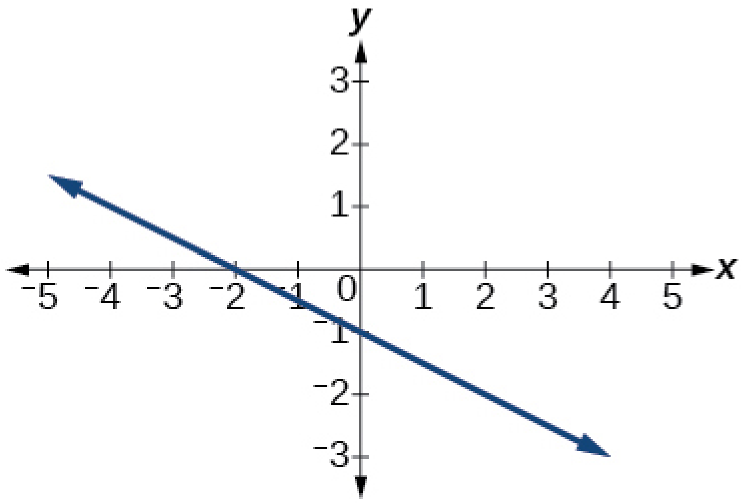


Solution:

5

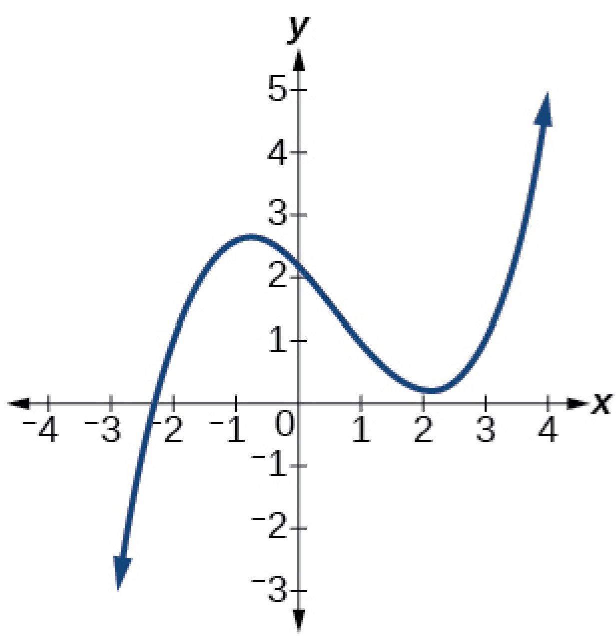
Exercise:

Problem:



Exercise:

Problem:

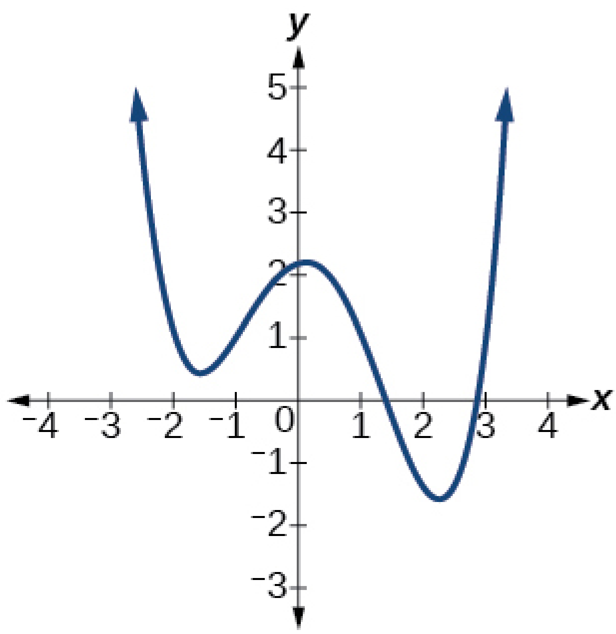


Solution:

3

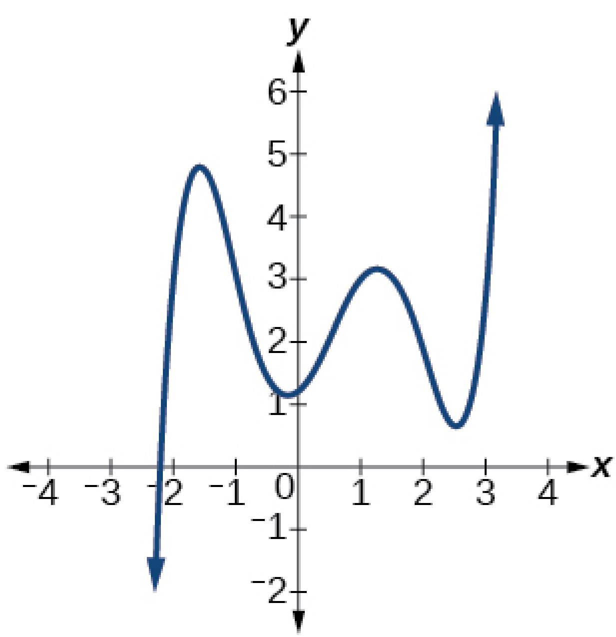
Exercise:

Problem:



Exercise:

Problem:

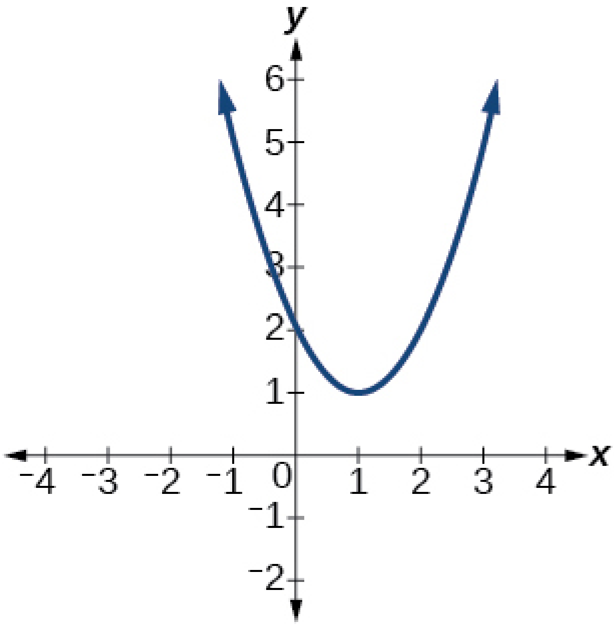


Solution:

5

Exercise:

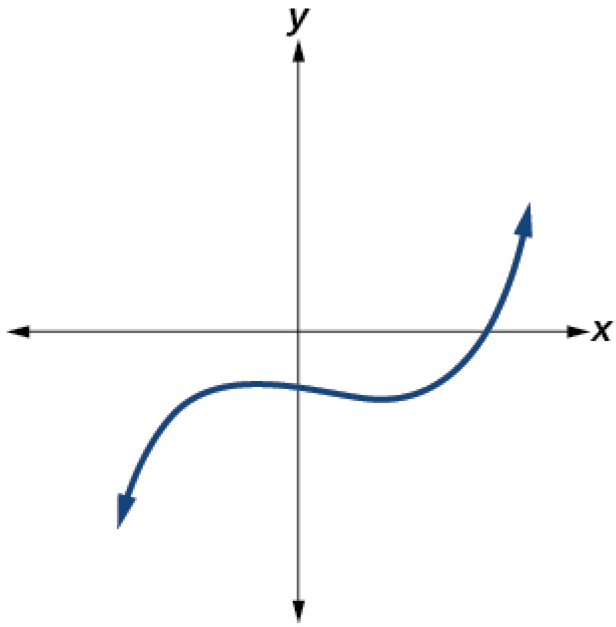
Problem:



For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

Exercise:

Problem:

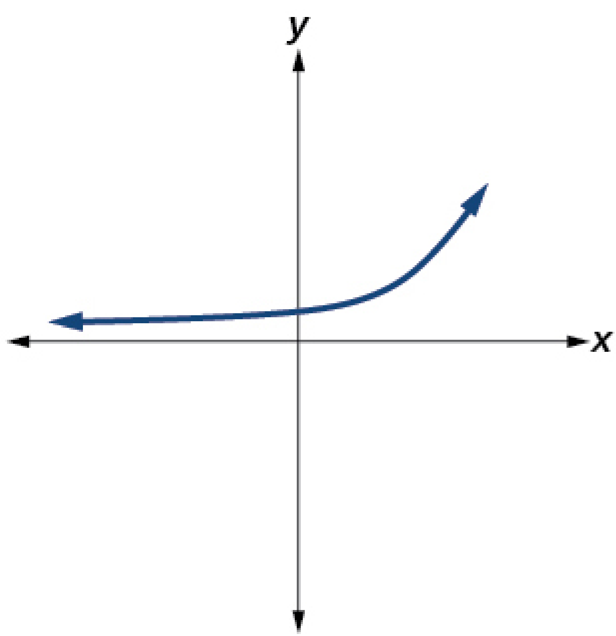


Solution:

Yes. Number of turning points is 2. Least possible degree is 3.

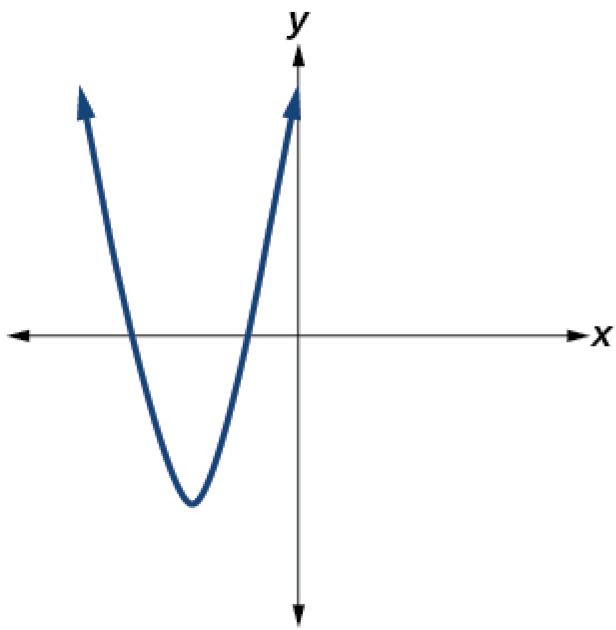
Exercise:

Problem:



Exercise:

Problem:

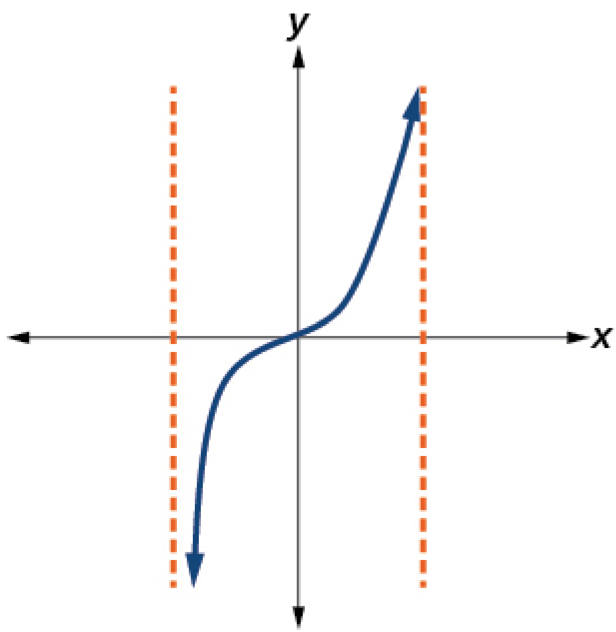


Solution:

Yes. Number of turning points is 1. Least possible degree is 2.

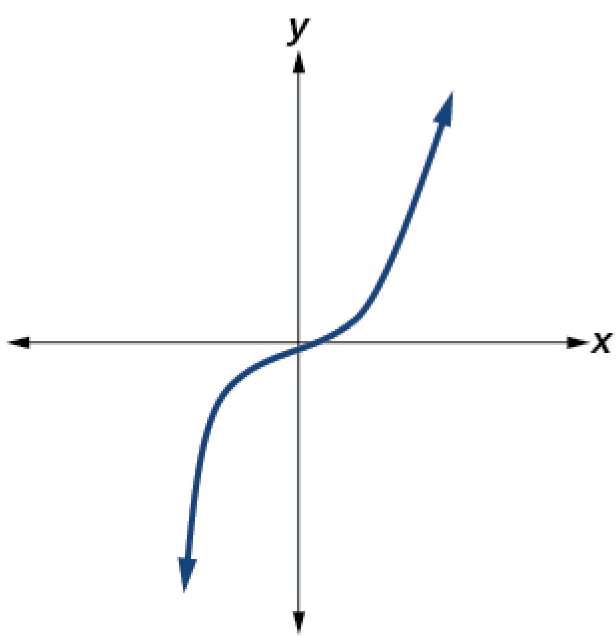
Exercise:

Problem:



Exercise:

Problem:

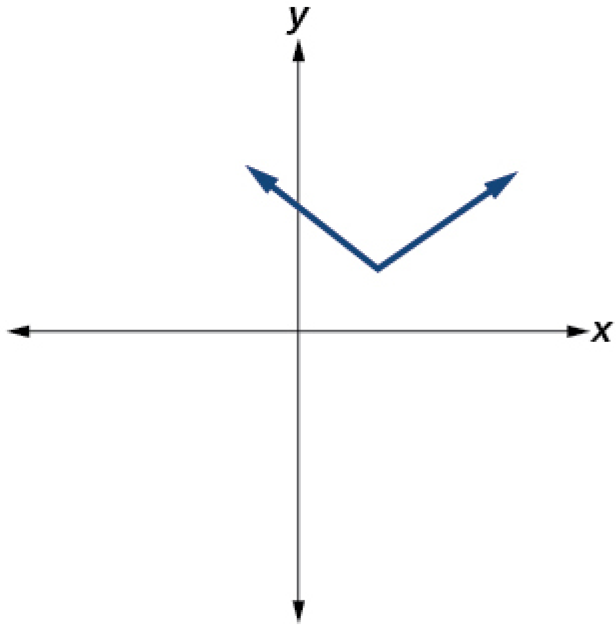


Solution:

Yes. Number of turning points is 0. Least possible degree is 1.

Exercise:

Problem:

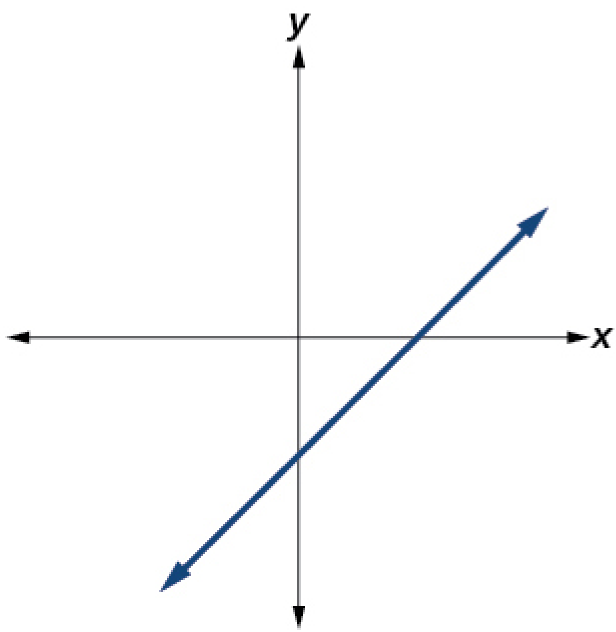


Solution:

No.

Exercise:

Problem:



Solution:

Yes. Number of turning points is 0. Least possible degree is 1.

Numeric

For the following exercises, make a table to confirm the end behavior of the function.

Exercise:

Problem: $f(x) = -x^3$

Exercise:

Problem: $f(x) = x^4 - 5x^2$

Solution:

x	$f\left(x\right)$
10	9,500
100	99,950,000
–10	9,500
–100	99,950,000

$\text{as}\,x\to -\infty,\ \ f(x)\to \infty,\,\text{as}\,x\to \infty,\ f(x)\to \infty$

Exercise:

Problem: $f(x)={x}^{2}{(1-x)}^{2}$

Exercise:

Problem: $f(x)=(x-1)(x-2)(3-x)$

Solution:

x	$f\left(x\right)$
10	–504
100	–941,094
–10	1,716

x	$f\left(x\right)$
–100	1,061,106

$\text{as}\,x\to -\infty,\ f(x)\to \infty,\,\text{as}\,x\to \infty,\ f(x)\to -\infty$

Exercise:

Problem: $f(x)=\frac{x^5}{10}-x^4$

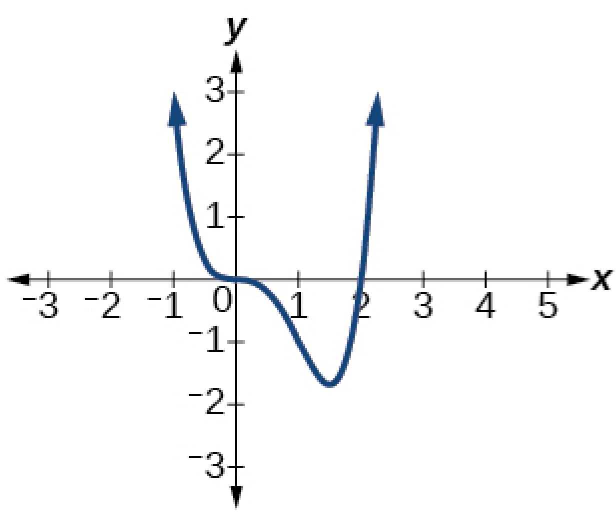
Technology

For the following exercises, graph the polynomial functions using a calculator. Based on the graph, determine the intercepts and the end behavior.

Exercise:

Problem: $f(x)=x^3(x-2)$

Solution:



The y-intercept is $(0,\ 0)$. The x-intercepts are $(0,\ 0),\ (2,\ 0)$.
As $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$

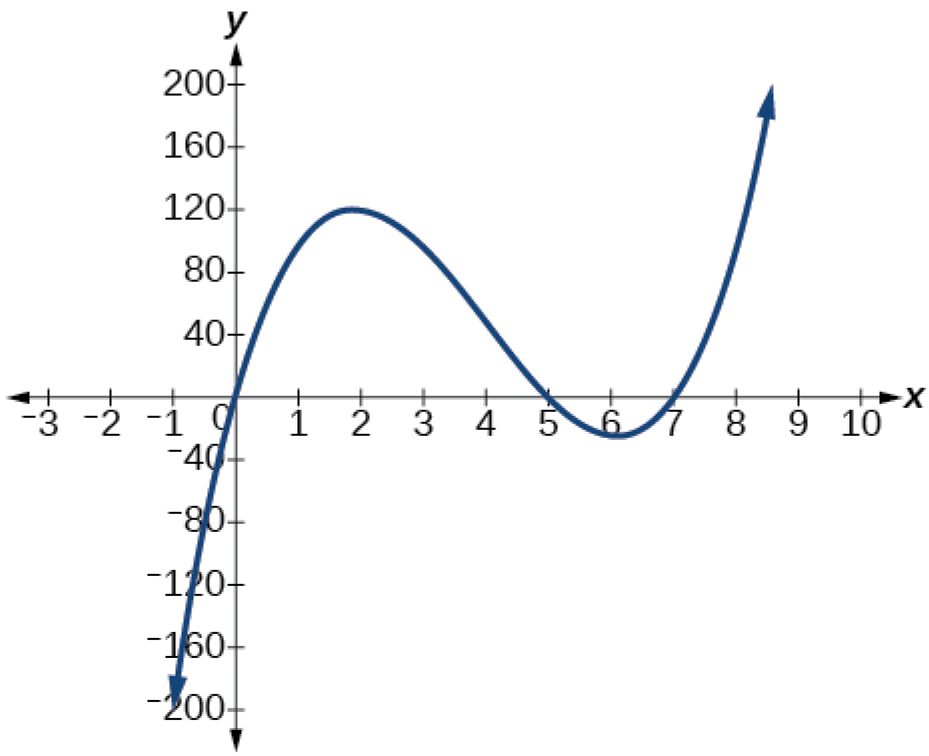
Exercise:

Problem: $f(x) = x(x-3)(x+3)$

Exercise:

Problem: $f(x) = x(14-2x)(10-2x)$

Solution:



The y-intercept is $(0, 0)$. The x-intercepts are $(0,\ 0),\ (5,\ 0),\ (7,\ 0)$.
As $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$

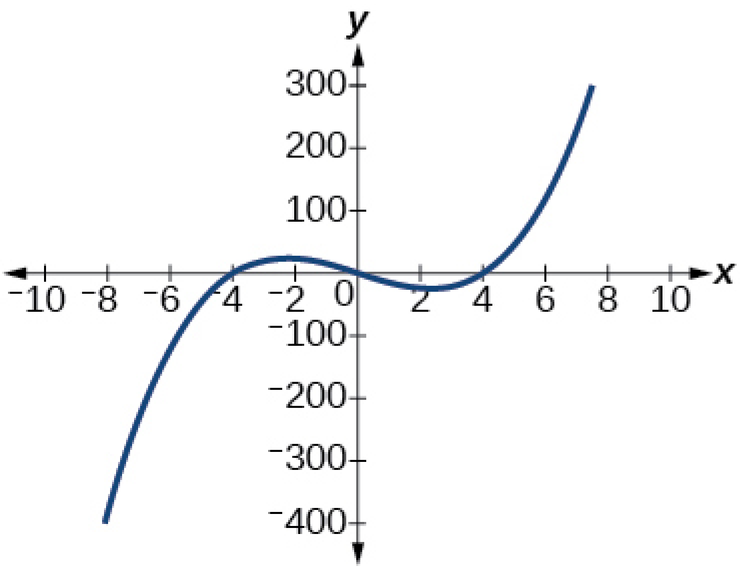
Exercise:

Problem: $f(x) = x(14-2x)(10-2x)^2$

Exercise:

Problem: $f(x) = x^3 - 16x$

Solution:



The y-intercept is $(0,\ 0)$. The x-intercept is $(-4,\ 0),\ (0,\ 0),\ (4,\ 0)$.
$As\, x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$

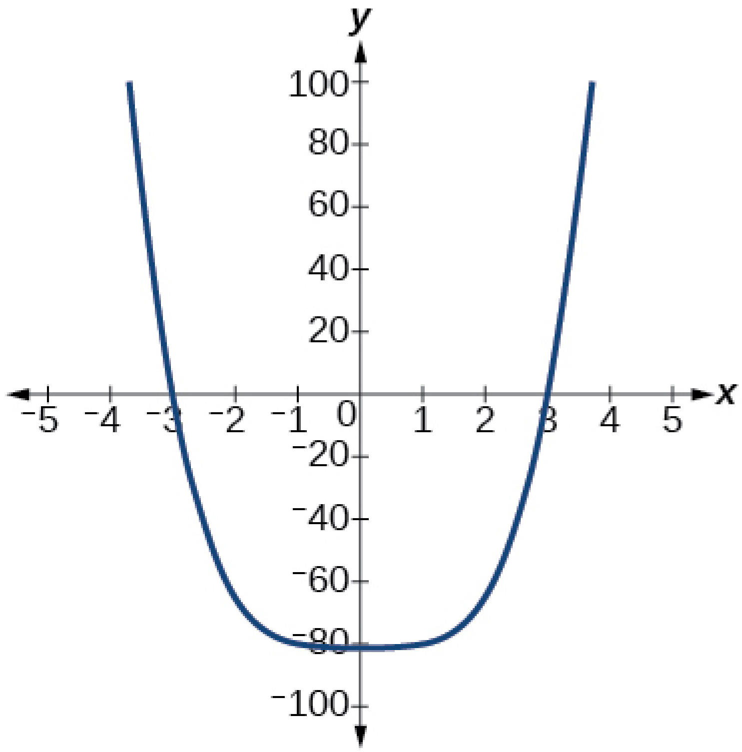
Exercise:

Problem: $f(x) = x^3 - 27$

Exercise:

Problem: $f(x) = x^4 - 81$

Solution:



The y-intercept is $(0,\ -81)$. The x-intercept are $(3,\ 0),\ (-3,\ 0)$.
As $x\to-\infty,\ f(x)\to\infty$, as $x\to\infty,\ f(x)\to\infty$

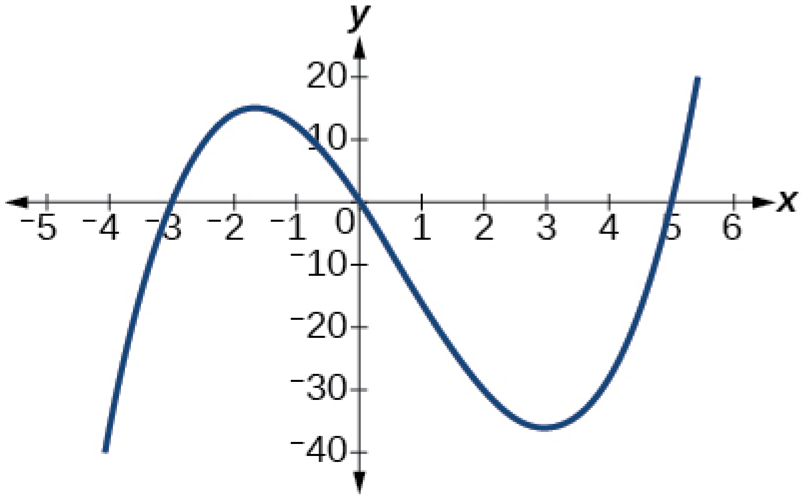
Exercise:

Problem: $f(x)=-x^3+x^2+2x$

Exercise:

Problem: $f(x)=x^3-2x^2-15x$

Solution:



The y-intercept is $(0,\ 0)$. The x-intercepts are $(-3,\ 0),\ (0,\ 0),\ (5,\ 0)$.
As $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$

Exercise:

Problem: $f(x) = x^3 - 0.01x$

Extensions

For the following exercises, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or –1. There may be more than one correct answer.

Exercise:

Problem:

The y-intercept is $(0, -4)$. The x-intercepts are $(-2, 0),\ (2, 0)$. Degree is 2.

End behavior: as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$.

Solution:

$f(x) = x^2 - 4$

Exercise:

Problem:

The y-intercept is $(0, 9)$. The x-intercepts are $(-3, 0)$, $(3, 0)$. Degree is 2.

End behavior:
as $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$.

Exercise:

Problem:

The y-intercept is $(0, 0)$. The x-intercepts are $(0, 0)$, $(2, 0)$. Degree is 3.

End behavior: as $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.

Solution:

$f(x) = x^3 - 4x^2 + 4x$

Exercise:

Problem:

The y-intercept is $(0, 1)$. The x-intercept is $(1, 0)$. Degree is 3.

End behavior: as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$.

Exercise:

Problem:

The y-intercept is $(0, 1)$. There is no x-intercept. Degree is 4.

End behavior: as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$.

Solution:

$$f(x) = x^4 + 1$$

Real-World Applications

For the following exercises, use the written statements to construct a polynomial function that represents the required information.

Exercise:

Problem:

An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of 20 meters per day. Express the area of the circle as a function of $d,$ the number of days elapsed.

Exercise:

Problem:

A cube has an edge of 3 feet. The edge is increasing at the rate of 2 feet per minute. Express the volume of the cube as a function of $m,$ the number of minutes elapsed.

Solution:

$$V(m) = 8m^3 + 36m^2 + 54m + 27$$

Exercise:

Problem:

A rectangle has a length of 10 inches and a width of 6 inches. If the length is increased by x inches and the width increased by twice that amount, express the area of the rectangle as a function of x.

Exercise:

Problem:

An open box is to be constructed by cutting out square corners of x-inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of x.

Solution:

$V(x) = 4x^3 - 32x^2 + 64x$

Exercise:

Problem:

A rectangle is twice as long as it is wide. Squares of side 2 feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width (x).

Glossary

coefficient
: a nonzero real number multiplied by a variable raised to an exponent

continuous function
: a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

degree
: the highest power of the variable that occurs in a polynomial

end behavior
: the behavior of the graph of a function as the input decreases without bound and increases without bound

leading coefficient
: the coefficient of the leading term

leading term

the term containing the highest power of the variable

polynomial function
a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

power function
a function that can be represented in the form $f(x) = kx^p$ where k is a constant, the base is a variable, and the exponent, p, is a constant

smooth curve
a graph with no sharp corners

term of a polynomial function
any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + a_{n\text{-}1} x^{n\text{-}1} + ... + a_2 x^2 + a_1 x + a_1$

turning point
the location at which the graph of a function changes direction

Dividing Polynomials

In this section, you will:

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m. [footnote] We can easily find the volume using elementary geometry.

National Park Service. "Lincoln Memorial Building Statistics." http://www.nps.gov/linc/historyculture/lincoln-memorial-building-statistics.htm. Accessed 4/3/2014

Equation:

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= 61.5 \cdot 40 \cdot 30 \\ &= 73{,}800 \end{aligned}$$

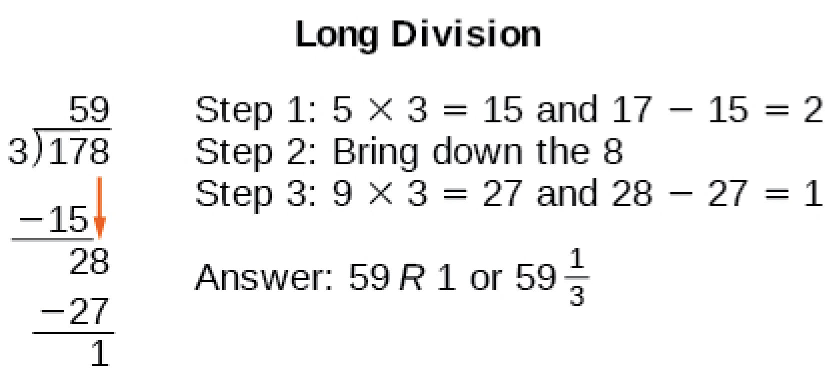
So the volume is 73,800 cubic meters $\left(\text{m}^3\right)$. Suppose we knew the volume, length, and width. We could divide to find the height.
Equation:

$$\begin{aligned} h &= \frac{V}{l \cdot w} \\ &= \frac{73{,}800}{61.5 \cdot 40} \\ &= 30 \end{aligned}$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$; the width is given by $x - 2$. To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

We are familiar with the long division algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let’s divide 178 by 3 using long division.



Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

Equation:

$$\begin{aligned} \text{dividend} &= (\text{divisor} \cdot \text{quotient}) + \text{remainder} \\ 178 &= (3 \cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this:

$x+2\ \overline{)\,2x^3-3x^2+4x+5}$	Set up the division problem.
$\begin{array}{r} 2x^2 \\ x+2\ \overline{)\,2x^3-3x^2+4x+5} \end{array}$	$2x^3$ divided by x is $2x^2$.
$\begin{array}{r} 2x^2 \\ x+2\ \overline{)\,2x^3-3x^2+4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+4x \end{array}$	Multiply $x+2$ by $2x^2$. Subtract.
$\begin{array}{r} 2x^2-7x \\ x+2\ \overline{)\,2x^3-3x^2+4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+4x \\ \underline{-(-7x^2+14x)} \\ 18x+5 \end{array}$	Bring down the next term. $-7x^2$ divided by x is $-7x$. Multiply $x+2$ by $-7x$. Subtract. Bring down the next term.
$\begin{array}{r} 2x^2-7x+18 \\ x+2\ \overline{)\,2x^3-3x^2+4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+4x \\ \underline{-(-7x^2+14x)} \\ 18x+5 \\ \underline{-18x+36} \\ -31 \end{array}$	$18x$ divided by x is 18. Multiply $x+2$ by 18. Subtract.

We have found
Equation:

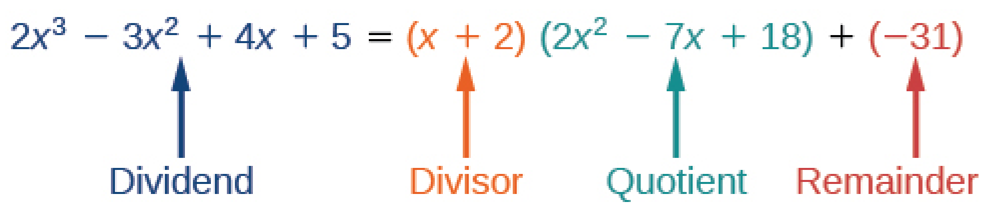
$$\frac{2x^3 - 3x^2 + 4x + 5}{x + 2} = 2x^2 - 7x + 18 - \frac{31}{x + 2}$$

or

Equation:

$$\frac{2x^3 - 3x^2 + 4x + 5}{x + 2} = (x + 2)(2x^2 - 7x + 18) - 31$$

We can identify the dividend, the divisor, the quotient, and the remainder.



Writing the result in this manner illustrates the Division Algorithm.

Note:
The Division Algorithm
The **Division Algorithm** states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

Equation:

$$f(x) = d(x)q(x) + r(x)$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.
If $r(x) = 0,$ then $d(x)$ divides evenly into $f(x).$ This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x).$

Note: Given a polynomial and a binomial, use long division to divide the polynomial by the binomial.

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom binomial from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

Example:
Exercise:

Problem:
Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

Solution:

$x+1\,\overline{)\,5x^2+3x-2}$	Set up division problem.
$\begin{array}{r} 5x \\ x+1\,\overline{)\,5x^2+3x-2} \end{array}$	$5x^2$ divided by x is $5x$.
$\begin{array}{r} 5x \\ x+1\,\overline{)\,5x^2+3x-2} \\ \underline{-(5x^2+5x)} \\ -2x-2 \end{array}$	Multiply $x+1$ by $5x$. Subtract.
$\begin{array}{r} 5x-2 \\ x+1\,\overline{)\,5x^2+3x-2} \\ \underline{-(5x^2+5x)} \\ -2x-2 \\ \underline{-(-2x-2)} \\ 0 \end{array}$	Bring down the next term. $-2x$ divided by x is -2. Multiply $x+1$ by -2. Subtract.

The quotient is $5x - 2$. The remainder is 0. We write the result as

Equation:

$$\frac{5x^2+3x-2}{x+1}=5x-2$$

or

Equation:

$$5x^2+3x-2=(x+1)(5x-2)$$

Analysis

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

Example:

Exercise:

Problem:

Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3+11x^2-31x+15$ by $3x-2$.

Solution:

$2x^2 + 5x - 7$	
$3x - 2 \overline{)\,6x^3 + 11x^2 - 31x + 15}$	$6x^3$ divided by $3x$ is $2x^2$.
$-(6x^3 - 4x^2)$	Multiply $3x - 2$ by $2x^2$.
$15x^2 - 31x$	Subtract. Bring down the next term. $15x^2$ divided by $3x$ is $5x$.
$-(15x^2 - 10x)$	Multiply $3x - 2$ by $5x$.
$-21x + 15$	Subtract. Bring down the next term. $-21x$ divided by $3x$ is -7.
$-(-21x + 14)$	Multiply $3x - 2$ by -7.
1	Subtract. The remainder is 1.

There is a remainder of 1. We can express the result as:

Equation:

$$\frac{6x^3+11x^2-31x+15}{3x-2}=2x^2+5x-7+\frac{1}{3x-2}$$

Analysis

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.
Equation:

$$(3x-2)(2x^2+5x-7)+1=6x^3+11x^2-31x+15$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is 1

Note:
Exercise:

Problem: Divide $16x^3-12x^2+20x-3$ by $4x+5$.

Solution:

$$4x^2-8x+15-\frac{78}{4x+5}$$

Using Synthetic Division to Divide Polynomials

As we’ve seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm.

The final form of the process looked like this:

$$
\begin{array}{r}
2x^2 + \quad x + 18 \\
x+2\overline{)\,2x^3 - 3x^2 + 4x + 5} \\
\underline{-(2x^3 + 4x^2)} \\
-7x^2 + 4x \\
\underline{-(-7x^2 - 14x)} \\
18x + 5 \\
\underline{-(18x + 36)} \\
-31
\end{array}
$$

There is a lot of repetition in the table. If we don’t write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$
\begin{array}{rrrrr}
2\,\big) & 2 & -3 & 4 & 5 \\
 & \underline{-2} & \underline{-4} & & \\
 & & \underline{-7} & \underline{14} & \\
 & & & \underline{18} & \underline{-36} \\
 & & & & -31
\end{array}
$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of

the “divisor” to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & -36 \\ \hline & 2 & -7 & 18 & -31 \end{array}$$

We then multiply it by the “divisor” and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2 – 7x + 18$ and the remainder is $–31.$ The process will be made more clear in [link].

Note:
Synthetic Division
Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x - k.$ In **synthetic division**, only the coefficients are used in the division process.

Note:
Given two polynomials, use synthetic division to divide.

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by $k.$ Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by $k.$ Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

Example:
Exercise:

Problem:
Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

Solution:

Begin by setting up the synthetic division. Write k and the coefficients.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & & \\ \hline \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by k.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by k. Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x + 12$. The remainder is 0. So $x - 3$ is a factor of the original polynomial.

Analysis

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x-3)(5x+12)+0=5x^2-3x-36$$

Example:
Exercise:

Problem:
Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3+10x^2-6x-20$ by $x+2$.

Solution:

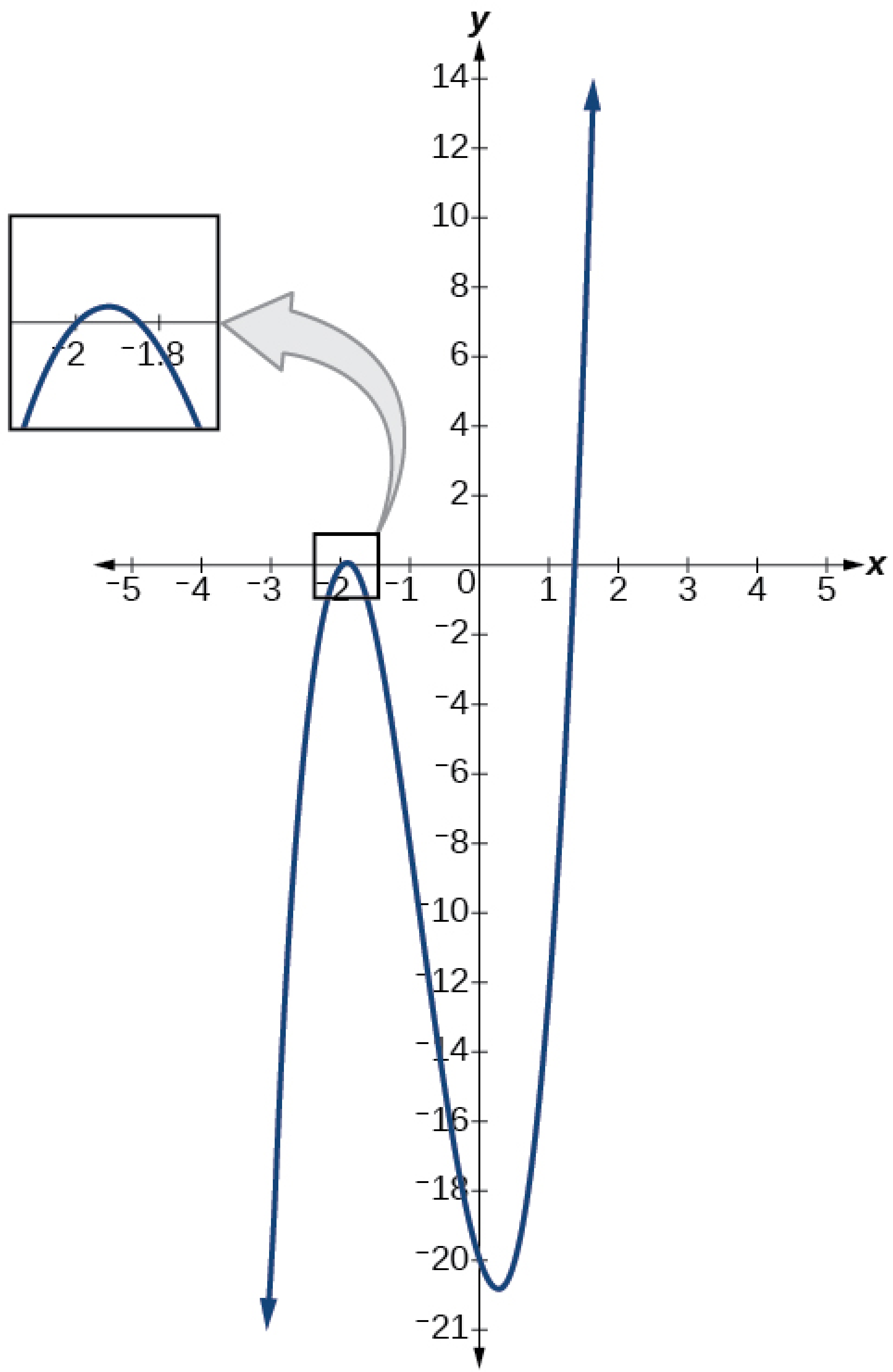
The binomial divisor is $x+2$ so $k=-2$. Add each column, multiply the result by –2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & 0 \end{array}$$

The result is $4x^2+2x-10$. The remainder is 0. Thus, $x+2$ is a factor of $4x^3+10x^2-6x-20$.

Analysis

The graph of the polynomial function $f(x)=4x^3+10x^2-6x-20$ in [link] shows a zero at $x=k=-2$. This confirms that $x+2$ is a factor of $4x^3+10x^2-6x-20$.



Example:
Exercise:

Problem:
Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4+10x^3+7x^2-6$ by $x-1$.

Solution:

Notice there is no *x*-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3+x^2+8x+8+\frac{2}{x-1}$.

Note:
Exercise:

Problem:

Use synthetic division to divide $3x^4+18x^3-3x+40$ by $x+7$.

Solution:

$3x^3-3x^2+21x-150+\frac{1{,}090}{x+7}$

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at

the beginning of this section. Now we will solve that problem in the following example.

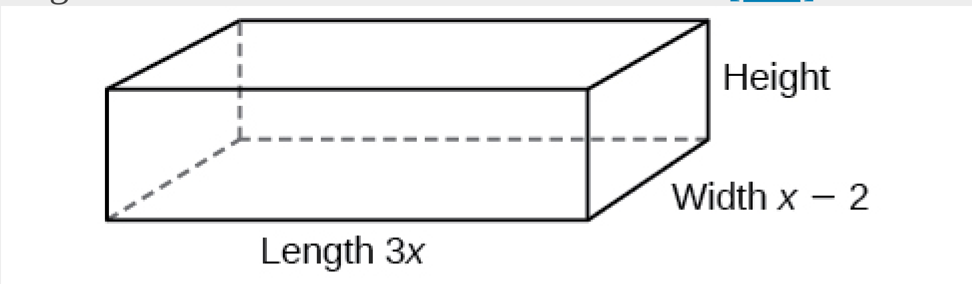
Example:
Exercise:

Problem:
Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x.$ The length of the solid is given by $3x$ and the width is given by $x - 2.$ Find the height of the solid.

Solution:

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in [link].



We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

Equation:

$$\begin{aligned} V &= l \cdot w \cdot h \\ 3x^4 - 3x^3 - 33x^2 + 54x &= 3x \cdot (x - 2) \cdot h \end{aligned}$$

To solve for $h,$ first divide both sides by $3x.$

Equation:

$$\frac{3x\cdot(x-2)\cdot h}{3x} = \frac{3x^4-3x^3-33x^2+54x}{3x}$$

$$(x-2)h = x^3 - x^2 - 11x + 18$$

Now solve for h using synthetic division.

Equation:

$$h = \frac{x^3 - x^2 - 11x + 18}{x - 2}$$

Equation:

$$\begin{array}{r|rrrr} & 1 & -1 & -11 & 18 \\ 2 & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The quotient is $x^2 + x - 9$ and the remainder is 0. The height of the solid is $x^2 + x - 9$.

Note:
Exercise:

Problem:

The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6.$ The width of the rectangle is given by $x + 6.$ Find an expression for the length of the rectangle.

Solution:

$3x^2 - 4x + 1$

Note:
Access these online resources for additional instruction and practice with polynomial division.

- Dividing a Trinomial by a Binomial Using Long Division
- Dividing a Polynomial by a Binomial Using Long Division
- Ex 2: Dividing a Polynomial by a Binomial Using Synthetic Division
- Ex 4: Dividing a Polynomial by a Binomial Using Synthetic Division

Key Equations

Division Algorithm	$f(x) = d(x)q(x) + r(x)$ where $q(x) \neq 0$

Key Concepts

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree. See [link] and [link].
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form $x - k$. See [link], [link], and [link].
- Polynomial division can be used to solve application problems, including area and volume. See [link].

Section Exercises

Verbal

Exercise:

Problem:

If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?

Solution:

The binomial is a factor of the polynomial.

Exercise:

Problem:

If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

Algebraic

For the following exercises, use long division to divide. Specify the quotient and the remainder.

Exercise:

Problem: $\left(x^2+5x-1\right) \div (x-1)$

Solution:

$x+6+\frac{5}{x-1}$, quotient: $x+6$, remainder: 5

Exercise:

Problem: $\left(2x^2-9x-5\right) \div (x-5)$

Exercise:

Problem: $\left(3x^2+23x+14\right) \div (x+7)$

Solution:

$3x + 2$, quotient: $3x + 2$, remainder: 0

Exercise:

Problem: $\left(4x^2 - 10x + 6\right) \div (4x + 2)$

Exercise:

Problem: $\left(6x^2 - 25x - 25\right) \div (6x + 5)$

Solution:

$x - 5$, quotient: $x - 5$, remainder: 0

Exercise:

Problem: $\left(-x^2 - 1\right) \div (x + 1)$

Exercise:

Problem: $\left(2x^2 - 3x + 2\right) \div (x + 2)$

Solution:

$2x - 7 + \frac{16}{x+2}$, quotient: $2x - 7$, remainder: 16

Exercise:

Problem: $\left(x^3 - 126\right) \div (x - 5)$

Exercise:

Problem: $\left(3x^2 - 5x + 4\right) \div (3x + 1)$

Solution:

$x - 2 + \frac{6}{3x+1}$, quotient: $x - 2$, remainder: 6

Exercise:

Problem: $\left(x^3 - 3x^2 + 5x - 6\right) \div (x - 2)$

Exercise:

Problem: $\left(2x^3 + 3x^2 - 4x + 15\right) \div (x + 3)$

Solution:

$2x^2 - 3x + 5$, quotient: $2x^2 - 3x + 5$, remainder: 0

For the following exercises, use synthetic division to find the quotient.

Exercise:

Problem: $\left(3x^3 - 2x^2 + x - 4\right) \div (x + 3)$

Exercise:

Problem: $\left(2x^3 - 6x^2 - 7x + 6\right) \div (x - 4)$

Solution:

$2x^2 + 2x + 1 + \frac{10}{x-4}$

Exercise:

Problem: $\left(6x^3 - 10x^2 - 7x - 15\right) \div (x + 1)$

Exercise:

Problem: $\left(4x^3 - 12x^2 - 5x - 1\right) \div (2x + 1)$

Solution:

$2x^2 - 7x + 1 - \frac{2}{2x+1}$

Exercise:

Problem: $\left(9x^3 - 9x^2 + 18x + 5\right) \div (3x - 1)$

Exercise:

Problem: $\left(3x^3 - 2x^2 + x - 4\right) \div (x + 3)$

Solution:

$3x^2 - 11x + 34 - \frac{106}{x+3}$

Exercise:

Problem: $\left(-6x^3 + x^2 - 4\right) \div (2x - 3)$

Exercise:

Problem: $\left(2x^3 + 7x^2 - 13x - 3\right) \div (2x - 3)$

Solution:

$x^2 + 5x + 1$

Exercise:

Problem: $\left(3x^3 - 5x^2 + 2x + 3\right) \div (x + 2)$

Exercise:

Problem: $\left(4x^3 - 5x^2 + 13\right) \div (x + 4)$

Solution:

$4x^2 - 21x + 84 - \frac{323}{x+4}$

Exercise:

Problem: $\left(x^3 - 3x + 2\right) \div (x + 2)$

Exercise:

Problem: $\left(x^3 - 21x^2 + 147x - 343\right) \div (x - 7)$

Solution:

$x^2 - 14x + 49$

Exercise:

Problem: $\left(x^3 - 15x^2 + 75x - 125\right) \div (x - 5)$

Exercise:

Problem: $\left(9x^3 - x + 2\right) \div (3x - 1)$

Solution:

$3x^2 + x + \frac{2}{3x-1}$

Exercise:

Problem: $\left(6x^3 - x^2 + 5x + 2\right) \div (3x + 1)$

Exercise:

Problem: $\left(x^4 + x^3 - 3x^2 - 2x + 1\right) \div (x + 1)$

Solution:

$x^3 - 3x + 1$

Exercise:

Problem: $\left(x^4 - 3x^2 + 1\right) \div (x - 1)$

Exercise:

Problem: $\left(x^4 + 2x^3 - 3x^2 + 2x + 6\right) \div (x + 3)$

Solution:

$x^3 - x^2 + 2$

Exercise:

Problem: $\left(x^4 - 10x^3 + 37x^2 - 60x + 36\right) \div (x - 2)$

Exercise:

Problem: $\left(x^4 - 8x^3 + 24x^2 - 32x + 16\right) \div (x - 2)$

Solution:

$x^3 - 6x^2 + 12x - 8$

Exercise:

Problem: $\left(x^4 + 5x^3 - 3x^2 - 13x + 10\right) \div (x + 5)$

Exercise:

Problem: $\left(x^4 - 12x^3 + 54x^2 - 108x + 81\right) \div (x - 3)$

Solution:

$x^3 - 9x^2 + 27x - 27$

Exercise:

Problem: $\left(4x^4-2x^3-4x+2\right)\div(2x-1)$

Exercise:

Problem: $\left(4x^4+2x^3-4x^2+2x+2\right)\div(2x+1)$

Solution:

$2x^3-2x+2$

For the following exercises, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

Exercise:

Problem: $x-2,\ 4x^3-3x^2-8x+4$

Exercise:

Problem: $x-2,\ 3x^4-6x^3-5x+10$

Solution:

Yes $(x-2)(3x^3-5)$

Exercise:

Problem: $x+3,\ -4x^3+5x^2+8$

Exercise:

Problem: $x-2,\ 4x^4-15x^2-4$

Solution:

Yes $(x-2)(4x^3+8x^2+x+2)$

Exercise:

Problem:$x-\frac{1}{2}$, $2x^4-x^3+2x-1$

Exercise:

Problem:$x+\frac{1}{3}$, $3x^4+x^3-3x+1$

Solution:

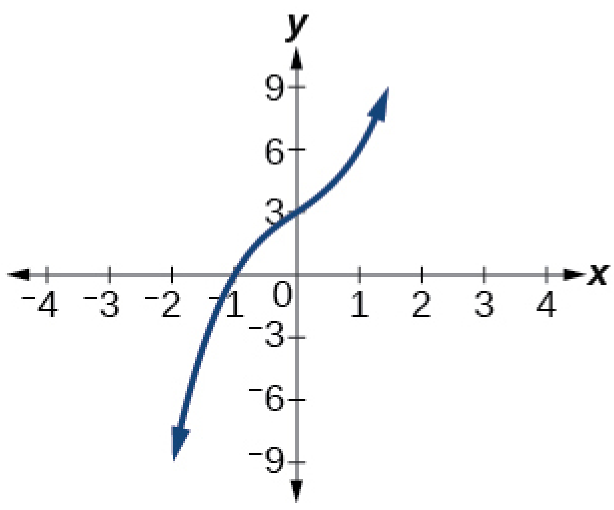
No

Graphical

For the following exercises, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

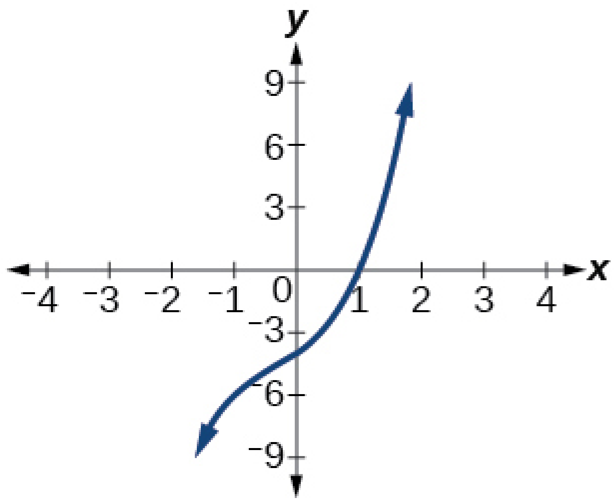
Exercise:

Problem: Factor is x^2-x+3



Exercise:

Problem: Factor is (x^2+2x+4)

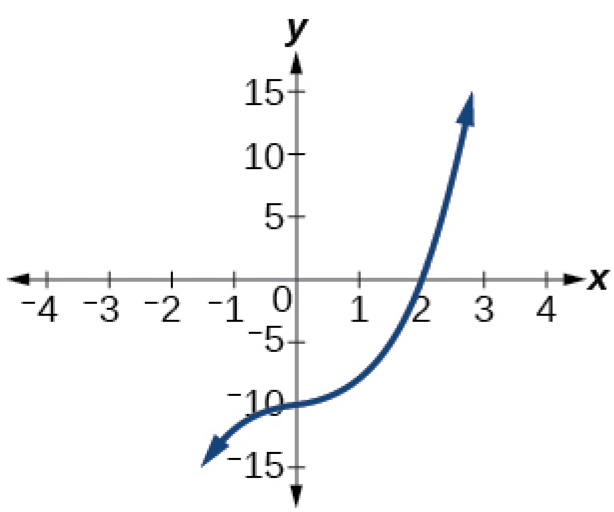


Solution:

$(x-1)(x^2+2x+4)$

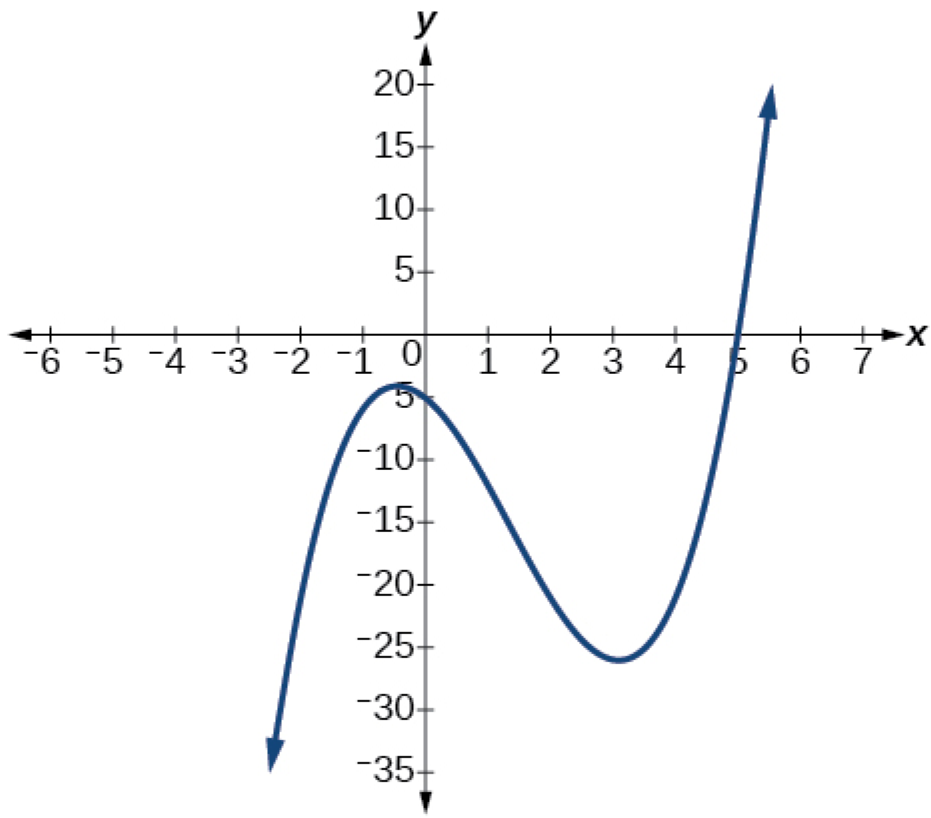
Exercise:

Problem: Factor is x^2+2x+5



Exercise:

Problem: Factor is $x^2 + x + 1$

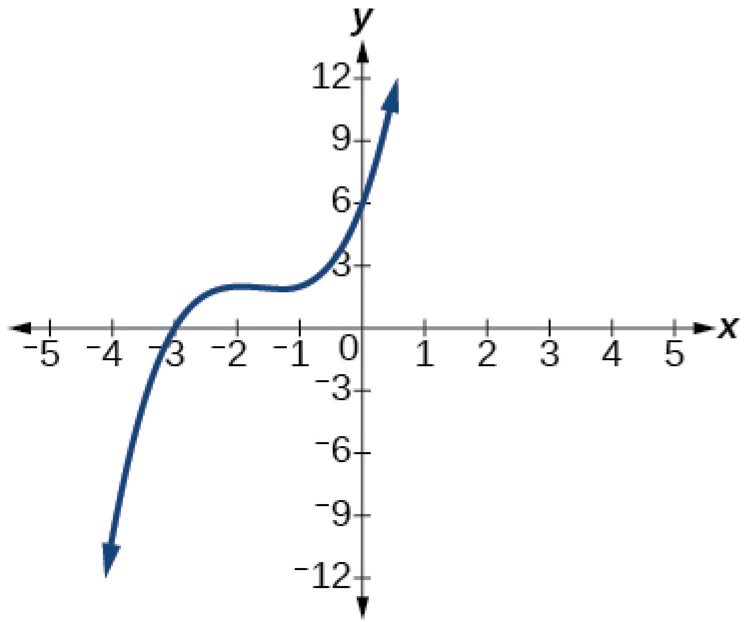


Solution:

$(x - 5)(x^2 + x + 1)$

Exercise:

Problem: Factor is$x^2 + 2x + 2$



For the following exercises, use synthetic division to find the quotient and remainder.

Exercise:

Problem: $\frac{4x^3-33}{x-2}$

Solution:

$\text{Quotient: } 4x^2 + 8x + 16, \text{remainder: } -1$

Exercise:

Problem: $\frac{2x^3+25}{x+3}$

Exercise:

Problem: $\frac{3x^3+2x-5}{x-1}$

Solution:

$\text{Quotient: } 3x^2 + 3x + 5, \text{remainder: } 0$

Exercise:

Problem: $\frac{-4x^3-x^2-12}{x+4}$

Exercise:

Problem: $\frac{x^4-22}{x+2}$

Solution:

$\text{Quotient:} \, x^3 - 2x^2 + 4x - 8, \text{remainder:} \, -6$

Technology

For the following exercises, use a calculator with CAS to answer the questions.

Exercise:

Problem:

Consider $\frac{x^k-1}{x-1}$ with $k = 1, \; 2, \; 3$. What do you expect the result to be if $k = 4$?

Exercise:

Problem:

Consider $\frac{x^k+1}{x+1}$ for $k = 1, \; 3, \; 5$. What do you expect the result to be if $k = 7$?

Solution:

$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$

Exercise:

Problem:

Consider $\frac{x^4-k^4}{x-k}$ for $k=1,\ 2,\ 3.$ What do you expect the result to be if $k=4$?

Exercise:

Problem:

Consider $\frac{x^k}{x+1}$ with $k=1,\ 2,\ 3.$ What do you expect the result to be if $k=4$?

Solution:

$x^3-x^2+x-1+\frac{1}{x+1}$

Exercise:

Problem:

Consider $\frac{x^k}{x-1}$ with $k=1,\ 2,\ 3.$ What do you expect the result to be if $k=4$?

Extensions

For the following exercises, use synthetic division to determine the quotient involving a complex number.

Exercise:

Problem: $\frac{x+1}{x-i}$

Solution:

$1+\frac{1+i}{x-i}$

Exercise:

Problem: $\frac{x^2+1}{x-i}$

Exercise:

Problem: $\frac{x+1}{x+i}$

Solution:

$1+\frac{1-i}{x+i}$

Exercise:

Problem: $\frac{x^2+1}{x+i}$

Exercise:

Problem: $\frac{x^3+1}{x-i}$

Solution:

$x^2-ix-1+\frac{1-i}{x-i}$

Real-World Applications

For the following exercises, use the given length and area of a rectangle to express the width algebraically.

Exercise:

Problem: Length is $x+5,$ area is $2x^2+9x-5.$

Exercise:

Problem: Length is $2x\ +\ 5,$ area is $4x^3+10x^2+6x+15$

Solution:

$2x^2 + 3$

Exercise:

Problem: Length is $3x – 4$, area is $6x^4 - 8x^3 + 9x^2 - 9x - 4$

For the following exercises, use the given volume of a box and its length and width to express the height of the box algebraically.

Exercise:

Problem:

Volume is $12x^3 + 20x^2 - 21x - 36$, length is $2x + 3$, width is $3x - 4$.

Solution:

$2x + 3$

Exercise:

Problem:

Volume is $18x^3 - 21x^2 - 40x + 48$, length is $3x – 4$, width is $3x – 4$.

Exercise:

Problem:

Volume is $10x^3 + 27x^2 + 2x - 24$, length is $5x – 4$, width is $2x + 3$.

Solution:

$x + 2$

Exercise:

Problem:

Volume is $10x^3 + 30x^2 - 8x - 24,$ length is $2,$ width is $x + 3.$

For the following exercises, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.

Exercise:

Problem: Volume is $\pi(25x^3 - 65x^2 - 29x - 3),$ radius is $5x + 1.$

Solution:

$x - 3$

Exercise:

Problem: Volume is $\pi(4x^3 + 12x^2 - 15x - 50),$ radius is $2x + 5.$

Exercise:

Problem:

Volume is $\pi(3x^4 + 24x^3 + 46x^2 - 16x - 32),$ radius is $x + 4.$

Solution:

$3x^2 - 2$

Glossary

Division Algorithm

: given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x),$ there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x).$

synthetic division

a shortcut method that can be used to divide a polynomial by a binomial of the form $x - k$

Zeros of Polynomial Functions
In this section, you will:

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zero Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes' Rule of Signs.
- Solve real-world applications of polynomial equations.

A new bakery offers decorated sheet cakes for children's birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

This problem can be solved by writing a cubic function and solving a cubic equation for the volume of the cake. In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x-k,$ the remainder may be found quickly by evaluating the polynomial function at $k,$ that is, $f\left(k\right)$ Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x),$ there exist unique polynomials $q(x)$ and $r(x)$ such that
Equation:

$$f(x) = d(x)q(x) + r(x)$$

If the divisor, $d(x),$ is $x - k,$ this takes the form

Equation:

$$f(x) = (x - k)q(x) + r$$

Since the divisor $x - k$ is linear, the remainder will be a constant, $r.$ And, if we evaluate this for $x = k,$ we have

Equation:

$$\begin{aligned} f(k) &= (k - k)q(k) + r \\ &= 0 \cdot q(k) + r \\ &= r \end{aligned}$$

In other words, $f(k)$ is the remainder obtained by dividing $f(x)$ by $x - k.$

> **Note:**
> The Remainder Theorem
> If a polynomial $f(x)$ is divided by $x - k,$ then the remainder is the value $f(k).$

> **Note:**
> **Given a polynomial function f, evaluate $f(x)$ at $x = k$ using the Remainder Theorem.**
>
> 1. Use synthetic division to divide the polynomial by $x - k.$
> 2. The remainder is the value $f(k).$

Example:
Exercise:

Problem:
Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$ at $x = 2$.

Solution:

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x - 2$.

Equation:

$$\begin{array}{c|ccccc} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array}$$

The remainder is 25. Therefore, $f(2) = 25$.

Analysis

We can check our answer by evaluating $f(2)$.

Equation:

$$\begin{aligned} f(x) &= 6x^4 - x^3 - 15x^2 + 2x - 7 \\ f(2) &= 6(2)^4 - (2)^3 - 15(2)^2 + 2(2) - 7 \\ &= 25 \end{aligned}$$

Note:

Exercise:

Problem:

Use the Remainder Theorem to evaluate $f(x)=2x^5-3x^4-9x^3+8x^2+2$ at $x=-3.$

Solution:

$f(-3)=-412$

Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm tells us

Equation:

$$f(x)=(x-k)q(x)+r.$$

If k is a zero, then the remainder r is $f(k)=0$ and $f(x)=(x-k)q(x)+0$ or $f(x)=(x-k)q(x).$

Notice, written in this form, $x-k$ is a factor of $f(x).$ We can conclude if k is a zero of $f(x),$ then $x-k$ is a factor of $f(x).$

Similarly, if $x-k$ is a factor of $f(x),$ then the remainder of the Division Algorithm $f(x)=(x-k)q(x)+r$ is 0. This tells us that k is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree n in the complex number system will have n zeros. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

Note:
The Factor Theorem
According to the **Factor Theorem**, k is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$.

Note:
Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial.

1. Use synthetic division to divide the polynomial by $(x-k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x-k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

Example:
Exercise:

Problem:
Using the Factor Theorem to Solve a Polynomial Equation

Show that $(x+2)$ is a factor of x^3-6x^2-x+30. Find the remaining factors. Use the factors to determine the zeros of the polynomial.

Solution:

We can use synthetic division to show that $(x+2)$ is a factor of the polynomial.
Equation:

$$\begin{array}{r|rrrr} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

Equation:

$$(x+2)(x^2-8x+15)$$

We can factor the quadratic factor to write the polynomial as

Equation:

$$(x+2)(x-3)(x-5)$$

By the Factor Theorem, the zeros of x^3-6x^2-x+30 are –2, 3, and 5.

Note:

Exercise:

Problem:

Use the Factor Theorem to find the zeros of $f(x)=x^3+4x^2-4x-16$ given that $(x-2)$ is a factor of the polynomial.

Solution:

The zeros are 2, –2, and –4.

Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading coefficient of the polynomial

Consider a quadratic function with two zeros, $x = \frac{2}{5}$ and $x = \frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function absent its stretching factor.

$x - \frac{2}{5} = 0$ or $x - \frac{3}{4} = 0$	Set each factor equal to 0.
$5x - 2 = 0$ or $4x - 3 = 0$	Multiply both sides of the equation to eliminate fractions.
$f(x) = (5x - 2)(4x - 3)$	Create the quadratic function, multiplying the factors.
$f(x) = 20x^2 - 23x + 6$	Expand the polynomial.
$f(x) = (5 \cdot 4)x^2 - 23x + (2 \cdot 3)$	

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

Note:
The Rational Zero Theorem
The **Rational Zero Theorem** states that, if the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ has integer coefficients, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Note:
Given a polynomial function $f(x)$,use the Rational Zero Theorem to find rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of $f(\frac{p}{q})$.

Example:
Exercise:

Problem:
Listing All Possible Rational Zeros

List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.

Solution:

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, –4, and the factors of the leading coefficient, 2.

The constant term is –4; the factors of –4 are $p = \pm 1, \pm 2, \pm 4$.

The leading coefficient is 2; the factors of 2 are $q = \pm 1, \pm 2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of –4 divided by one of the factors of 2.

Equation:

$$\frac{p}{q} = \pm\frac{1}{1}, \ \pm\frac{1}{2} \qquad \frac{p}{q} = \pm\frac{2}{1}, \pm\frac{2}{2} \qquad \frac{p}{q} = \ \pm\frac{4}{1}, \pm\frac{4}{2}$$

Note that $\frac{2}{2} = 1$ and $\frac{4}{2} = 2,$ which have already been listed. So we can shorten our list.

Equation:

$$\frac{p}{q} = \frac{\text{Factors of the last}}{\text{Factors of the first}} = \pm 1, \pm 2, \pm 4, \pm\frac{1}{2}$$

Example:

Exercise:

Problem:

Using the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of $f(x) = 2x^3 + x^2 - 4x + 1.$

Solution:

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x),$ then p is a factor of 1 and q is a factor of 2.

Equation:

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of 1}}{\text{factor of 2}} \end{aligned}$$

The factors of 1 are ± 1 and the factors of 2 are ± 1 and ± 2. The possible values for $\frac{p}{q}$ are ± 1 and $\pm\frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for x in $f(x)$.

Equation:

$$\begin{aligned} f(-1) &= 2(-1)^3 + (-1)^2 - 4(-1) + 1 = 4 \\ f(1) &= 2(1)^3 + (1)^2 - 4(1) + 1 = 0 \\ f\left(-\tfrac{1}{2}\right) &= 2\left(-\tfrac{1}{2}\right)^3 + \left(-\tfrac{1}{2}\right)^2 - 4\left(-\tfrac{1}{2}\right) + 1 = 3 \\ f\left(\tfrac{1}{2}\right) &= 2\left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^2 - 4\left(\tfrac{1}{2}\right) + 1 = -\tfrac{1}{2} \end{aligned}$$

Of those, $-1, -\frac{1}{2},$ and $\frac{1}{2}$ are not zeros of $f(x)$. 1 is the only rational zero of $f(x)$.

Note:
Exercise:

Problem:

Use the Rational Zero Theorem to find the rational zeros of $f(x) = x^3 - 5x^2 + 2x + 1$.

Solution:

There are no rational zeros.

Finding the Zeros of Polynomial Functions

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can

use synthetic division repeatedly to determine all of the **zeros** of a polynomial function.

Note:
Given a polynomial function f,use synthetic division to find its zeros.

1. Use the Rational Zero Theorem to list all possible rational zeros of the function.
2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0, the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found with synthetic division. If possible, continue until the quotient is a quadratic.
4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.

Example:
Exercise:

Problem:
Finding the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of $f(x) = 4x^3 - 3x - 1.$

Solution:

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x),$ then p is a factor of –1 and q is a factor of 4.
Equation:

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of } -1}{\text{factor of } 4} \end{aligned}$$

The factors of -1 are ± 1 and the factors of 4 are $\pm 1, \pm 2,$ and ± 4. The possible values for $\frac{p}{q}$ are $\pm 1,\ \pm \frac{1}{2}$, and $\pm \frac{1}{4}$. These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

Equation:

$$\begin{array}{r|rrrr} 1 & 4 & 0 & -3 & -1 \\ & & 4 & 4 & 1 \\ \hline & 4 & 4 & 1 & 0 \end{array}$$

Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

Equation:

$$(x-1)(4x^2+4x+1).$$

The quadratic is a perfect square. $f(x)$ can be written as

Equation:

$$(x-1)(2x+1)^2.$$

We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

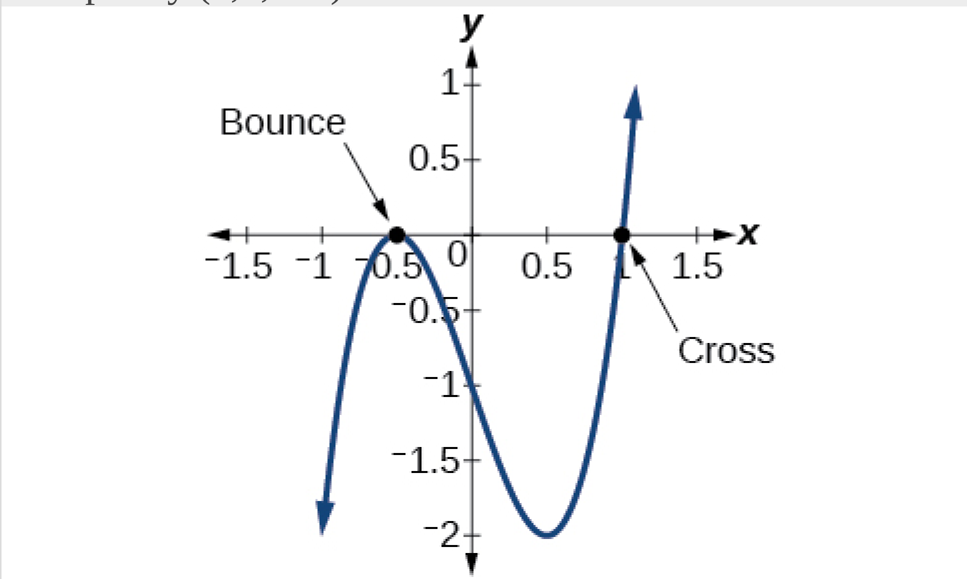
Equation:

$$2x+1=0$$
$$x=-\frac{1}{2}$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

Analysis

Look at the graph of the function f in [link]. Notice, at $x=-0.5,$ the graph bounces off the *x*-axis, indicating the even multiplicity (2,4,6…) for the zero $-0.5.$ At $x=1,$ the graph crosses the *x*-axis, indicating the odd multiplicity (1,3,5…) for the zero $x=1.$



Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose f is a polynomial function of degree four, and $f(x)=0$. The Fundamental Theorem of Algebra states that there is at least one complex

solution, call it c_1. By the Factor Theorem, we can write $f(x)$ as a product of $x - c_1$ and a polynomial quotient. Since $x - c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it c_2. So we can write the polynomial quotient as a product of $x - c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

Note:
The **Fundamental Theorem of Algebra** states that, if *f(x)* is a polynomial of degree *n > 0*, then *f(x)* has at least one complex zero.
We can use this theorem to argue that, if $f(x)$ is a polynomial of degree $n > 0$, and a is a non-zero real number, then $f(x)$ has exactly n linear factors
Equation:

$$f(x) = a(x - c_1)(x - c_2)...(x - c_n)$$

where $c_1, c_2, ..., c_n$ are complex numbers. Therefore, $f(x)$ has n roots if we allow for multiplicities.

Note:
Does every polynomial have at least one imaginary zero?
No. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

Example:
Exercise:

Problem:
Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of $f(x) = 3x^3 + 9x^2 + x + 3$.

Solution:

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 3 and q is a factor of 3.

Equation:

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of 3}}{\text{factor of 3}} \end{aligned}$$

The factors of 3 are ± 1 and ± 3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are $\pm 3, \pm 1, \text{and } \pm \frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with –3.

Equation:

$$\begin{array}{r|rrrr} -3 & 3 & 9 & 1 & 3 \\ & & -9 & 0 & -3 \\ \hline & 3 & 0 & 1 & 0 \end{array}$$

Dividing by $(x+3)$ gives a remainder of 0, so –3 is a zero of the function. The polynomial can be written as

Equation:

$$(x+3)(3x^2+1)$$

We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

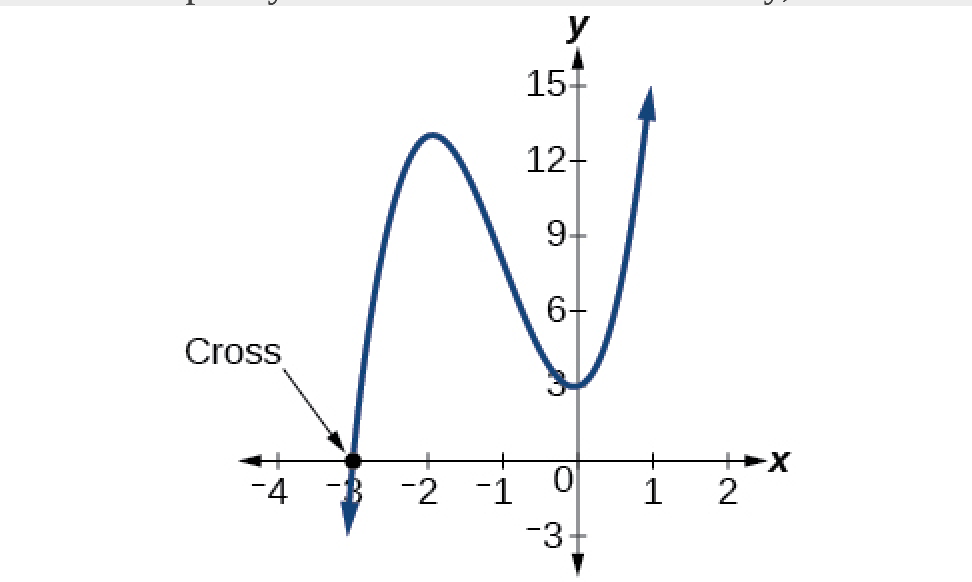
Equation:

$$3x^2+1=0$$
$$x^2=-\frac{1}{3}$$
$$x=\pm\sqrt{-\frac{1}{3}}=\pm\frac{i\sqrt{3}}{3}$$

The zeros of $f(x)$ are –3 and $\pm\frac{i\sqrt{3}}{3}$.

Analysis

Look at the graph of the function f in [link]. Notice that, at $x=-3$, the graph crosses the *x*-axis, indicating an odd multiplicity (1) for the zero $x=-3$. Also note the presence of the two turning points. This means that, since there is a 3rd degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the *x*-intercepts for the function are shown. So either the multiplicity of $x=-3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x=-3$ is three. Either way, our result is correct.



Note:
Exercise:

Problem: Find the zeros of $f(x) = 2x^3 + 5x^2 - 11x + 4.$

Solution:

The zeros are $-4,\ \frac{1}{2},$ and 1.

Using the Linear Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the Fundamental Theorem of Algebra, as we stated above, is that a polynomial function of degree n will have n zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into n factors. The **Linear Factorization Theorem** tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x - c),$ where c is a complex number.

Let f be a polynomial function with real coefficients, and suppose $a + bi, b \neq 0,$ is a zero of $f(x).$ Then, by the Factor Theorem, $x - (a + bi)$ is a factor of $f(x).$ For f to have real coefficients, $x - (a - bi)$ must also be a factor of $f(x).$ This is true because any factor other than $x - (a - bi),$ when multiplied by $x - (a + bi),$ will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function f with real coefficients has a complex zero $a + bi,$ then the complex conjugate $a - bi$ must also be a zero of $f(x).$ This is called the Complex Conjugate Theorem.

Note:
Complex Conjugate Theorem

According to the **Linear Factorization Theorem,** a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x - c)$, where c is a complex number.
If the polynomial function f has real coefficients and a complex zero in the form $a + bi$, then the complex conjugate of the zero, $a - bi$, is also a zero.

Note:
Given the zeros of a polynomial function f and a point *(c, f(c))* on the graph of f, use the Linear Factorization Theorem to find the polynomial function.

1. Use the zeros to construct the linear factors of the polynomial.
2. Multiply the linear factors to expand the polynomial.
3. Substitute $(c, f(c))$ into the function to determine the leading coefficient.
4. Simplify.

Example:
Exercise:

Problem:
Using the Linear Factorization Theorem to Find a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of –3, 2, i, such that $f(-2) = 100$.

Solution:

Because $x = i$ is a zero, by the Complex Conjugate Theorem $x = -i$ is also a zero. The polynomial must have factors of $(x + 3)$, $(x - 2)$, $(x - i)$, and $(x + i)$. Since we are looking for a

degree 4 polynomial, and now have four zeros, we have all four factors. Let’s begin by multiplying these factors.

Equation:

$$\begin{aligned} f(x) &= a(x+3)(x-2)(x-i)(x+i) \\ f(x) &= a(x^2+x-6)(x^2+1) \\ f(x) &= a(x^4+x^3-5x^2+x-6) \end{aligned}$$

We need to find *a* to ensure $f(-2)=100$. Substitute $x=-2$ and $f(2)=100$ into $f(x)$.

Equation:

$$\begin{aligned} 100 &= a((-2)^4+(-2)^3-5(-2)^2+(-2)-6) \\ 100 &= a(-20) \\ -5 &= a \end{aligned}$$

So the polynomial function is

Equation:

$$f(x)=-5(x^4+x^3-5x^2+x-6)$$

or

Equation:

$$f(x)=-5x^4-5x^3+25x^2-5x+30$$

Analysis

We found that both i and $-i$ were zeros, but only one of these zeros needed to be given. If i is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of i.

Note:
If $2 + 3i$ were given as a zero of a polynomial with real coefficients, would $2 - 3i$ also need to be a zero?
Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

Note:
Exercise:

Problem:

Find a third degree polynomial with real coefficients that has zeros of 5 and $-2i$ such that $f(1) = 10$.

Solution:

$f(x) = -\frac{1}{2}x^3 + \frac{5}{2}x^2 - 2x + 10$

Using Descartes' Rule of Signs

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.

$$f(x) = x^4 + x^3 + x^2 + x - 1$$

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.

$$f(-x) = (-x)^4 + (-x)^3 + (-x)^2 + (-x) - 1$$
$$f(-x) = +x^4 - x^3 + x^2 - x - 1$$

In this case, $f(-\mathrm{x})$ has 3 sign changes. This tells us that $f(x)$ could have 3 or 1 negative real zeros.

Note:
Descartes' Rule of Signs
According to **Descartes' Rule of Signs**, if we let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.

Example:
Exercise:

Problem:
Using Descartes' Rule of Signs

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12.$

Solution:

Begin by determining the number of sign changes.

$$f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$$

There are two sign changes, so there are either 2 or 0 positive real roots. Next, we examine $f(-x)$ to determine the number of negative real roots.

$$f(-x) = -(-x)^4 - 3(-x)^3 + 6(-x)^2 - 4(-x) - 12$$
$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12$$

$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12$$

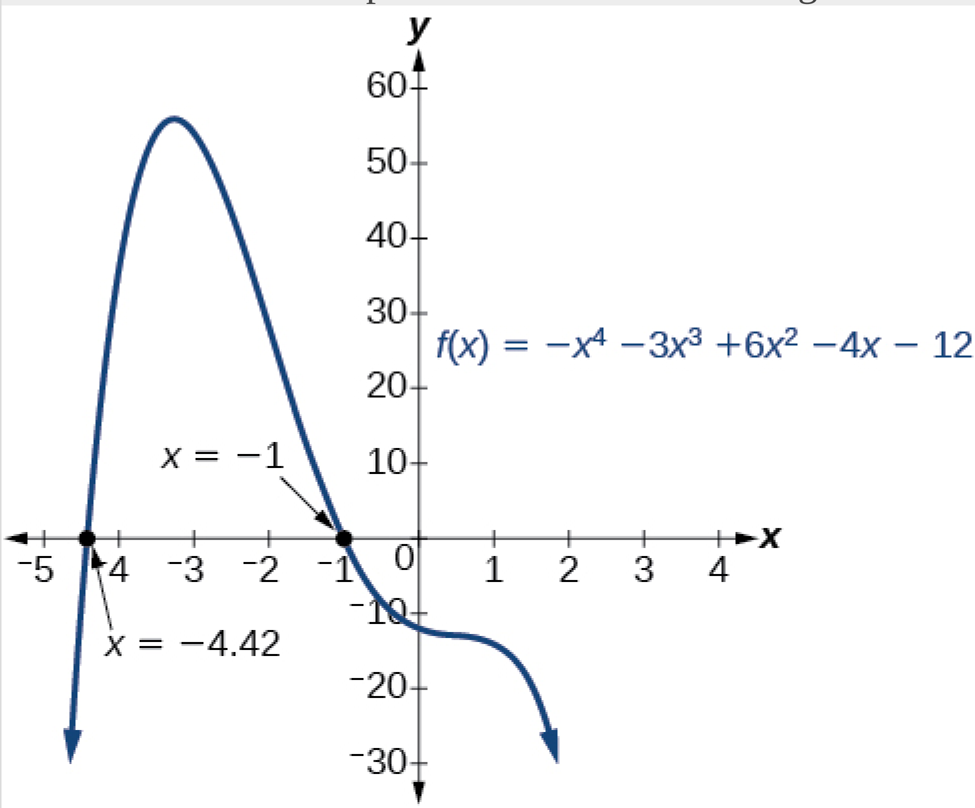
Again, there are two sign changes, so there are either 2 or 0 negative real roots.

There are four possibilities, as we can see in [link].

Positive Real Zeros	Negative Real Zeros	Complex Zeros	Total Zeros
2	2	0	4
2	0	2	4
0	2	2	4
0	0	4	4

Analysis

We can confirm the numbers of positive and negative real roots by examining a graph of the function. See [link]. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.



Note:
Exercise:

Problem:

Use Descartes' Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for $f(x)=2x^4-10x^3+11x^2-15x+12.$ Use a graph to verify the numbers of positive and negative real zeros for the function.

Solution:

There must be 4, 2, or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.

Solving Real-World Applications

We have now introduced a variety of tools for solving polynomial equations. Let's use these tools to solve the bakery problem from the beginning of the section.

Example:
Exercise:

Problem:
Solving Polynomial Equations

A new bakery offers decorated sheet cakes for children's birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

Solution:

Begin by writing an equation for the volume of the cake. The volume of a rectangular solid is given by $V = lwh.$ We were given that the length must be four inches longer than the width, so we can express the length of the cake as $l = w + 4.$ We were given that the height of the cake is one-third of the width, so we can express the height of the cake as $h = \frac{1}{3}w.$ Let's write the volume of the cake in terms of width of the cake.

Equation:

$$V = (w+4)(w)\left(\tfrac{1}{3}w\right)$$
$$V = \tfrac{1}{3}w^3 + \tfrac{4}{3}w^2$$

Substitute the given volume into this equation.

Equation:

$$\begin{array}{rl} 351 = \frac{1}{3}w^3 + \frac{4}{3}w^2 & \text{Substitute 351 for } V. \\ 1053 = w^3 + 4w^2 & \text{Multiply both sides by 3.} \\ 0 = w^3 + 4w^2 - 1053 & \text{Subtract 1053 from both sides.} \end{array}$$

Descartes' rule of signs tells us there is one positive solution. The Rational Zero Theorem tells us that the possible rational zeros are $\pm 1, \ \pm 3, \ \pm 9, \ \pm 13, \ \pm 27, \ \pm 39, \ \pm 81, \ \pm 117, \ \pm 351,$ and $\pm 1053.$ We can use synthetic division to test these possible zeros. Only positive numbers make sense as dimensions for a cake, so we need not test any negative values. Let’s begin by testing values that make the most sense as dimensions for a small sheet cake. Use synthetic division to check $x = 1$.

Equation:

$$\begin{array}{r|rrrr} 1 & 1 & 4 & 0 & -1053 \\ & & 1 & 5 & 5 \\ \hline & 1 & 5 & 5 & -1048 \end{array}$$

Since 1 is not a solution, we will check $x = 3$.

$$\begin{array}{r|rrrr} 3 & 1 & 4 & 0 & -1053 \\ & & 3 & 21 & 63 \\ \hline & 1 & 7 & 21 & -990 \end{array}$$

Since 3 is not a solution either, we will test $x = 9$.

$$\begin{array}{r|rrrr} 9 & 1 & 4 & 0 & -1053 \\ & & 9 & 117 & 1053 \\ \hline & 1 & 13 & 117 & 0 \end{array}$$

Synthetic division gives a remainder of 0, so 9 is a solution to the equation. We can use the relationships between the width and the other dimensions to determine the length and height of the sheet cake pan.

Equation:

$$l = w + 4 = 9 + 4 = 13 \text{ and } h = \frac{1}{3}w = \frac{1}{3}(9) = 3$$

The sheet cake pan should have dimensions 13 inches by 9 inches by 3 inches.

Note:

Exercise:

Problem:

A shipping container in the shape of a rectangular solid must have a volume of 84 cubic meters. The client tells the manufacturer that, because of the contents, the length of the container must be one meter longer than the width, and the height must be one meter greater than twice the width. What should the dimensions of the container be?

Solution:

3 meters by 4 meters by 7 meters

Note:

Access these online resources for additional instruction and practice with zeros of polynomial functions.

Key Concepts

- To find $f(k)$, determine the remainder of the polynomial $f(x)$ when it is divided by $x - k$. See [link].
- k is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$.See [link].
- Each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient. See [link] and [link].
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function. See [link].
- According to the Fundamental Theorem, every polynomial function has at least one complex zero. See [link].
- Every polynomial function with degree greater than 0 has at least one complex zero.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form $(x - c)$, where c is a complex number. See [link].
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of $f(-x)$ or less than the number of sign changes by an even integer. See [link].
- Polynomial equations model many real-world scenarios. Solving the equations is easiest done by synthetic division. See [link].

Section Exercises

Verbal

Exercise:

Problem: Describe a use for the Remainder Theorem.

Solution:

The theorem can be used to evaluate a polynomial.

Exercise:

Problem:

Explain why the Rational Zero Theorem does not guarantee finding zeros of a polynomial function.

Exercise:

Problem: What is the difference between rational and real zeros?

Solution:

Rational zeros can be expressed as fractions whereas real zeros include irrational numbers.

Exercise:

Problem:

If Descartes' Rule of Signs reveals a no change of signs or one sign of changes, what specific conclusion can be drawn?

Exercise:

Problem:

If synthetic division reveals a zero, why should we try that value again as a possible solution?

Solution:

Polynomial functions can have repeated zeros, so the fact that number is a zero doesn't preclude it being a zero again.

Algebraic

For the following exercises, use the Remainder Theorem to find the remainder.

Exercise:

Problem: $\left(x^4 - 9x^2 + 14\right) \div (x - 2)$

Exercise:

Problem: $\left(3x^3 - 2x^2 + x - 4\right) \div (x + 3)$

Solution:

-106

Exercise:

Problem: $\left(x^4 + 5x^3 - 4x - 17\right) \div (x + 1)$

Exercise:

Problem: $\left(-3x^2 + 6x + 24\right) \div (x - 4)$

Solution:

0

Exercise:

Problem: $\left(5x^5 - 4x^4 + 3x^3 - 2x^2 + x - 1\right) \div (x + 6)$

Exercise:

Problem: $\left(x^4 - 1\right) \div (x - 4)$

Solution:

255

Exercise:

Problem: $\left(3x^3 + 4x^2 - 8x + 2\right) \div (x - 3)$

Exercise:

Problem: $\left(4x^3 + 5x^2 - 2x + 7\right) \div (x + 2)$

Solution:

-1

For the following exercises, use the Factor Theorem to find all real zeros for the given polynomial function and one factor.

Exercise:

Problem: $f(x) = 2x^3 - 9x^2 + 13x - 6;\ x - 1$

Exercise:

Problem: $f(x) = 2x^3 + x^2 - 5x + 2;\ x + 2$

Solution:

$-2,\ 1,\ \frac{1}{2}$

Exercise:

Problem:$f(x) = 3x^3 + x^2 - 20x + 12;\ x + 3$

Exercise:

Problem:$f(x) = 2x^3 + 3x^2 + x + 6;\quad x + 2$

Solution:

-2

Exercise:

Problem:$f(x) = -5x^3 + 16x^2 - 9;\quad x - 3$

Exercise:

Problem:$x^3 + 3x^2 + 4x + 12;\ x + 3$

Solution:

-3

Exercise:

Problem:$4x^3 - 7x + 3;\ x - 1$

Exercise:

Problem:$2x^3 + 5x^2 - 12x - 30,\ 2x + 5$

Solution:

$-\frac{5}{2},\ \sqrt{6},\ -\sqrt{6}$

For the following exercises, use the Rational Zero Theorem to find all real zeros.

Exercise:

Problem: $x^3 - 3x^2 - 10x + 24 = 0$

Exercise:

Problem: $2x^3 + 7x^2 - 10x - 24 = 0$

Solution:

$2, \ -4, \ -\frac{3}{2}$

Exercise:

Problem: $x^3 + 2x^2 - 9x - 18 = 0$

Exercise:

Problem: $x^3 + 5x^2 - 16x - 80 = 0$

Solution:

$4, \ -4, \ -5$

Exercise:

Problem: $x^3 - 3x^2 - 25x + 75 = 0$

Exercise:

Problem: $2x^3 - 3x^2 - 32x - 15 = 0$

Solution:

$5, \ -3, \ -\frac{1}{2}$

Exercise:

Problem: $2x^3 + x^2 - 7x - 6 = 0$

Exercise:

Problem: $2x^3 - 3x^2 - x + 1 = 0$

Solution:

$\frac{1}{2}, \ \frac{1+\sqrt{5}}{2}, \ \frac{1-\sqrt{5}}{2}$

Exercise:

Problem: $3x^3 - x^2 - 11x - 6 = 0$

Exercise:

Problem: $2x^3 - 5x^2 + 9x - 9 = 0$

Solution:

$\frac{3}{2}$

Exercise:

Problem: $2x^3 - 3x^2 + 4x + 3 = 0$

Exercise:

Problem: $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$

Solution:

$2, \ 3, \ -1, \ -2$

Exercise:

Problem: $x^4 + 2x^3 - 9x^2 - 2x + 8 = 0$

Exercise:

Problem:$4x^4+4x^3-25x^2-x+6=0$

Solution:

$\frac{1}{2},\ -\frac{1}{2},\ 2,\ -3$

Exercise:

Problem:$2x^4-3x^3-15x^2+32x-12=0$

Exercise:

Problem:$x^4+2x^3-4x^2-10x-5=0$

Solution:

$-1,\ -1,\ \sqrt{5},\ -\sqrt{5}$

Exercise:

Problem:$4x^3-3x+1=0$

Exercise:

Problem:$8x^4+26x^3+39x^2+26x+6$

Solution:

$-\frac{3}{4},\ -\frac{1}{2}$

For the following exercises, find all complex solutions (real and non-real).

Exercise:

Problem:$x^3+x^2+x+1=0$

Exercise:

Problem:$x^3 - 8x^2 + 25x - 26 = 0$

Solution:

$2,\ 3 + 2i,\ 3 - 2i$

Exercise:

Problem:$x^3 + 13x^2 + 57x + 85 = 0$

Exercise:

Problem:$3x^3 - 4x^2 + 11x + 10 = 0$

Solution:

$-\frac{2}{3},\ 1 + 2i,\ 1 - 2i$

Exercise:

Problem:$x^4 + 2x^3 + 22x^2 + 50x - 75 = 0$

Exercise:

Problem:$2x^3 - 3x^2 + 32x + 17 = 0$

Solution:

$-\frac{1}{2},\ 1 + 4i,\ 1 - 4i$

Graphical

For the following exercises, use Descartes' Rule to determine the possible number of positive and negative solutions. Then graph to confirm which of

those possibilities is the actual combination.

Exercise:

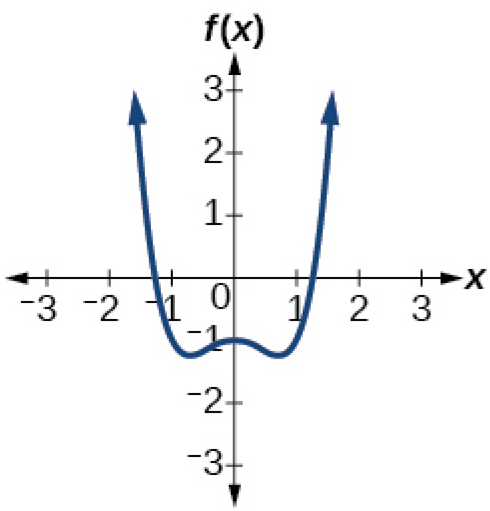
Problem: $f(x) = x^3 - 1$

Exercise:

Problem: $f(x) = x^4 - x^2 - 1$

Solution:

1 positive, 1 negative



Exercise:

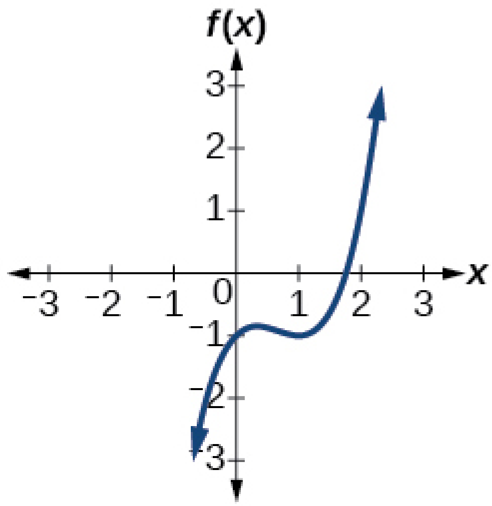
Problem: $f(x) = x^3 - 2x^2 - 5x + 6$

Exercise:

Problem: $f(x) = x^3 - 2x^2 + x - 1$

Solution:

3 or 1 positive, 0 negative



Exercise:

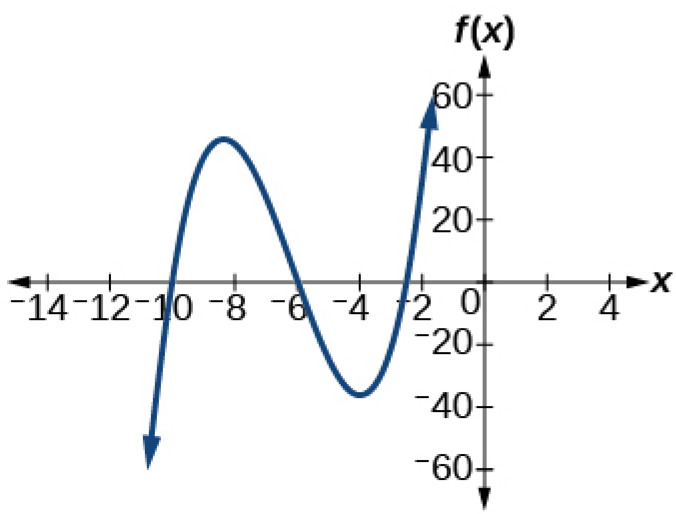
Problem: $f(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$

Exercise:

Problem: $f(x) = 2x^3 + 37x^2 + 200x + 300$

Solution:

0 positive, 3 or 1 negative



Exercise:

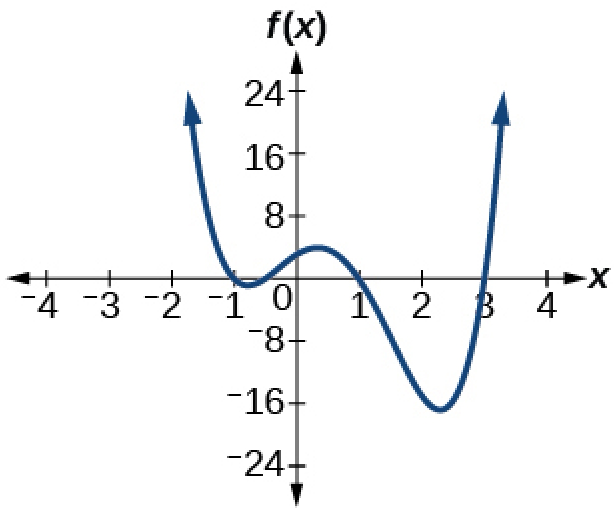
Problem: $f(x) = x^3 - 2x^2 - 16x + 32$

Exercise:

Problem: $f(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$

Solution:

2 or 0 positive, 2 or 0 negative



Exercise:

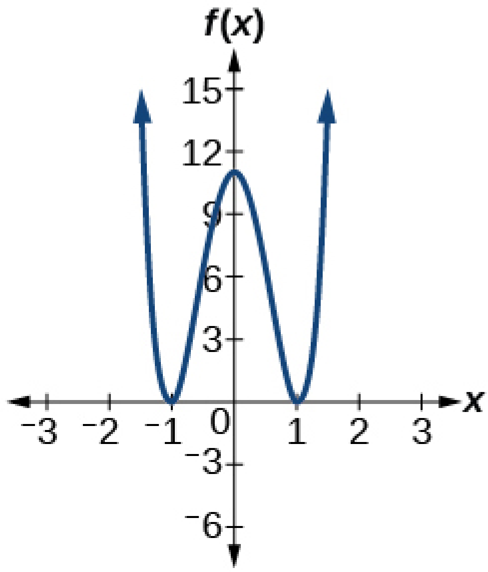
Problem: $f(x) = 2x^4 - 5x^3 - 14x^2 + 20x + 8$

Exercise:

Problem: $f(x) = 10x^4 - 21x^2 + 11$

Solution:

2 or 0 positive, 2 or 0 negative



Numeric

For the following exercises, list all possible rational zeros for the functions.

Exercise:

Problem: $f(x) = x^4 + 3x^3 - 4x + 4$

Exercise:

Problem: $f(x) = 2x^3 + 3x^2 - 8x + 5$

Solution:

$\pm 5, \ \pm 1, \ \pm \frac{5}{2}$

Exercise:

Problem: $f(x) = 3x^3 + 5x^2 - 5x + 4$

Exercise:

Problem: $f(x) = 6x^4 - 10x^2 + 13x + 1$

Solution:

$\pm 1, \ \pm \frac{1}{2}, \ \pm \frac{1}{3}, \ \pm \frac{1}{6}$

Exercise:

Problem: $f(x) = 4x^5 - 10x^4 + 8x^3 + x^2 - 8$

Technology

For the following exercises, use your calculator to graph the polynomial function. Based on the graph, find the rational zeros. All real solutions are rational.

Exercise:

Problem: $f(x) = 6x^3 - 7x^2 + 1$

Solution:

$1, \ \frac{1}{2}, \ -\frac{1}{3}$

Exercise:

Problem: $f(x) = 4x^3 - 4x^2 - 13x - 5$

Exercise:

Problem: $f(x) = 8x^3 - 6x^2 - 23x + 6$

Solution:

$2, \ \frac{1}{4}, \ -\frac{3}{2}$

Exercise:

Problem: $f(x)=12x^4+55x^3+12x^2-117x+54$

Exercise:

Problem: $f(x)=16x^4-24x^3+x^2-15x+25$

Solution:

$\frac{5}{4}$

Extensions

For the following exercises, construct a polynomial function of least degree possible using the given information.

Exercise:

Problem: Real roots: –1, 1, 3 and $(2, f(2))=(2,4)$

Exercise:

Problem:

Real roots: –1 (with multiplicity 2 and 1) and $(2, f(2))=(2,4)$

Solution:

$f(x)=\frac{4}{9}\left(x^3+x^2-x-1\right)$

Exercise:

Problem:

Real roots: –2, $\frac{1}{2}$ (with multiplicity 2) and $(-3, f(-3))=(-3,5)$

Exercise:

Problem: Real roots: $-\frac{1}{2}$, 0, $\frac{1}{2}$ and $(-2, f(-2)) = (-2, 6)$

Solution:

$f(x) = -\frac{1}{5}\left(4x^3 - x\right)$

Exercise:

Problem: Real roots: –4, –1, 1, 4 and $(-2, f(-2)) = (-2, 10)$

Real-World Applications

For the following exercises, find the dimensions of the box described.

Exercise:

Problem:

The length is twice as long as the width. The height is 2 inches greater than the width. The volume is 192 cubic inches.

Solution:

8 by 4 by 6 inches

Exercise:

Problem:

The length, width, and height are consecutive whole numbers. The volume is 120 cubic inches.

Exercise:

Problem:

The length is one inch more than the width, which is one inch more than the height. The volume is 86.625 cubic inches.

Solution:

5.5 by 4.5 by 3.5 inches

Exercise:

Problem:

The length is three times the height and the height is one inch less than the width. The volume is 108 cubic inches.

Exercise:

Problem:

The length is 3 inches more than the width. The width is 2 inches more than the height. The volume is 120 cubic inches.

Solution:

8 by 5 by 3 inches

For the following exercises, find the dimensions of the right circular cylinder described.

Exercise:

Problem:

The radius is 3 inches more than the height. The volume is 16π cubic meters.

Exercise:

Problem:

The height is one less than one half the radius. The volume is 72π cubic meters.

Solution:

Radius = 6 meters, Height = 2 meters

Exercise:

Problem:

The radius and height differ by one meter. The radius is larger and the volume is 48π cubic meters.

Exercise:

Problem:

The radius and height differ by two meters. The height is greater and the volume is 28.125π cubic meters.

Solution:

Radius = 2.5 meters, Height = 4.5 meters

Exercise:

Problem:

80. The radius is $\frac{1}{3}$ meter greater than the height. The volume is $\frac{98}{9}\pi$ cubic meters.

Glossary

Descartes' Rule of Signs
: a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $f(x)$ and $f(-x)$

Factor Theorem
: k is a zero of polynomial function $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$

Fundamental Theorem of Algebra
: a polynomial function with degree greater than 0 has at least one complex zero

Linear Factorization Theorem

allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x - c)$, where c is a complex number

Rational Zero Theorem

the possible rational zeros of a polynomial function have the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient.

Remainder Theorem

if a polynomial $f(x)$ is divided by $x - k$, then the remainder is equal to the value $f(k)$

Rational Functions
In this section, you will:

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, $x,$ produced. This is given by the equation $C(x)=15{,}000x-0.1x^2+1000.$ If we want to know the average cost for producing x items, we would divide the cost function by the number of items, $x.$

The average cost function, which yields the average cost per item for x items produced, is
Equation:

$$f(x)=\frac{15{,}000x-0.1x^2+1000}{x}$$

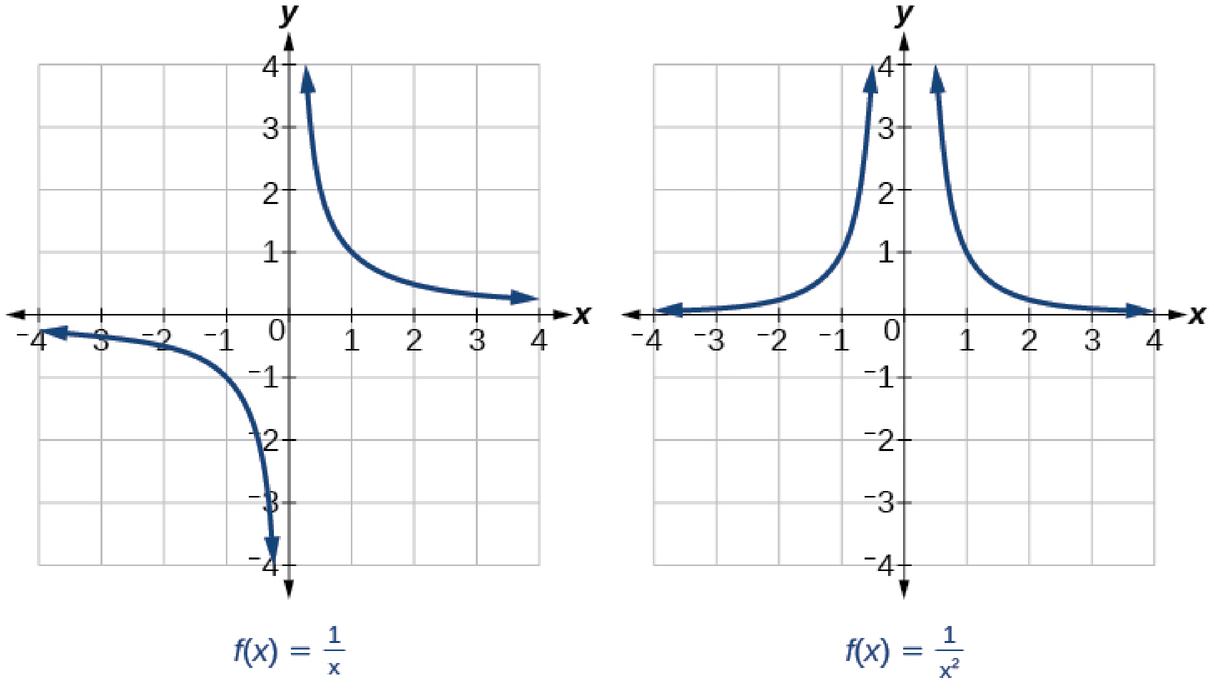
Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

Using Arrow Notation

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in [link], and notice some of their features.

Graphs of Toolkit Functions



Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

1. On the left branch of the graph, the curve approaches the *x*-axis $(y = 0)$ as $x \to -\infty$.
2. As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
3. Finally, on the right branch of the graph, the curves approaches the *x*-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that x or $f(x)$ is approaching a particular value. See [link].

Symbol	Meaning

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Arrow Notation

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in [link].

x	–0.1	–0.01	–0.001	–0.0001
$f(x)=\frac{1}{x}$	–10	–100	–1000	–10,000

We write in arrow notation
Equation:

$$\text{as } x \to 0^{-}, f(x) \to -\infty$$

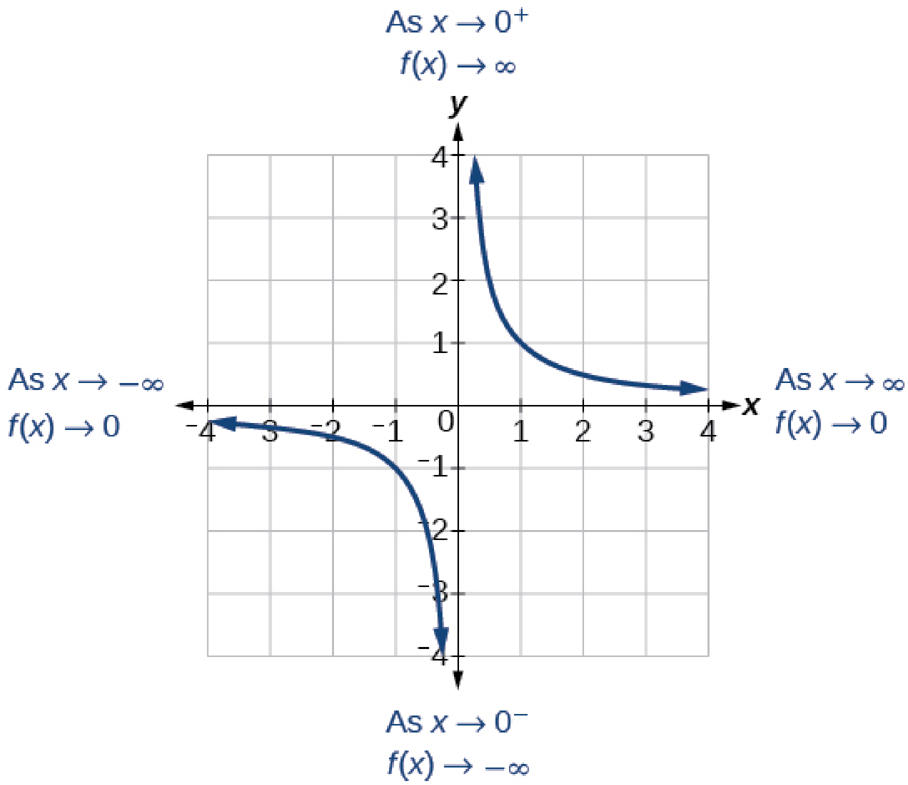
As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in [link].

x	0.1	0.01	0.001	0.0001
$f(x)=\frac{1}{x}$	10	100	1000	10,000

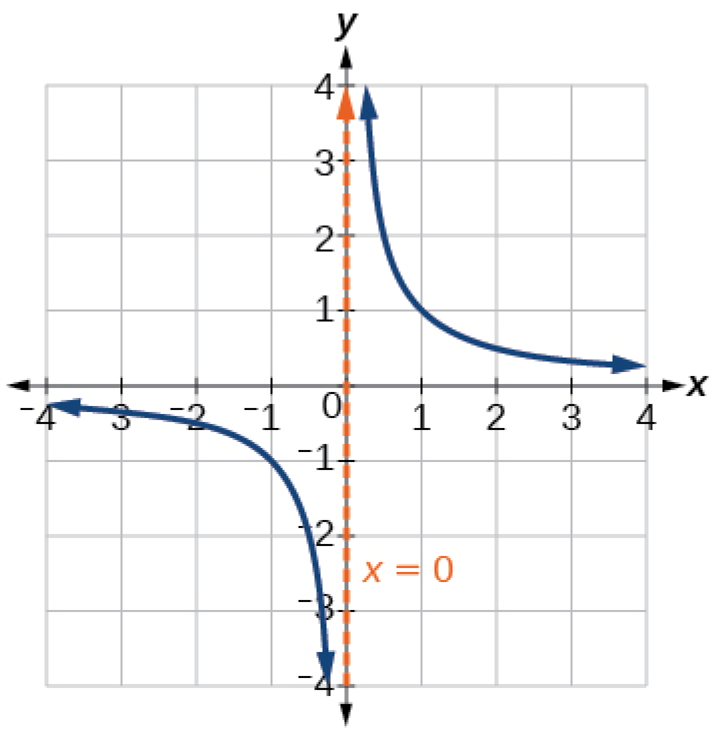
We write in arrow notation
Equation:

$$\text{As } x \to 0^{+},\ f(x) \to \infty.$$

See [link].



This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero. See [link].



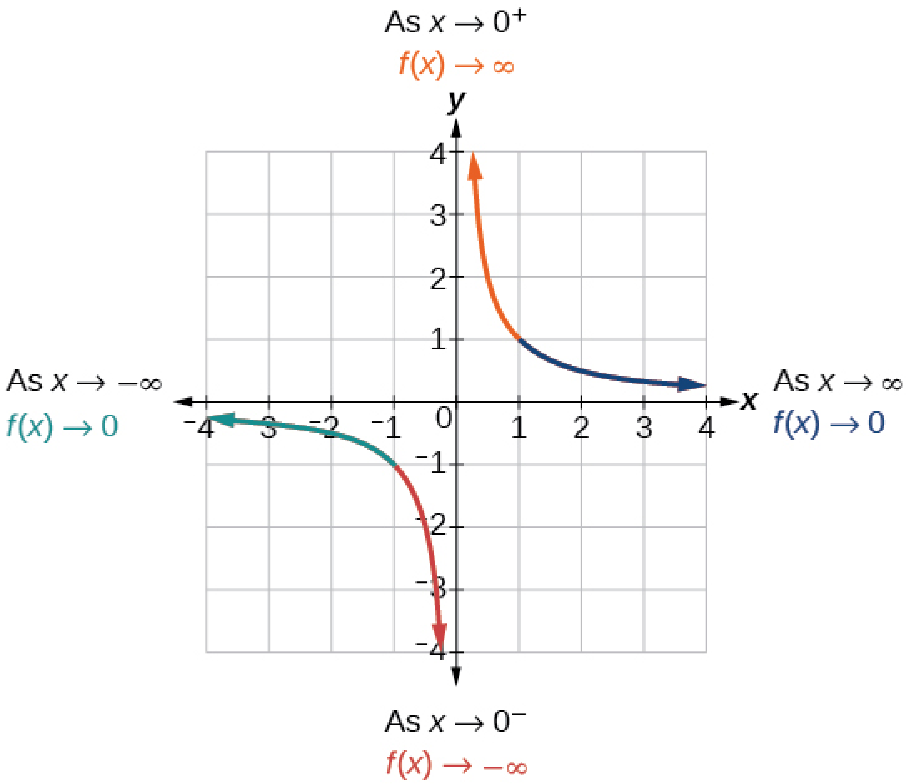
Note:
Vertical Asymptote
A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a. We write
Equation:

$$\text{As } x \to a, f(x) \to \infty, \text{ or as } x \to a, f(x) \to -\infty.$$

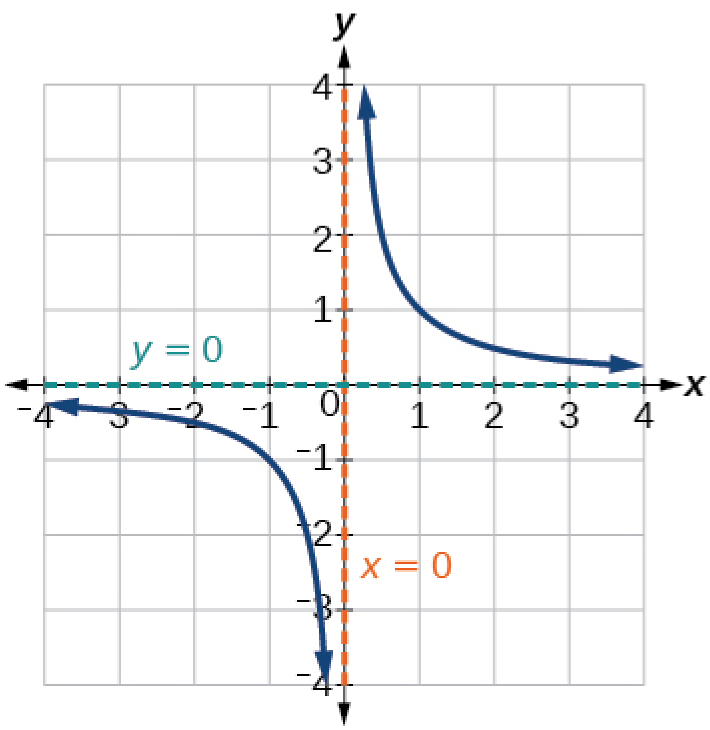
End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. As the values of x approach negative infinity, the function values approach 0. See [link]. Symbolically, using arrow notation

$$\text{As } x \to \infty, f(x) \to 0, \text{and as } x \to -\infty, f(x) \to 0.$$



Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See [link].



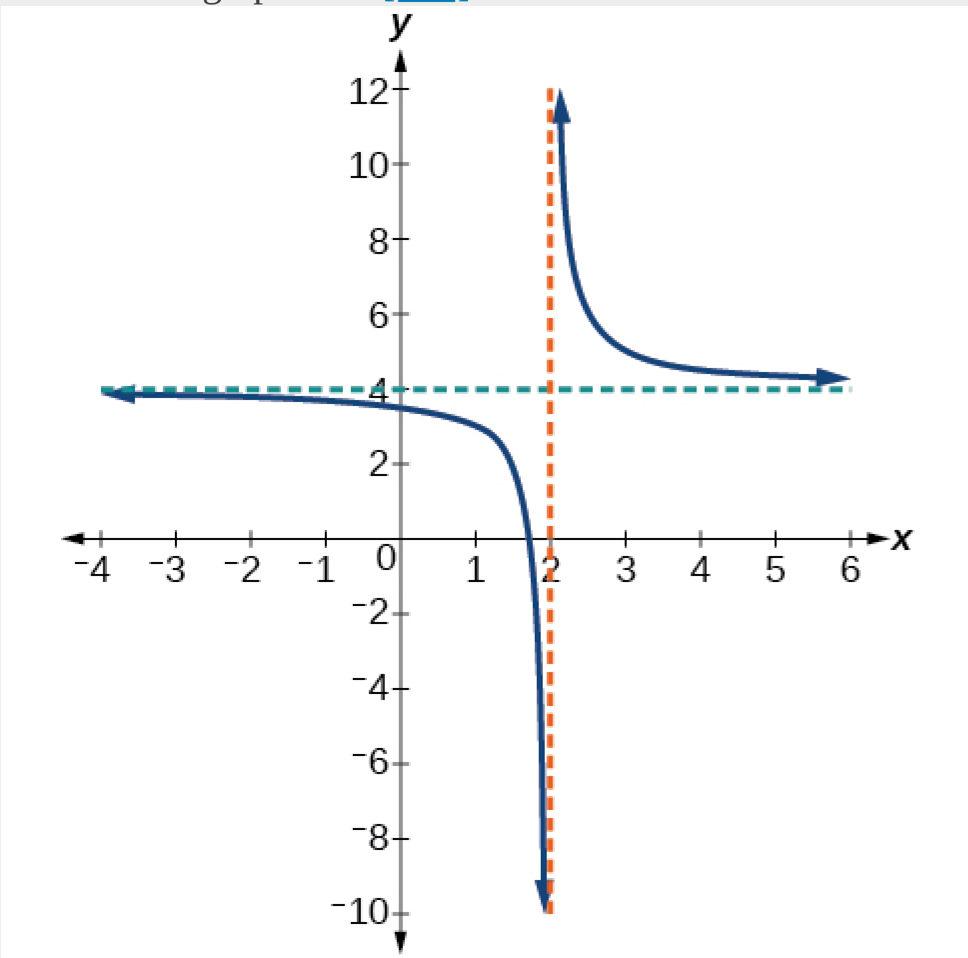
Note:
Horizontal Asymptote
A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write
Equation:

$$\text{As } x \to \infty \text{ or } x \to -\infty, \; f(x) \to b.$$

Example:
Exercise:

Problem:
Using Arrow Notation

Use arrow notation to describe the end behavior and local behavior of the function graphed in [link].



Solution:

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$.

Equation:

$$\text{As } x \to 2^{-}, f(x) \to -\infty, \text{ and as } x \to 2^{+}, \; f(x) \to \infty.$$

And as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4$. As the inputs increase without bound, the graph levels off at 4.

Equation:

As $x \to \infty,\ f(x) \to 4$ and as $x \to -\infty,\ f(x) \to 4.$

Note:
Exercise:

Problem:

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

Solution:

End behavior: as $x \to \pm\infty,\ f(x) \to 0;$ Local behavior: as $x \to 0,\ f(x) \to \infty$ (there are no *x*- or *y*-intercepts)

Example:
Exercise:

Problem:
Using Transformations to Graph a Rational Function

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution:

Shifting the graph left 2 and up 3 would result in the function
Equation:

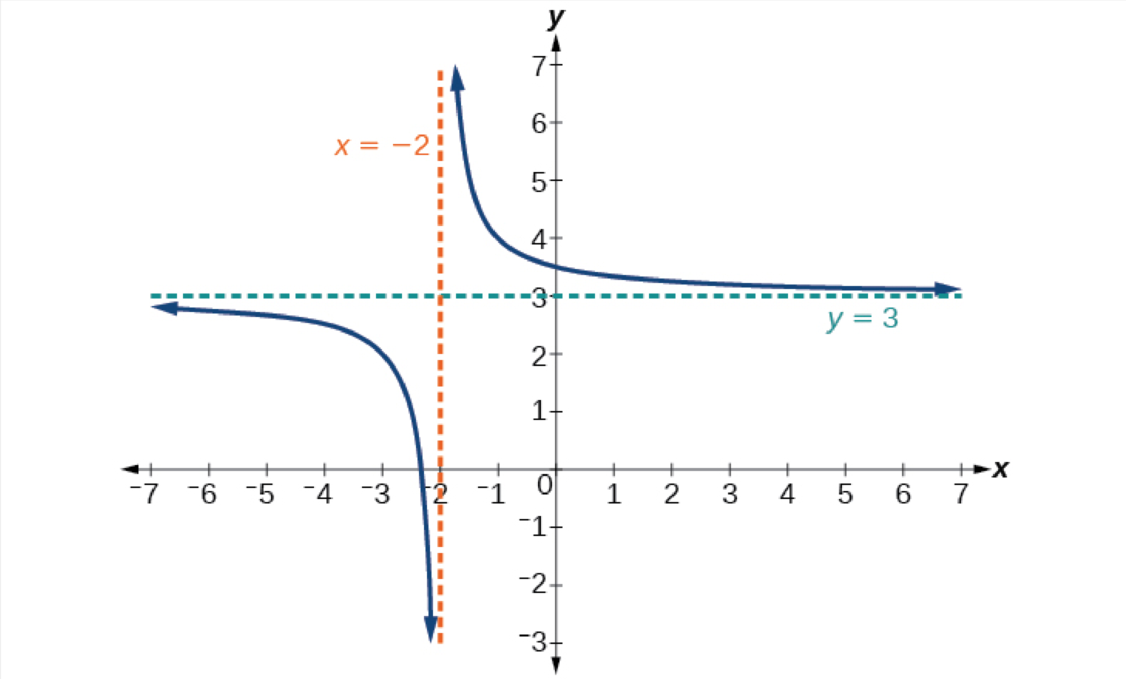
$$f(x) = \frac{1}{x+2} + 3$$

or equivalently, by giving the terms a common denominator,

Equation:

$$f(x) = \frac{3x+7}{x+2}$$

The graph of the shifted function is displayed in [link].



Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

Equation:

$$\text{As } x \to -2^-,\ f(x) \to -\infty, \text{and as } x \to -2^+,\ f(x) \to \infty.$$

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

Equation:

$$\text{As } x \to \pm\infty,\ f(x) \to 3.$$

Analysis

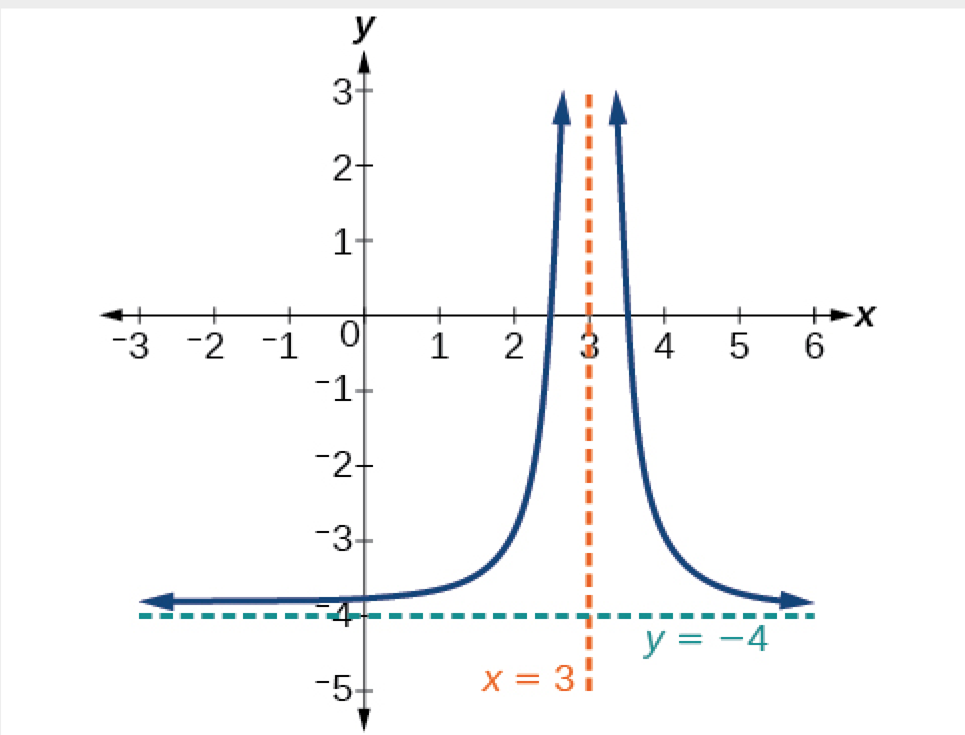
Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

Note:
Exercise:

Problem:

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Solution:



The function and the asymptotes are shifted 3 units right and 4 units down. As $x \to 3, f(x) \to \infty,$ and as $x \to \pm\infty, f(x) \to -4.$

The function is $f(x) = \frac{1}{(x-3)^2} - 4.$

Solving Applied Problems Involving Rational Functions

In [link], we shifted a toolkit function in a way that resulted in the function $f(x) = \frac{3x+7}{x+2}.$ This is an example of a rational function. A **rational function** is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

Note:
Rational Function
A **rational function** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.
Equation:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1} x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + \ldots + b_1 x + b_0}, Q(x) \neq 0$$

Example:
Exercise:

Problem:
Solving an Applied Problem Involving a Rational Function

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

Solution:

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:
Equation:

$$\begin{aligned}&\text{water: } W(t) = 100 + 10t \text{ in gallons}\\&\text{sugar: } S(t) = 5 + 1t \text{ in pounds}\end{aligned}$$

The concentration, C, will be the ratio of pounds of sugar to gallons of water
Equation:

$$C(t) = \frac{5+t}{100+10t}$$

The concentration after 12 minutes is given by evaluating $C(t)$ at $t = 12$.
Equation:

$$\begin{aligned}C(12) &= \tfrac{5+12}{100+10(12)}\\&= \tfrac{17}{220}\end{aligned}$$

This means the concentration is 17 pounds of sugar to 220 gallons of water.

At the beginning, the concentration is

Equation:

$$\begin{aligned} C(0) &= \tfrac{5+0}{100+10(0)} \\ &= \tfrac{1}{20} \end{aligned}$$

Since $\frac{17}{220} \approx 0.08 > \frac{1}{20} = 0.05,$ the concentration is greater after 12 minutes than at the beginning.

Analysis

To find the horizontal asymptote, divide the leading coefficient in the numerator by the leading coefficient in the denominator:

Equation:

$$\frac{1}{10} = 0.1$$

Notice the horizontal asymptote is $y = \ 0.1.$ This means the concentration, $C,$ the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

Note:

Exercise:

Problem:

There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

Solution:

$\frac{12}{11}$

Finding the Domains of Rational Functions

A vertical asymptote represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Note:
Domain of a Rational Function
The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

Note:
Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Example:
Exercise:

Problem:
Finding the Domain of a Rational Function

Find the domain of $f(x) = \frac{x+3}{x^2-9}$.

Solution:

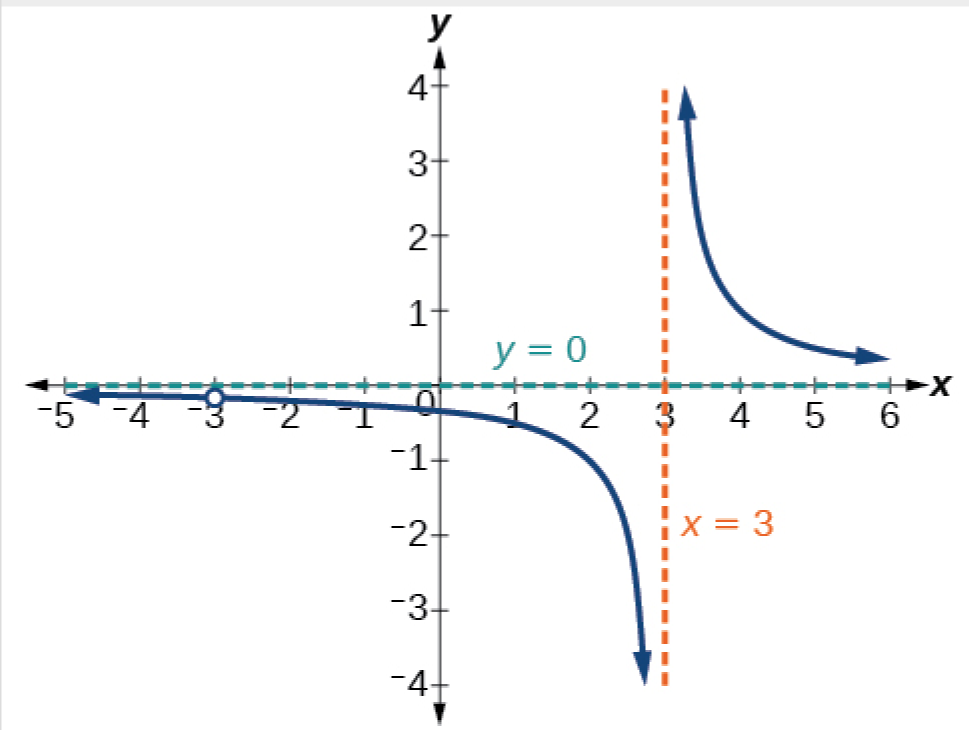
Begin by setting the denominator equal to zero and solving.
Equation:

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

Analysis

A graph of this function, as shown in [link], confirms that the function is not defined when $x = \pm 3$.



There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

Note:
Exercise:

Problem: Find the domain of $f(x) = \frac{4x}{5(x-1)(x-5)}$.

Solution:

The domain is all real numbers except $x = 1$ and $x = 5$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

Note:
Given a rational function, identify any vertical asymptotes of its graph.

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities.

Example:
Exercise:

Problem:
Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \frac{5+2x^2}{2-x-x^2}$.

Solution:

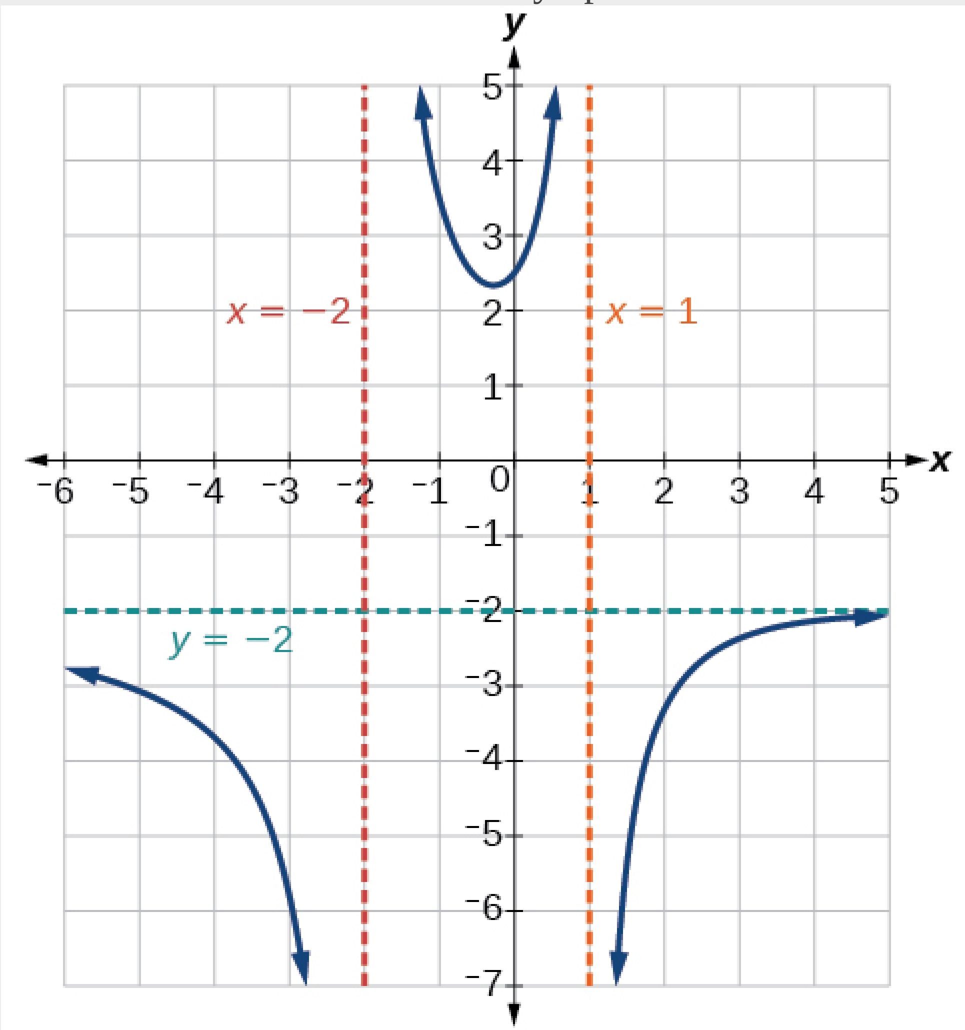
First, factor the numerator and denominator.
Equation:

$$\begin{aligned} k(x) &= \frac{5+2x^2}{2-x-x^2} \\ &= \frac{5+2x^2}{(2+x)(1-x)} \end{aligned}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:
Equation:

$$\begin{aligned} (2+x)(1-x) &= 0 \\ x &= -2, 1 \end{aligned}$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in [link] confirms the location of the two vertical asymptotes.



Removable Discontinuities

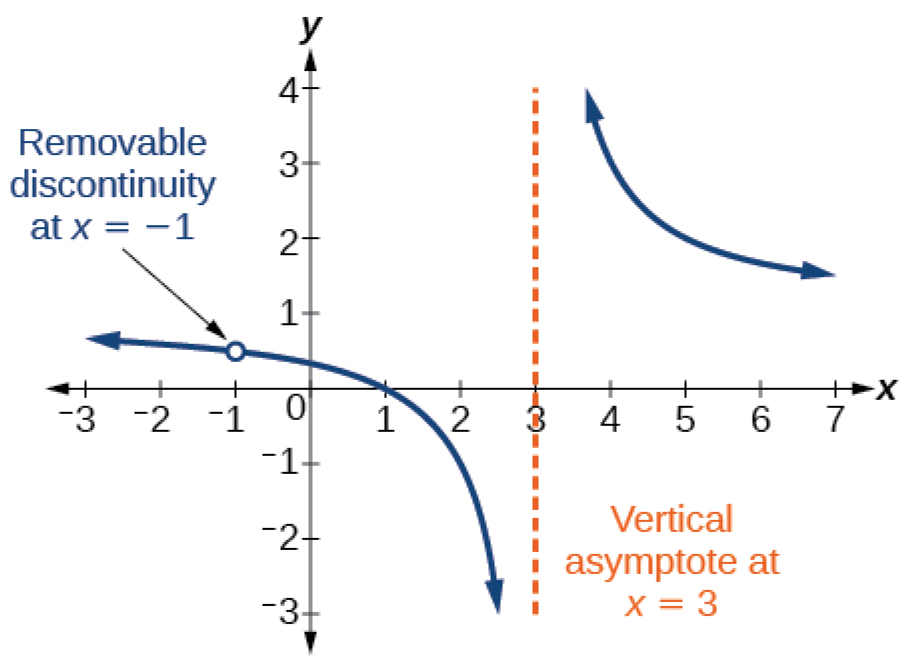
Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a **removable discontinuity**.

For example, the function $f(x) = \frac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

Equation:

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1,$ is the location of the removable discontinuity. Notice also that $x–3$ is not a factor in both the numerator and denominator. The zero of this factor, $x = 3,$ is the vertical asymptote. See [link].



Note:
Removable Discontinuities of Rational Functions
A **removable discontinuity** occurs in the graph of a rational function at $x = a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal

to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Example:
Exercise:

Problem:
Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x)=\frac{x-2}{x^2-4}$.

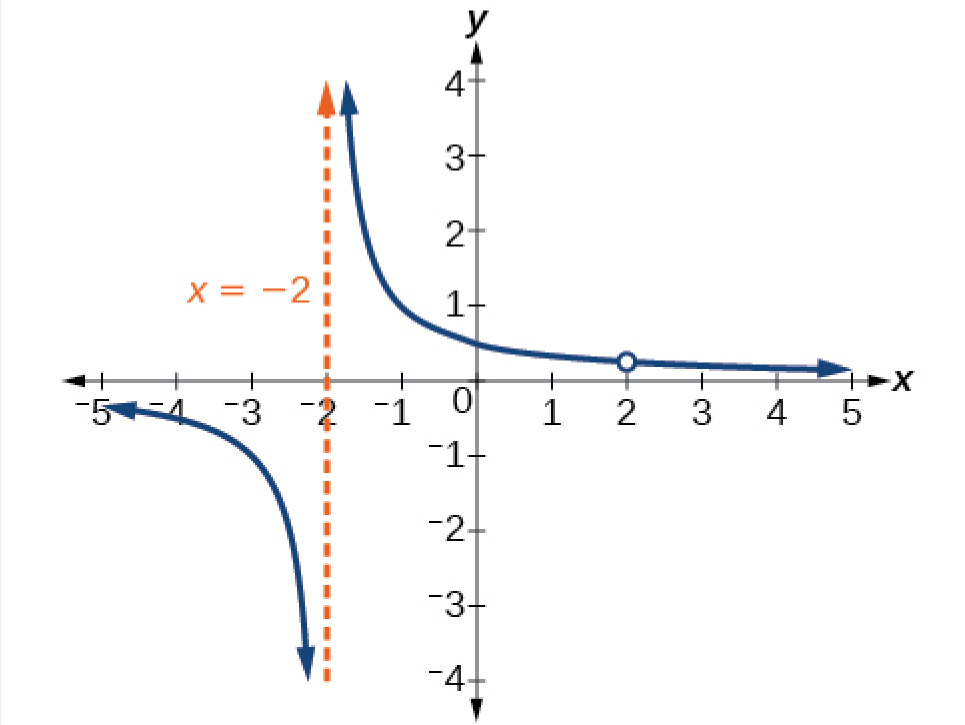
Solution:

Factor the numerator and the denominator.
Equation:

$$k(x)=\frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x–2$. The zero for this factor is $x=2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2$. The zero for this factor is $x=-2$. The vertical asymptote is $x=-2$. See [link].



The graph of this function will have the vertical asymptote at $x = -2,$ but at $x = 2$ the graph will have a hole.

Note:
Exercise:

Problem:

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \frac{x^2 - 25}{x^3 - 6x^2 + 5x}$.

Solution:

Removable discontinuity at $x = 5$. Vertical asymptotes: $x = 0,\ x = 1.$

Identifying Horizontal Asymptotes of Rational Functions

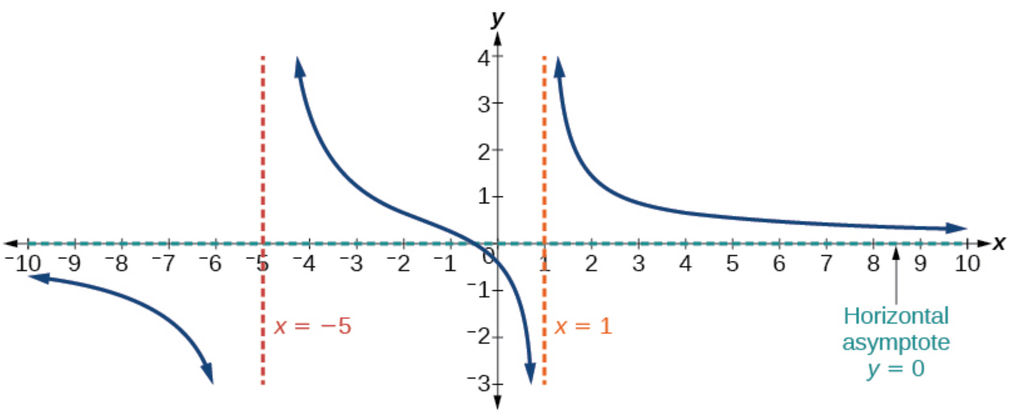
While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial’s end behavior will mirror that of the leading term. Likewise, a rational function’s end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.

There are three distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator > degree of the numerator, there is a horizontal asymptote at $y = 0$.
Equation:

$$\text{Example: } f(x) = \frac{4x+2}{x^2+4x-5}$$

In this case, the end behavior is $f(x) \approx \frac{4x}{x^2} = \frac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = \frac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y = 0$. See [link]. Note that this graph crosses the horizontal asymptote.



Horizontal Asymptote $y = 0$ when

$$f(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0 \text{ where degree of } p < degree\ of\ q.$$

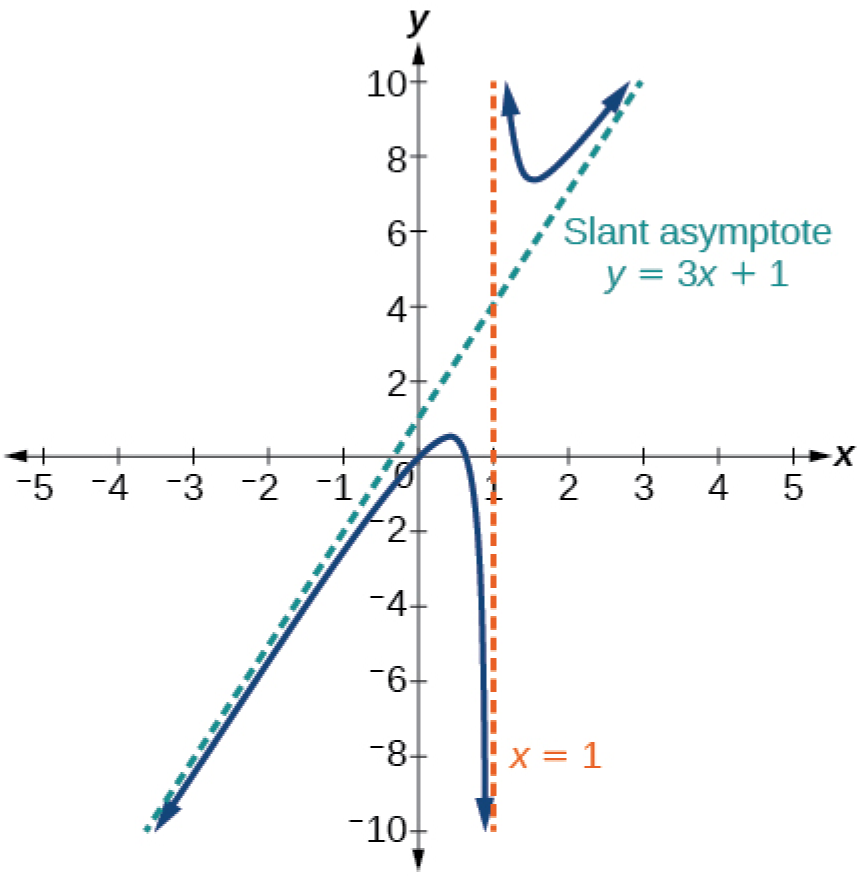
Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

Equation:

$$\text{Example: } f(x) = \frac{3x^2 - 2x + 1}{x - 1}$$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x} = 3x.$ This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = 3x.$ As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x) = 3x$ looks like a diagonal line, and since f will behave similarly to $g,$ it will approach a line close to $y = 3x.$ This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\frac{3x^2-2x+1}{x-1}.$ The quotient is $3x + 1,$ and the remainder is 2. The slant asymptote is the graph of the line $g(x) = 3x + 1$. See [link].



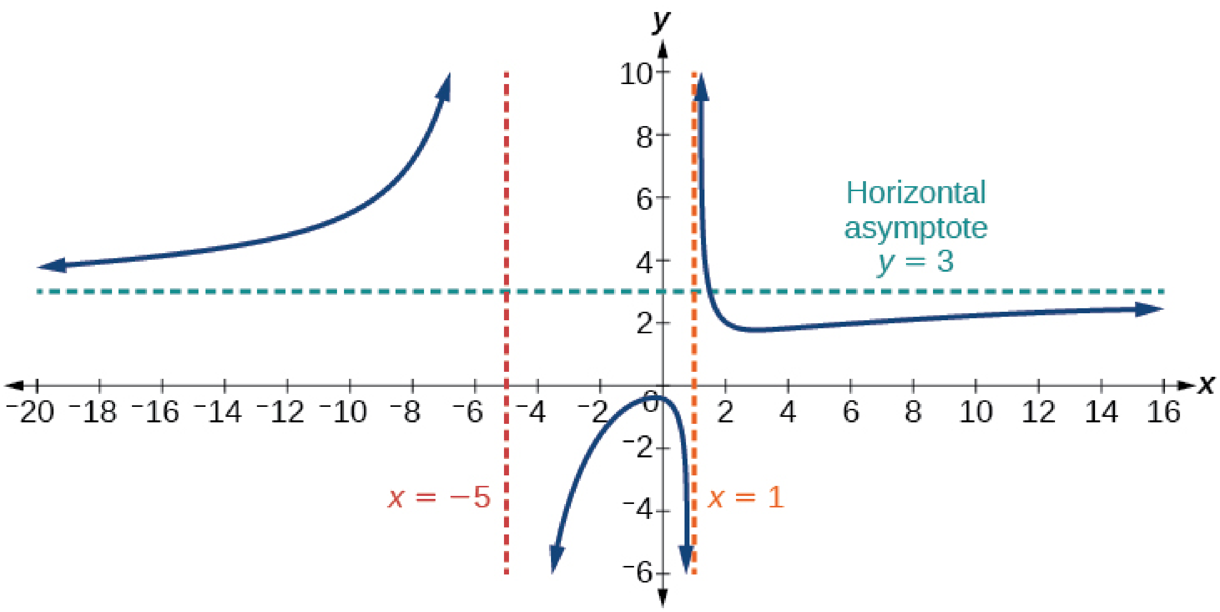
Slant Asymptote when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p >$ degree of q by 1.

Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \frac{a_n}{b_n}$, where a_n and b_n are the leading coefficients of $p\,(x)$ and $q\,(x)$ for $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$.

Equation:

$$\text{Example: } f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$. See [link]. Note that this graph crosses the horizontal asymptote.



Horizontal Asymptote when $f(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0$ where degree of $p =$ degree of q.

Notice that, while the graph of a rational function will never cross a vertical asymptote, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the end behavior of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

Equation:

$$f(x) = \frac{3x^5 - x^2}{x+3}$$

with end behavior

Equation:

$$f(x) \approx \frac{3x^5}{x} = 3x^4,$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

Equation:

$$x \to \pm\infty,\ f(x) \to \infty$$

Note:

Horizontal Asymptotes of Rational Functions

The horizontal asymptote of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator *is less than* degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator *is greater than degree of denominator by one*: no horizontal asymptote; slant asymptote.
- Degree of numerator *is equal to* degree of denominator: horizontal asymptote at ratio of leading coefficients.

Example:

Exercise:

Problem:
Identifying Horizontal and Slant Asymptotes

For the functions below, identify the horizontal or slant asymptote.

a. $g(x) = \frac{6x^3-10x}{2x^3+5x^2}$
b. $h(x) = \frac{x^2-4x+1}{x+2}$
c. $k(x) = \frac{x^2+4x}{x^3-8}$

Solution:

For these solutions, we will use $f(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0.$

a. $g(x) = \frac{6x^3-10x}{2x^3+5x^2}$: The degree of $p =$ degree of $q = 3,$ so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3.$
b. $h(x) = \frac{x^2-4x+1}{x+2}$: The degree of $p = 2$ and degree of $q = 1.$ Since $p > q$ by 1, there is a slant asymptote found at $\frac{x^2-4x+1}{x+2}.$
Equation:

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

The quotient is $x-6$ and the remainder is 13. There is a slant asymptote at $y = x-6.$
c. $k(x) = \frac{x^2+4x}{x^3-8}$: The degree of $p = 2\ <$ degree of $q = 3,$ so there is a horizontal asymptote $y = 0.$

Example:
Exercise:

Problem:
Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \frac{5+t}{100+10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution:

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

Equation:

$$t \to \infty,\ C(t) \to \frac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example:
Exercise:

Problem:
Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function
Equation:

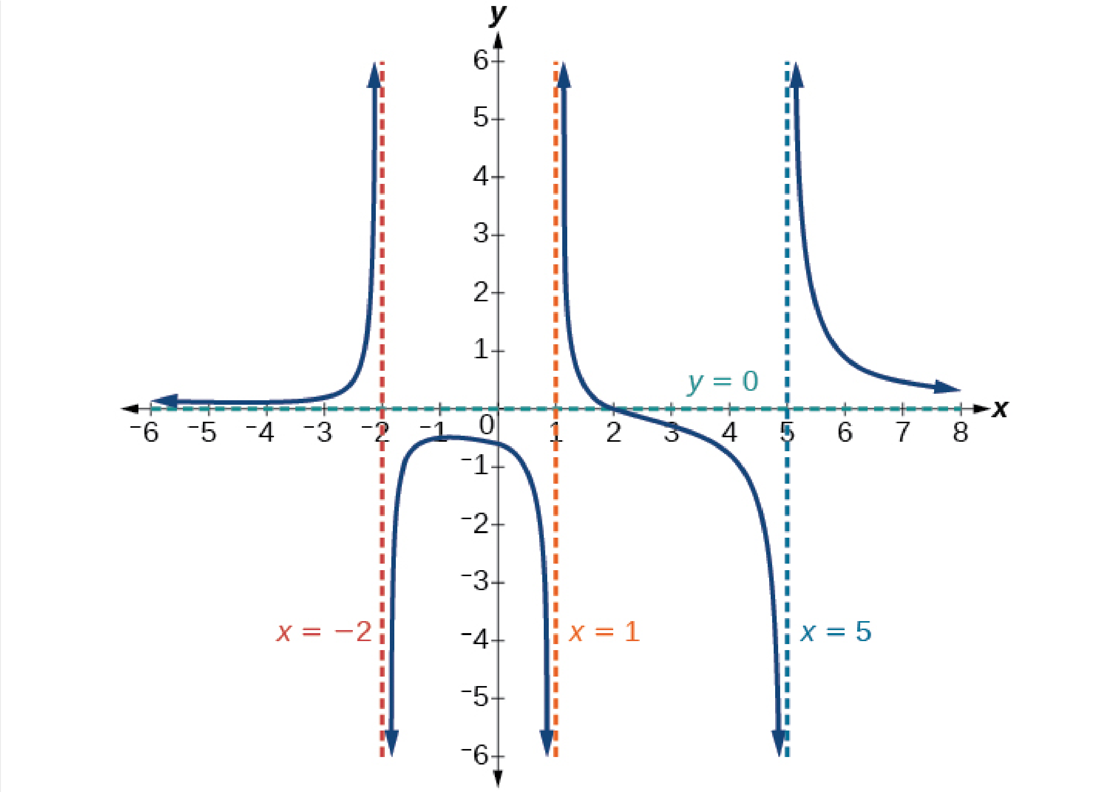
$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

Solution:

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2, \text{and } 5,$ indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty, \ f(x) \to 0.$ This function will have a horizontal asymptote at $y = 0.$ See [link].



Note:
Exercise:

Problem: Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Solution:

Vertical asymptotes at $x = 2$ and $x = -3$; horizontal asymptote at $y = 4$.

Note:

Intercepts of Rational Functions

A rational function will have a *y*-intercept when the input is zero, if the function is defined at zero. A rational function will not have a *y*-intercept if the function is not defined at zero.

Likewise, a rational function will have *x*-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, *x*-intercepts can only occur when the numerator of the rational function is equal to zero.

Example:
Exercise:

Problem:
Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution:

We can find the *y*-intercept by evaluating the function at zero

Equation:

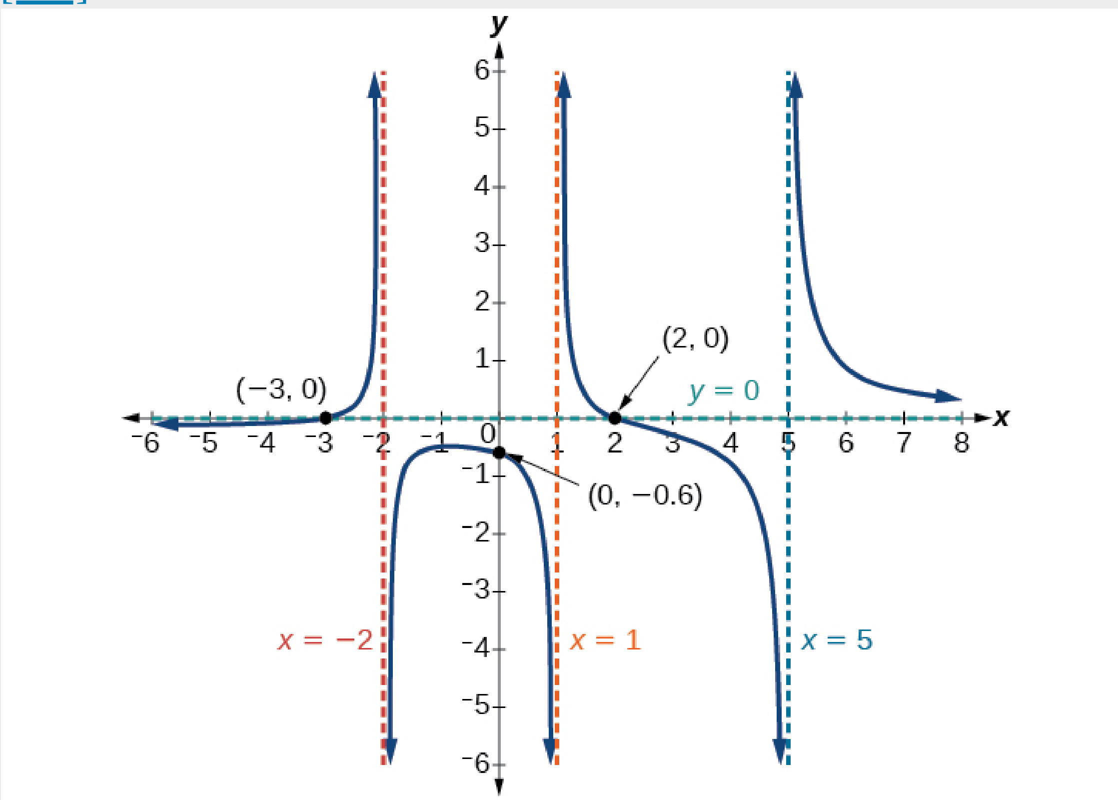
$$\begin{aligned} f(0) &= \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} \\ &= \frac{-6}{10} \\ &= -\frac{3}{5} \\ &= -0.6 \end{aligned}$$

The *x*-intercepts will occur when the function is equal to zero:

Equation:

$$\begin{aligned} 0 &= \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} && \text{This is zero when the numerator is zero.} \\ 0 &= (x-2)(x+3) \\ x &= 2, \ -3 \end{aligned}$$

The *y*-intercept is $(0,–0.6),$ the *x*-intercepts are $(2,0)$ and $(–3,0).$ See [link].



Note:
Exercise:

Problem:

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the *x*- and *y*-intercepts and the horizontal and vertical asymptotes.

Solution:

For the transformed reciprocal squared function, we find the rational form.

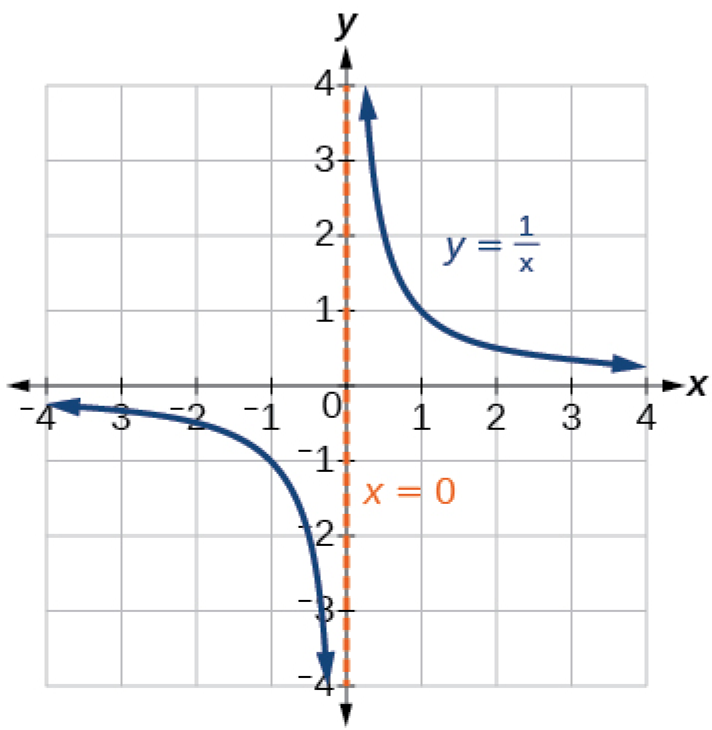
$$f(x) = \frac{1}{(x-3)^2} - 4 = \frac{1-4(x-3)^2}{(x-3)^2} = \frac{1-4(x^2-6x+9)}{(x-3)(x-3)} = \frac{-4x^2+24x-35}{x^2-6x+9}$$

Because the numerator is the same degree as the denominator we know that as $x \to \pm\infty,\ f(x) \to -4;$ so $y =- 4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x = 3,$ because as $x \to 3, f(x) \to \infty.$ We then set the numerator equal to 0 and find the *x*-intercepts are at $(2.5, 0)$ and $(3.5, 0).$ Finally, we evaluate the function at 0 and find the *y*-intercept to be at $\left(0, \frac{-35}{9}\right).$

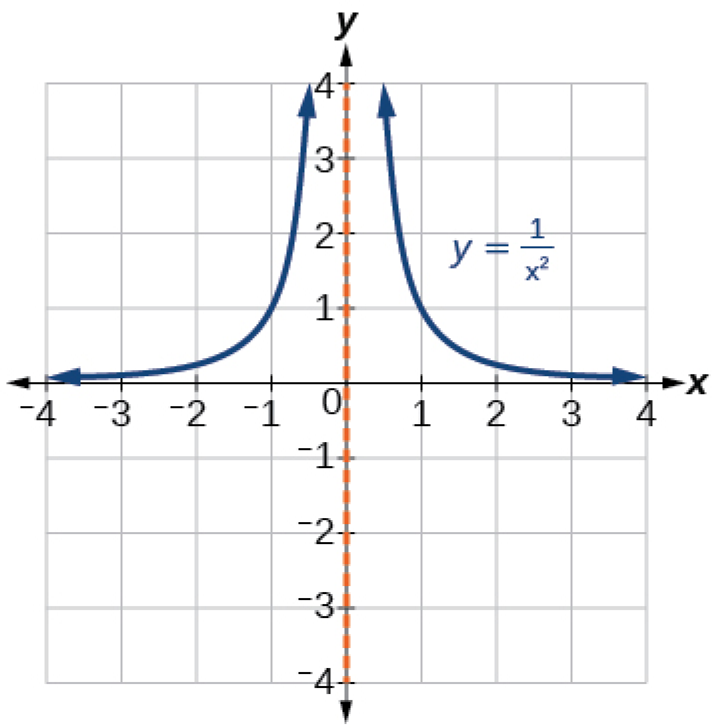
Graphing Rational Functions

In [link], we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

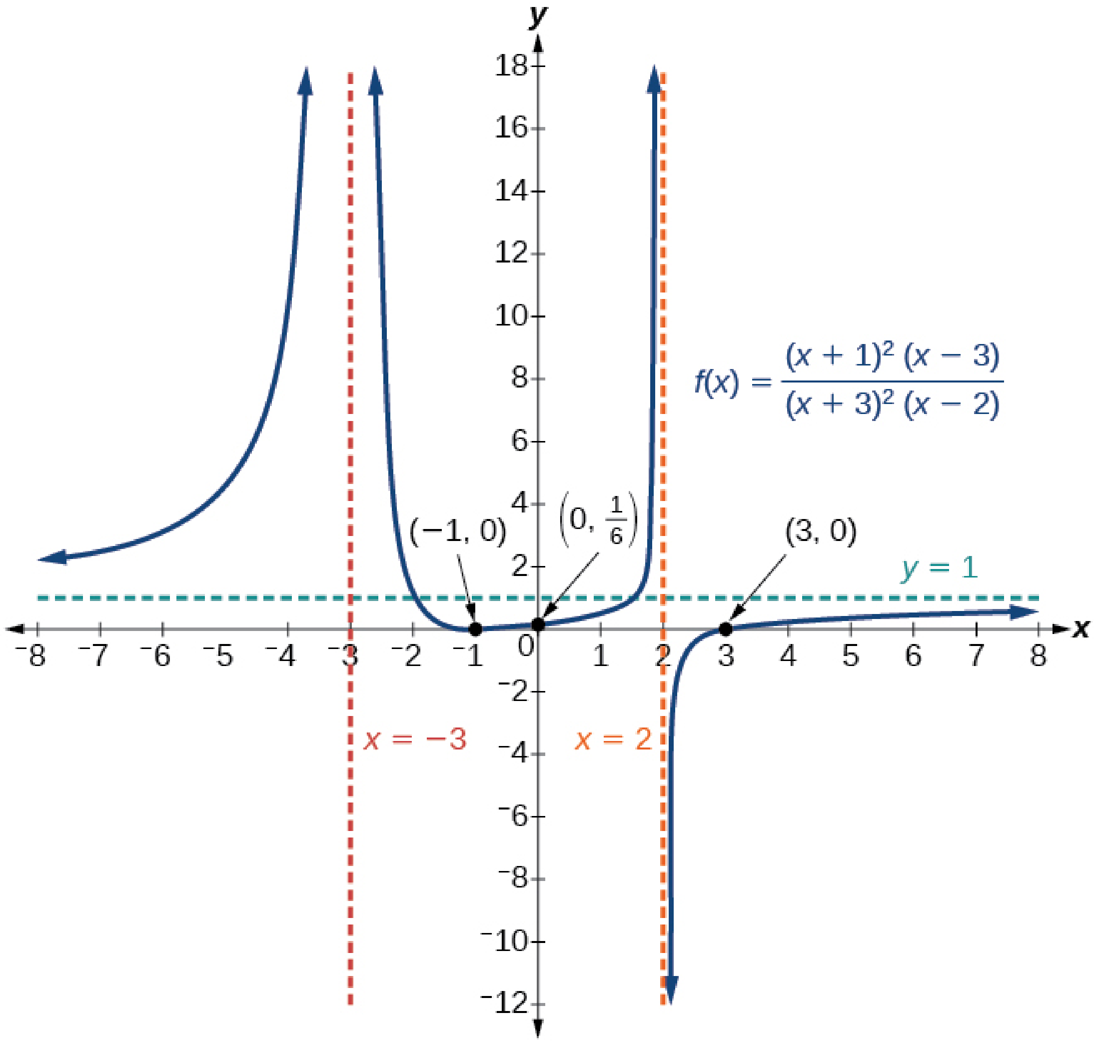
The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See [link].



When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See [link].



For example, the graph of $f(x)=\frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in [link].



- At the *x*-intercept $x=-1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph bounces, consistent with the quadratic nature of the factor.
- At the *x*-intercept $x=3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x=-3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x)=\frac{1}{x^2}$.
- At the vertical asymptote $x=2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the

left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x)=\frac{1}{x}$.

Note:
Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the *y*-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the *x*-intercepts.
4. Find the multiplicities of the *x*-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Example:
Exercise:

Problem:
Graphing a Rational Function

Sketch a graph of $f(x)=\frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution:

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the *y*-intercept:

Equation:

$$\begin{aligned} f(0) &= \frac{(0+2)(0-3)}{(0+1)^2(0-2)} \\ &= 3 \end{aligned}$$

To find the *x*-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find *x*-intercepts at $x = -2$ and $x = 3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

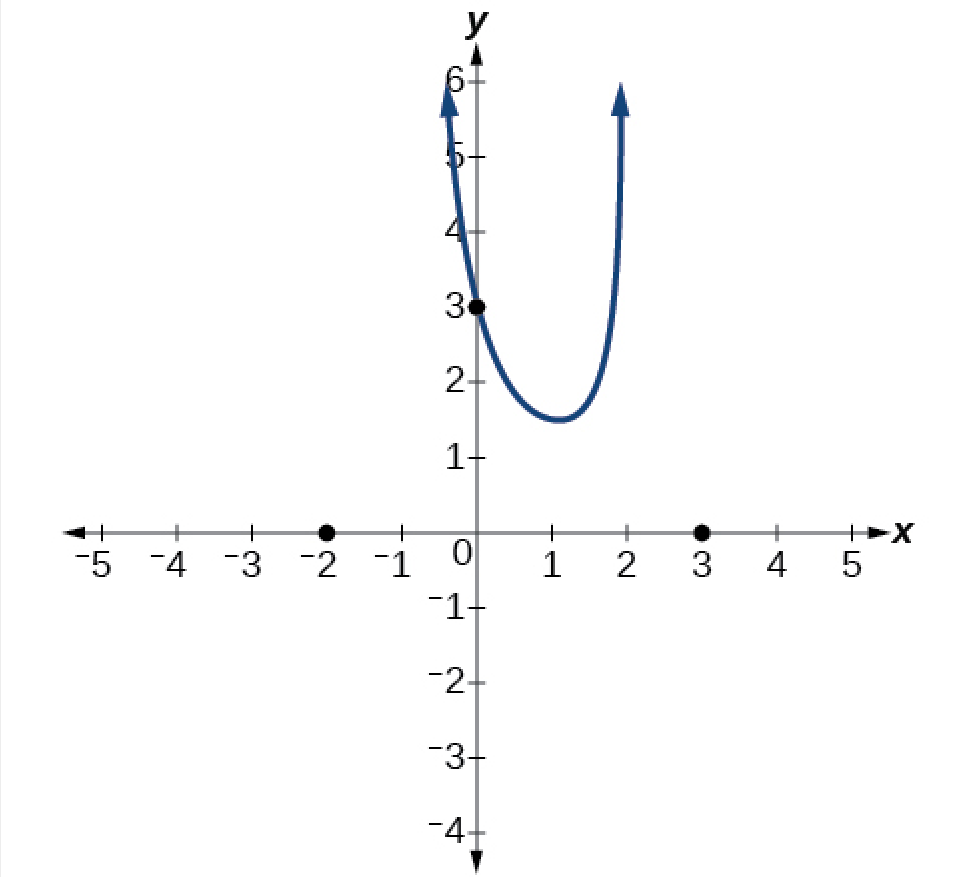
We have a *y*-intercept at $(0, 3)$ and *x*-intercepts at $(-2, 0)$ and $(3, 0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x + 1 = 0$ and when $x - 2 = 0$, giving us vertical asymptotes at $x = -1$ and $x = 2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

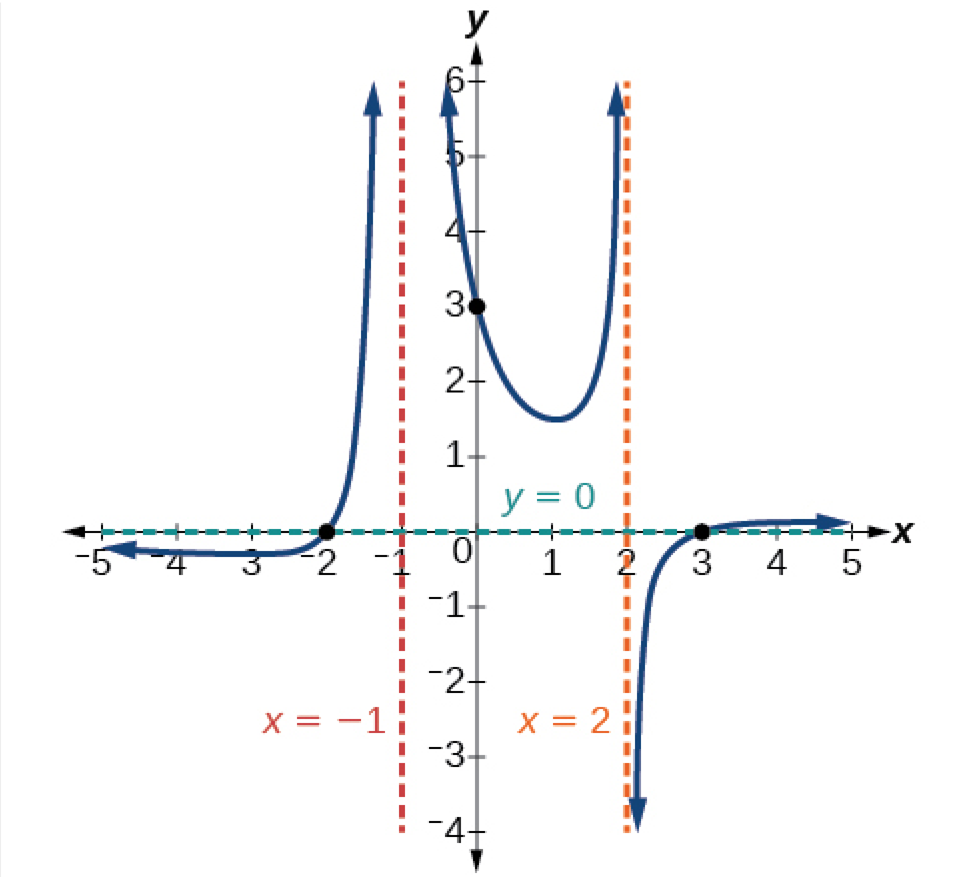
Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no *x*-intercepts between the vertical asymptotes, and the *y*-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in [link].



The factor associated with the vertical asymptote at $x = -1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See [link]. After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.



Note:
Exercise:

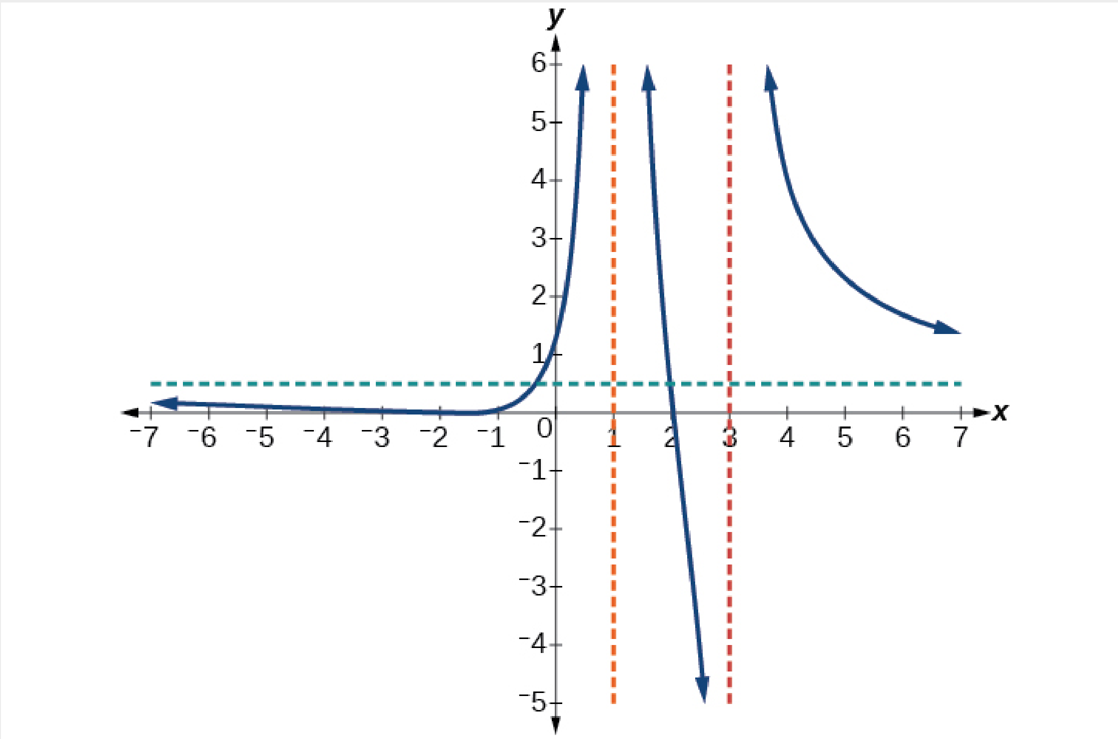
Problem:

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Solution:

Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ and $x = 3$. y-intercept at $\left(0, \frac{4}{3}.\right)$

x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity 2, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the axis at this point.



Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an x-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of x-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

Note:

Writing Rational Functions from Intercepts and Asymptotes

If a rational function has *x*-intercepts at $x = x_1,\ x_2,\ ...,\ x_n,$ vertical asymptotes at $x = v_1, v_2, \ldots, v_m,$ and no $x_i =$ any $v_j,$ then the function can be written in the form:

Equation:

$$f(x) = a\frac{(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}}{(x - v_1)^{q_1}(x - v_2)^{q_2} \cdots (x - v_m)^{q_n}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the *x*-intercept or by the horizontal asymptote if it is nonzero.

Note:

Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the *x*-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the "simplest" function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

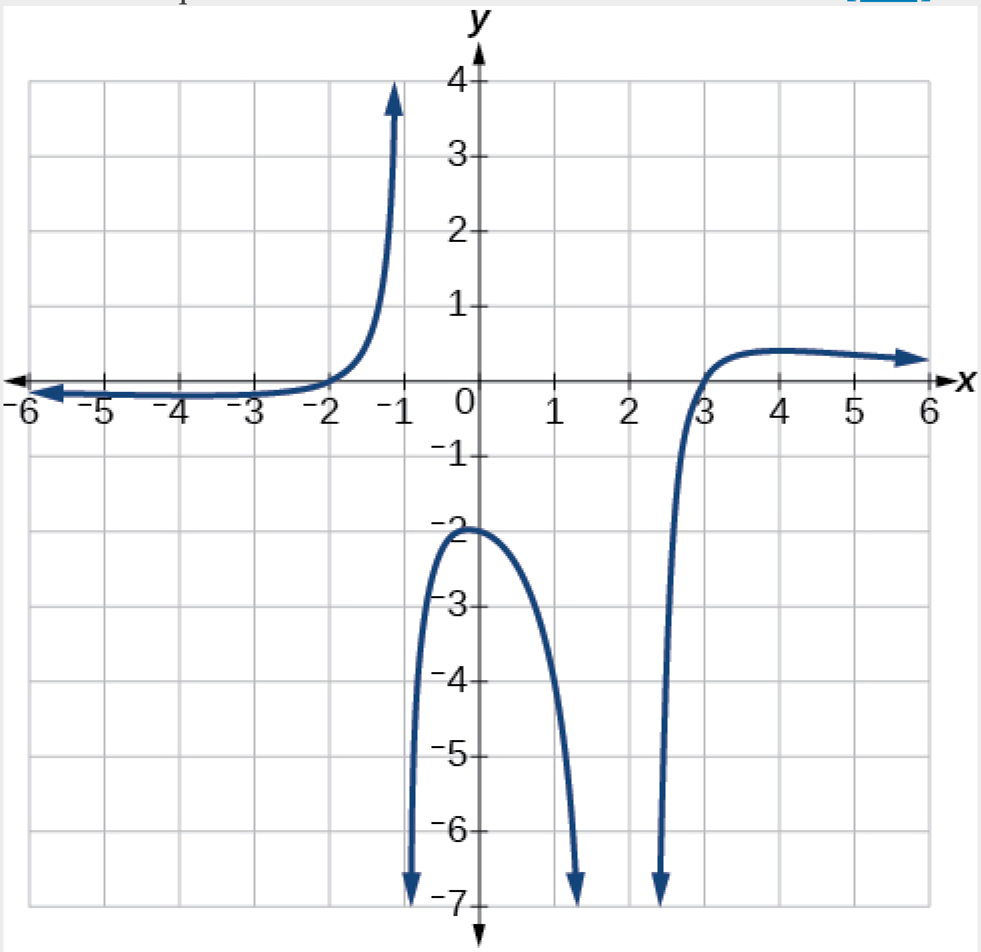
Example:

Exercise:

Problem:

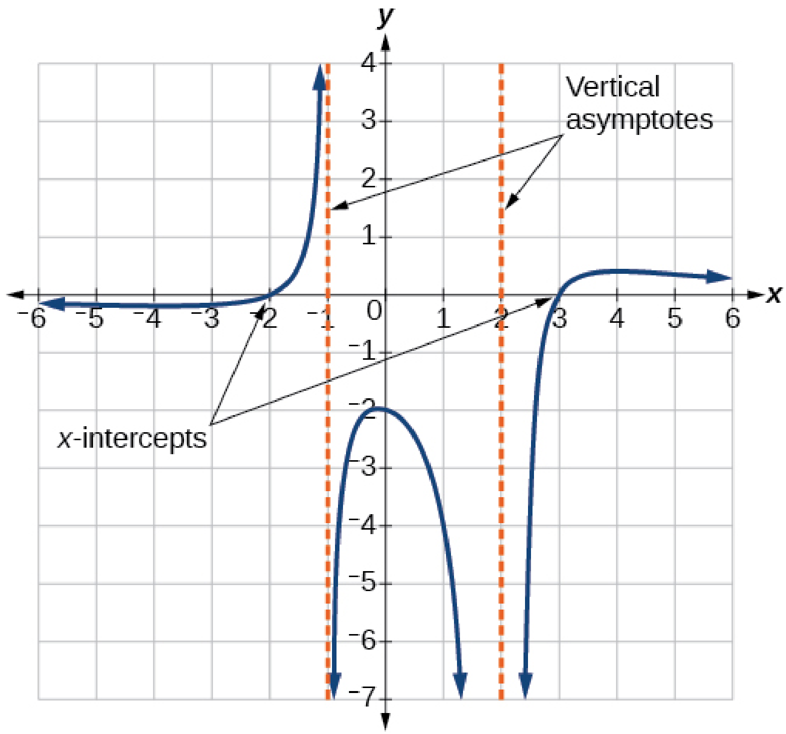
Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in [link].



Solution:

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See [link].



We can use this information to write a function of the form

Equation:

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}.$$

To find the stretch factor, we can use another clear point on the graph, such as the y-intercept $(0,-2)$.

Equation:

$$\begin{aligned} -2 &= a\tfrac{(0+2)(0-3)}{(0+1)(0-2)^2} \\ -2 &= a\tfrac{-6}{4} \\ a &= \tfrac{-8}{-6} = \tfrac{4}{3} \end{aligned}$$

This gives us a final function of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

Note:
Access these online resources for additional instruction and practice with rational functions.

- [Graphing Rational Functions]
- [Find the Equation of a Rational Function]
- [Determining Vertical and Horizontal Asymptotes]
- [Find the Intercepts, Asymptotes, and Hole of a Rational Function]

Key Equations

Rational Function	$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1} x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + \ldots + b_1 x + b_0},\ Q(x) \neq 0$

Key Concepts

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. See [link].
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See [link].
- Application problems involving rates and concentrations often involve rational functions. See [link].
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See [link].
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See [link].

- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See [link].
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See [link], [link], [link], and [link].
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See [link].
- If a rational function has *x*-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i = \text{any } v_j$, then the function can be written in the form

  Equation:

  $$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$$

  See [link].

Section Exercises

Verbal

Exercise:

Problem:

What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?

Solution:

The rational function will be represented by a quotient of polynomial functions.

Exercise:

Problem:

What is the fundamental difference in the graphs of polynomial functions and rational functions?

Exercise:

Problem:

If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?

Solution:

The numerator and denominator must have a common factor.

Exercise:

Problem:

Can a graph of a rational function have no vertical asymptote? If so, how?

Exercise:

Problem:

Can a graph of a rational function have no *x*-intercepts? If so, how?

Solution:

Yes. The numerator of the formula of the functions would have only complex roots and/or factors common to both the numerator and denominator.

Algebraic

For the following exercises, find the domain of the rational functions.

Exercise:

Problem: $f(x) = \frac{x-1}{x+2}$

Exercise:

Problem: $f(x) = \frac{x+1}{x^2-1}$

Solution:

All reals $x \neq -1,\ 1$

Exercise:

Problem: $f(x) = \frac{x^2+4}{x^2-2x-8}$

Exercise:

Problem: $f(x) = \frac{x^2+4x-3}{x^4-5x^2+4}$

Solution:

All reals $x \neq -1,\ -2,\ 1,\ 2$

For the following exercises, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.

Exercise:

Problem: $f(x) = \frac{4}{x-1}$

Exercise:

Problem: $f\left(x\right) = \frac{2}{5x+2}$

Solution:

V.A. at $x = -\frac{2}{5}$; H.A. at $y = 0$; Domain is all reals $x \neq -\frac{2}{5}$

Exercise:

Problem: $f(x) = \frac{x}{x^2-9}$

Exercise:

Problem: $f(x) = \frac{x}{x^2+5x-36}$

Solution:

V.A. at $x = 4,\ -9;$ H.A. at $y = 0;$ Domain is all reals $x \neq 4,\ -9$

Exercise:

Problem: $f\ (x) = \frac{3+x}{x^3-27}$

Exercise:

Problem: $f(x) = \frac{3x-4}{x^3-16x}$

Solution:

V.A. at $x = 0,\ 4,\ \ -4;$ H.A. at $y = 0;$Domain is all reals $x \neq 0, 4,\ -4$

Exercise:

Problem: $f(x) = \frac{x^2-1}{x^3+9x^2+14x}$

Exercise:

Problem: $f(x) = \frac{x+5}{x^2-25}$

Solution:

V.A. at $x = -5;$ H.A. at $y = 0;$ Domain is all reals $x \neq 5, -5$

Exercise:

Problem: $f(x) = \frac{x-4}{x-6}$

Exercise:

Problem: $f\left(x\right) = \frac{4-2x}{3x-1}$

Solution:

V.A. at $x = \frac{1}{3};$ H.A. at $y = -\frac{2}{3};$ Domain is all reals $x \neq \frac{1}{3}.$

For the following exercises, find the *x*- and *y*-intercepts for the functions.

Exercise:

Problem: $f(x) = \frac{x+5}{x^2+4}$

Exercise:

Problem: $f(x) = \frac{x}{x^2-x}$

Solution:

none

Exercise:

Problem: $f(x) = \frac{x^2+8x+7}{x^2+11x+30}$

Exercise:

Problem: $f(x) = \frac{x^2+x+6}{x^2-10x+24}$

Solution:

x-intercepts none, y-intercept $\left(0, \frac{1}{4}\right)$

Exercise:

Problem: $f(x) = \frac{94-2x^2}{3x^2-12}$

For the following exercises, describe the local and end behavior of the functions.

Exercise:

Problem: $f\left(x\right) = \frac{x}{2x+1}$

Solution:

Local behavior: $x \to -\frac{1}{2}^+, f(x) \to -\infty, x \to -\frac{1}{2}^-, f(x) \to \infty$

End behavior: $x \to \pm\infty, f(x) \to \frac{1}{2}$

Exercise:

Problem: $f\left(x\right) = \frac{2x}{x-6}$

Exercise:

Problem: $f\left(x\right) = \frac{-2x}{x-6}$

Solution:

Local behavior: $x \to 6^+, f(x) \to -\infty, x \to 6^-, f(x) \to \infty,$ End behavior: $x \to \pm\infty, f(x) \to -2$

Exercise:

Problem: $f\left(x\right) = \frac{x^2-4x+3}{x^2-4x-5}$

Exercise:

Problem: $f\left(x\right)=\frac{2x^2-32}{6x^2+13x-5}$

Solution:

Local behavior: $x\to -\frac{1}{3}^{+}, f(x)\to \infty, x\to -\frac{1}{3}^{-},$
$f(x)\to -\infty, x\to \frac{5}{2}^{-}, f(x)\to \infty, x\to \frac{5}{2}^{+}, f(x)\to -\infty$

End behavior: $x\to \pm\infty,\ f(x)\to \frac{1}{3}$

For the following exercises, find the slant asymptote of the functions.
Exercise:

Problem: $f(x)=\frac{24x^2+6x}{2x+1}$

Exercise:

Problem: $f(x)=\frac{4x^2-10}{2x-4}$

Solution:

$y=2x+4$

Exercise:

Problem: $f(x)=\frac{81x^2-18}{3x-2}$

Exercise:

Problem: $f(x)=\frac{6x^3-5x}{3x^2+4}$

Solution:

$y=2x$

Exercise:

Problem: $f(x) = \frac{x^2+5x+4}{x-1}$

Graphical

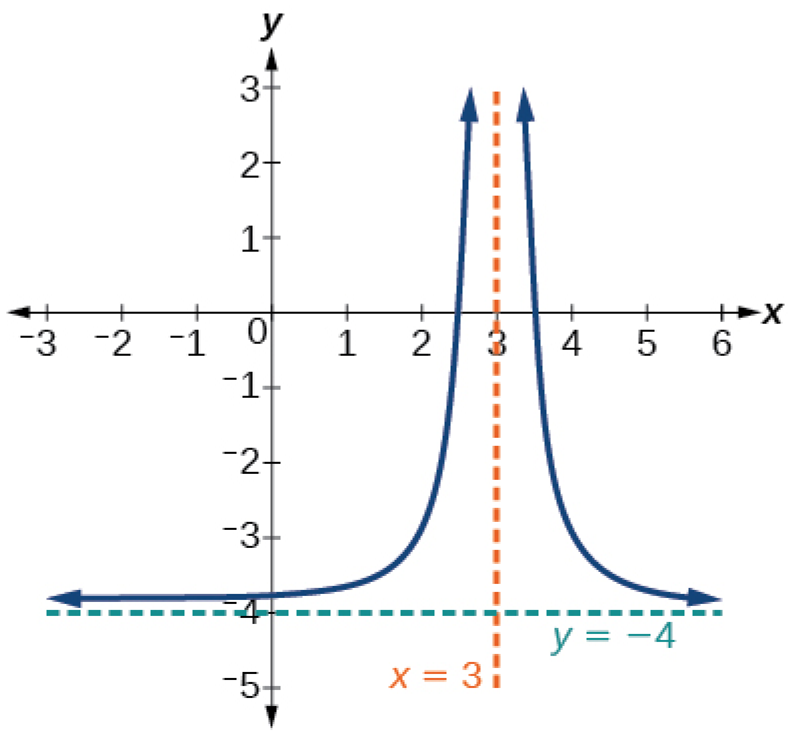
For the following exercises, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.

Exercise:

Problem: The reciprocal function shifted up two units.

Solution:

$V.A.\ x = 0, H.A.\ y = 2$



Exercise:

Problem:

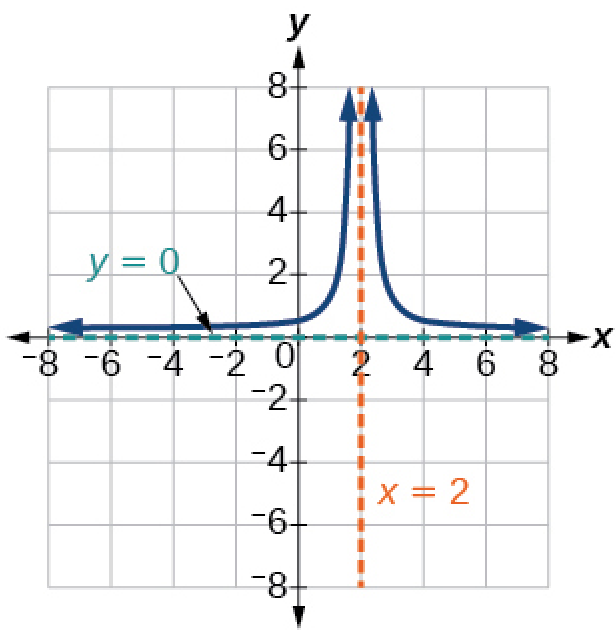
The reciprocal function shifted down one unit and left three units.

Exercise:

Problem: The reciprocal squared function shifted to the right 2 units.

Solution:

$V.A.\ x=2,\ H.A.\ y=0$



Exercise:

Problem:

The reciprocal squared function shifted down 2 units and right 1 unit.

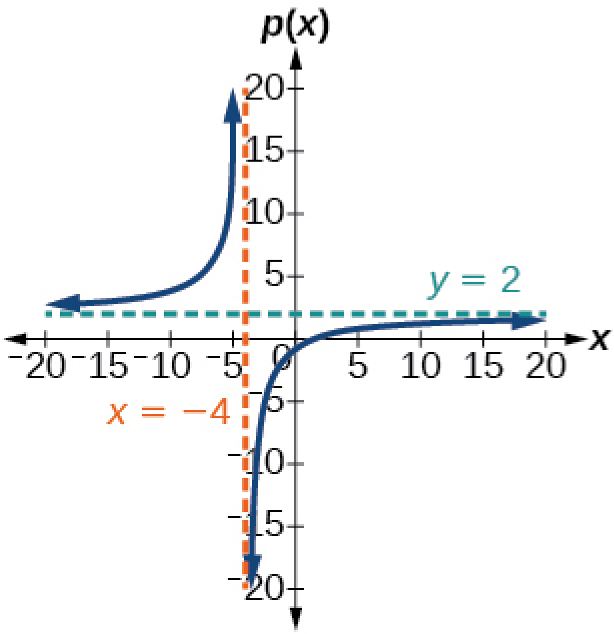
For the following exercises, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.

Exercise:

Problem: $p\left(x\right)=\frac{2x-3}{x+4}$

Solution:

$V.A.\ x=-4,\ H.A.\ y=2;\left(\frac{3}{2},0\right);\left(0,-\frac{3}{4}\right)$



Exercise:

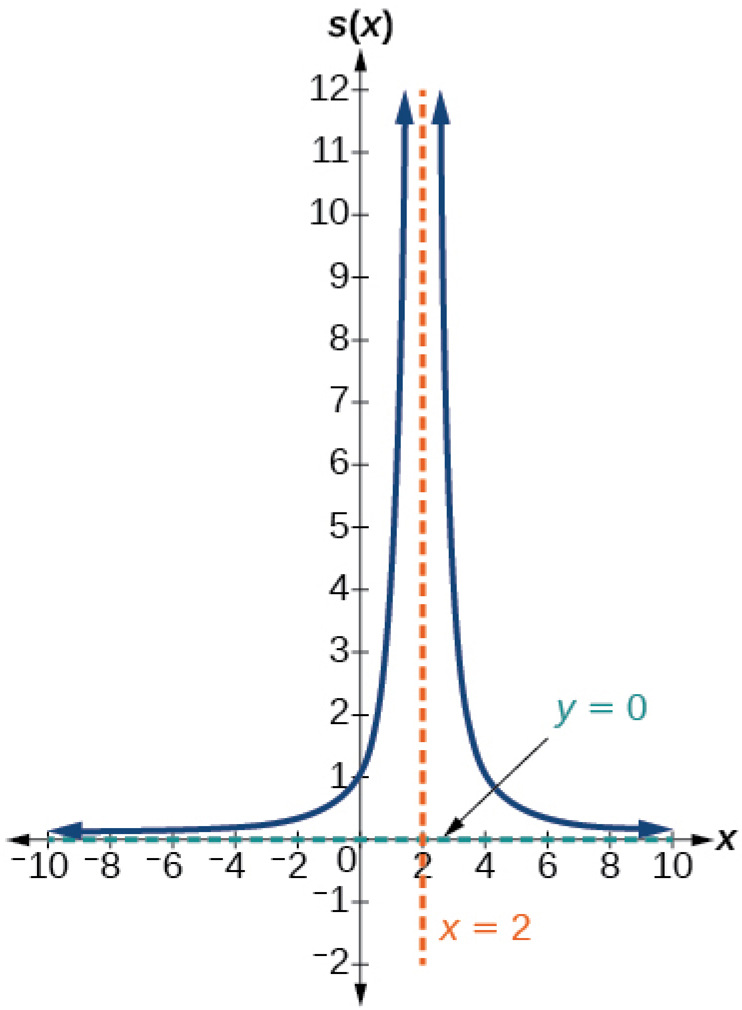
Problem: $q\left(x\right)=\frac{x-5}{3x-1}$

Exercise:

Problem: $s\left(x\right)=\frac{4}{{\left(x-2\right)}^{2}}$

Solution:

$V.A.\ x=2,\ H.A.\ y=0,\ \left(0,1\right)$



Exercise:

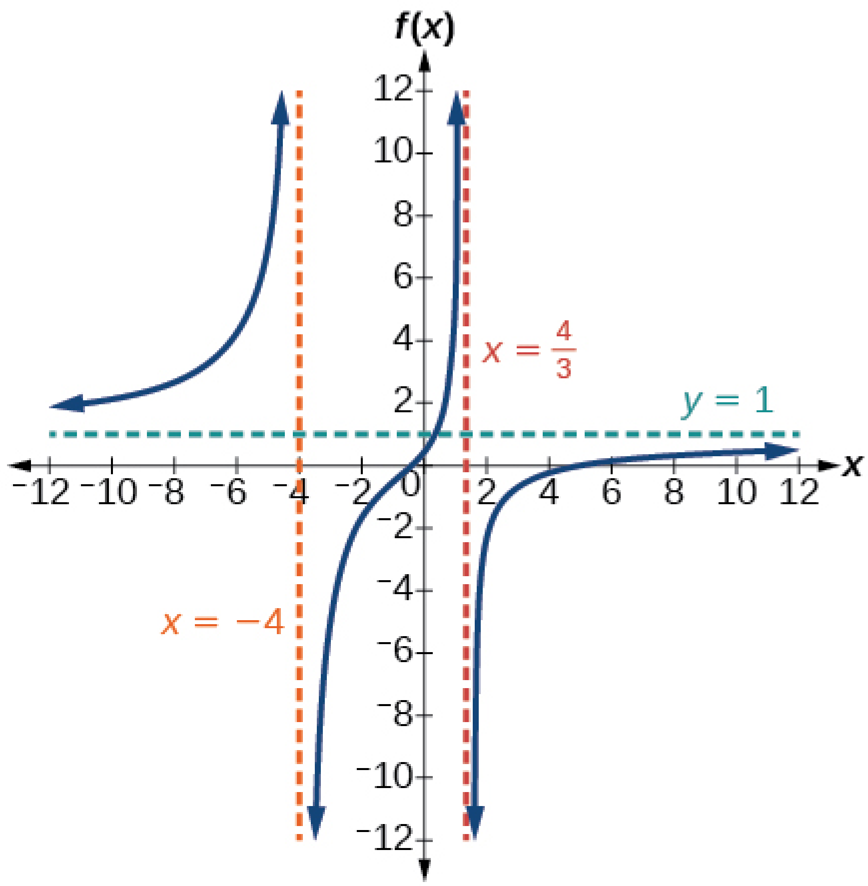
Problem: $r\left(x\right)=\frac{5}{{\left(x+1\right)}^{2}}$

Exercise:

Problem: $f\left(x\right)=\frac{3{x}^{2}-14x-5}{3{x}^{2}+8x-16}$

Solution:

$V.A.\text{ }x=-4,\text{ }x=\frac{4}{3},\text{ }H.A.\text{ }y=1;\left(5,0\right);\left(-\frac{1}{3},0\right);\left(0,\frac{5}{16}\right)$



Exercise:

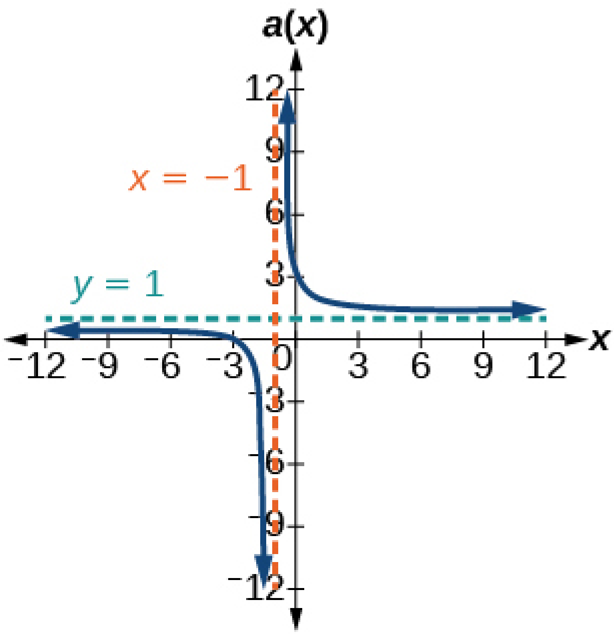
Problem: $g\left(x\right)=\frac{2{x}^{2}+7x-15}{3{x}^{2}-14+15}$

Exercise:

Problem: $a\left(x\right)=\frac{{x}^{2}+2x-3}{{x}^{2}-1}$

Solution:

$V.A.\ x=-1,\ H.A.\ y=1;\left(-3,0\right);\left(0,3\right)$



Exercise:

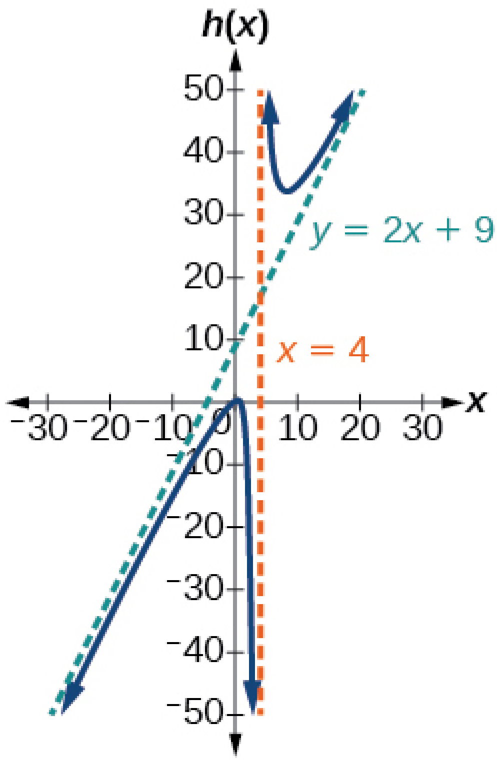
Problem: $b\left(x\right)=\frac{x^2-x-6}{x^2-4}$

Exercise:

Problem: $h\left(x\right)=\frac{2x^2+x-1}{x-4}$

Solution:

$V.A.\ x=4,\ S.A.\ y=2x+9;(-1,0);\left(\frac{1}{2},0\right);\left(0,\frac{1}{4}\right)$



Exercise:

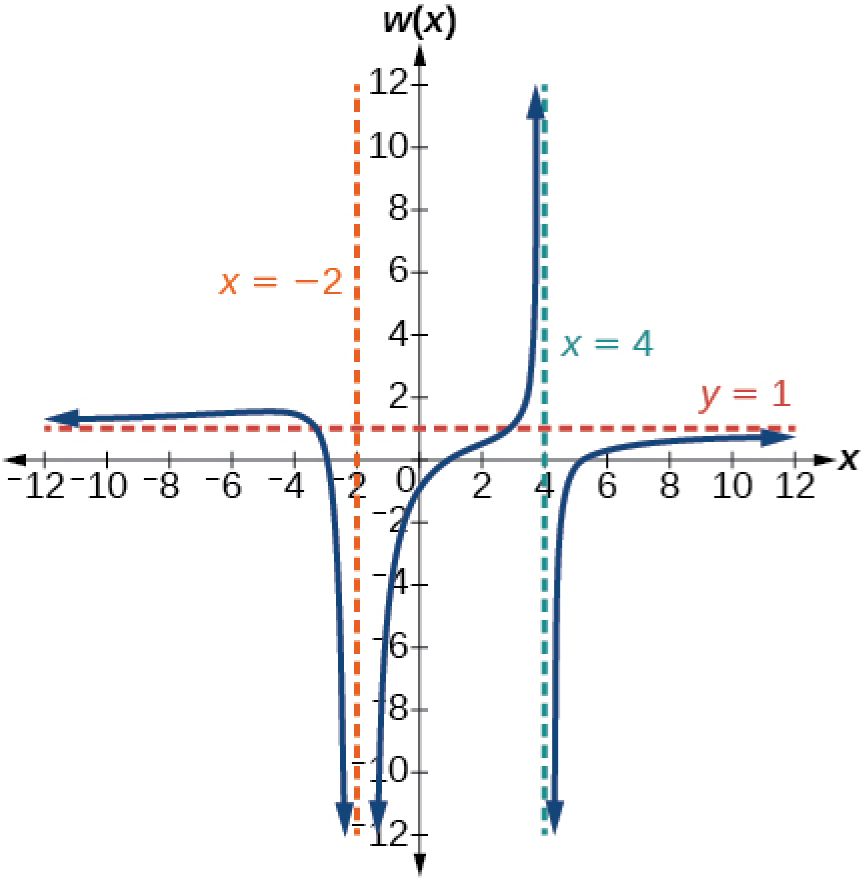
Problem:$k\left(x\right)=\frac{2x^2-3x-20}{x-5}$

Exercise:

Problem:$w\left(x\right)=\frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

Solution:

$V.A.\ x=-2,\ x=4,\ H.A.\ y=1,(1,0);(5,0);(-3,0);\left(0,-\frac{15}{16}\right)$



Exercise:

Problem: $z\left(x\right)=\frac{{\left(x+2\right)}^{2}\left(x-5\right)}{\left(x-3\right)\left(x+1\right)\left(x+4\right)}$

For the following exercises, write an equation for a rational function with the given characteristics.

Exercise:

Problem:

Vertical asymptotes at $x=5$ and $x=-5,$ x-intercepts at $\left(2,0\right)$ and $\left(-1,0\right),$ y-intercept at $\left(0,4\right)$

Solution:

$y=50\frac{x^{2}-x-2}{x^{2}-25}$

Exercise:

Problem:

Vertical asymptotes at $x = -4$ and $x = -1$, x-intercepts at $(1, 0)$ and $(5, 0)$, y-intercept at $(0, 7)$

Exercise:

Problem:

Vertical asymptotes at $x = -4$ and $x = -5$, x-intercepts at $(4, 0)$ and $(-6, 0)$, Horizontal asymptote at $y = 7$

Solution:

$y = 7\frac{x^2+2x-24}{x^2+9x+20}$

Exercise:

Problem:

Vertical asymptotes at $x = -3$ and $x = 6$, x-intercepts at $(-2, 0)$ and $(1, 0)$, Horizontal asymptote at $y = -2$

Exercise:

Problem:

Vertical asymptote at $x = -1$, Double zero at $x = 2$, y-intercept at $(0, 2)$

Solution:

$y = \frac{1}{2}\frac{x^2-4x+4}{x+1}$

Exercise:

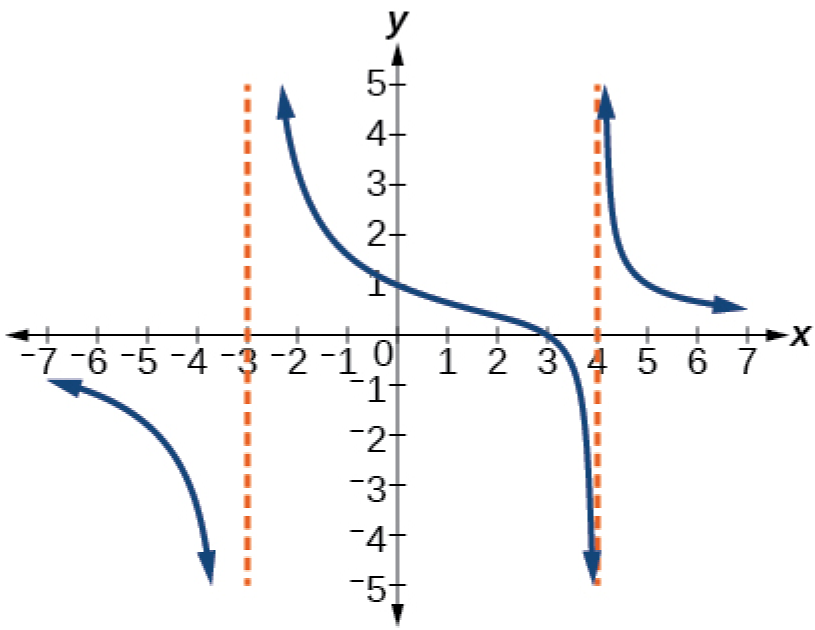
Problem:

Vertical asymptote at $x = 3$, Double zero at $x = 1$, y-intercept at $(0, 4)$

For the following exercises, use the graphs to write an equation for the function.

Exercise:

Problem:

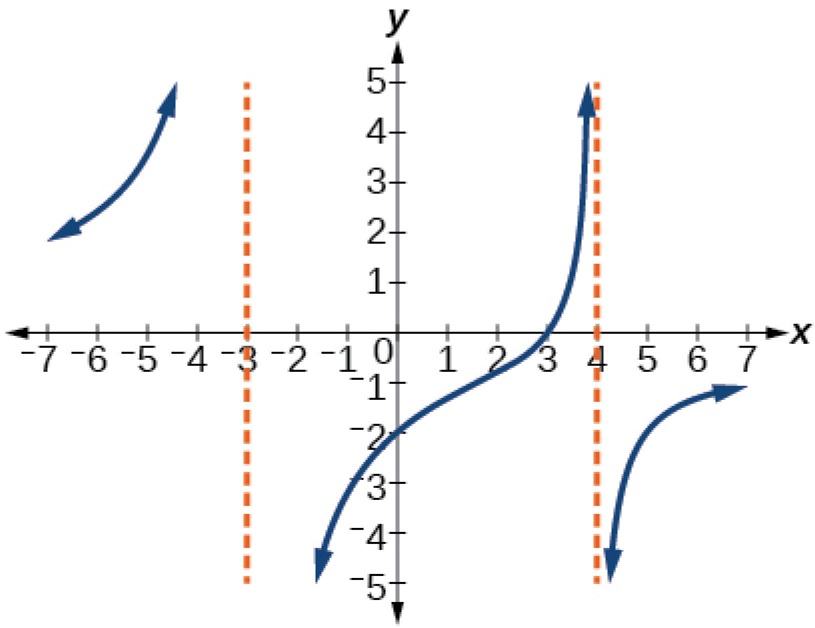


Solution:

$y = 4\frac{x-3}{x^2-x-12}$

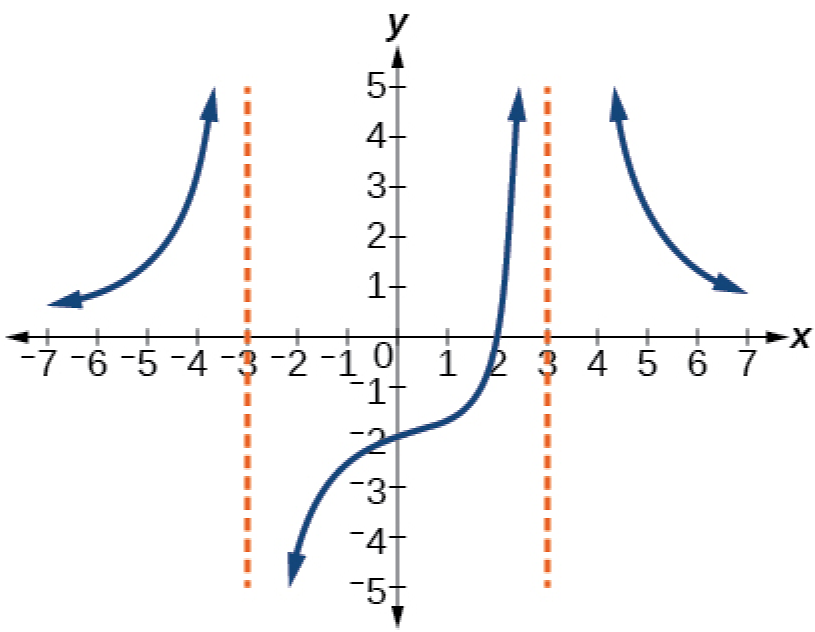
Exercise:

Problem:



Exercise:

Problem:

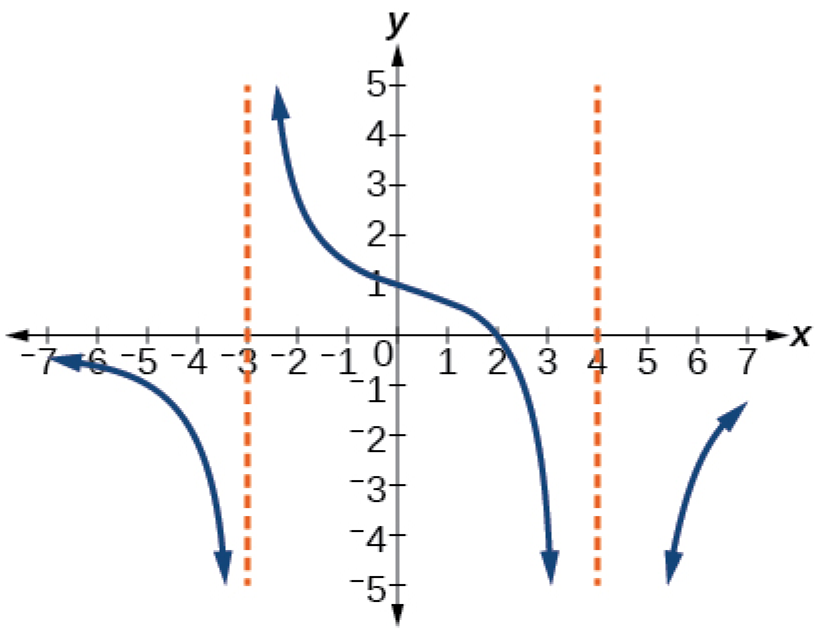


Solution:

$y = -9\frac{x-2}{x^2-9}$

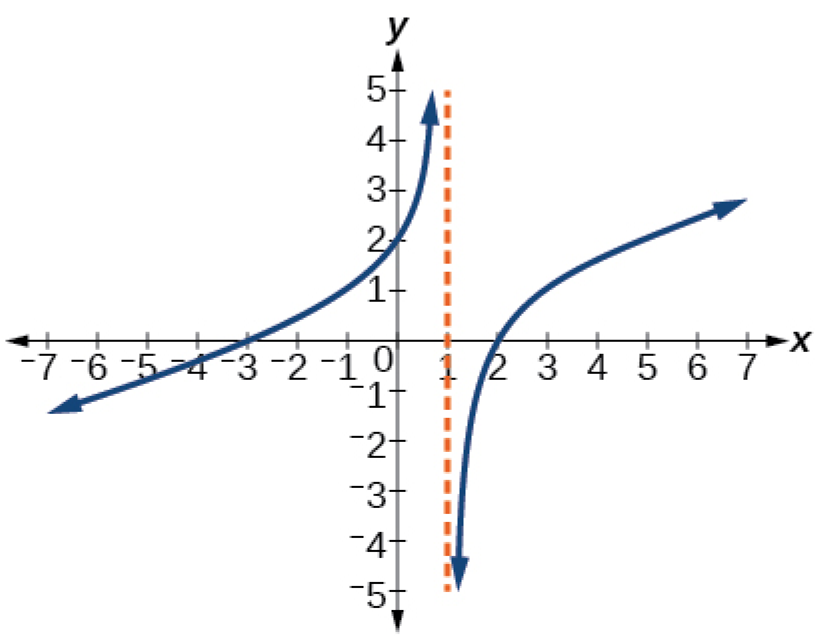
Exercise:

Problem:



Exercise:

Problem:

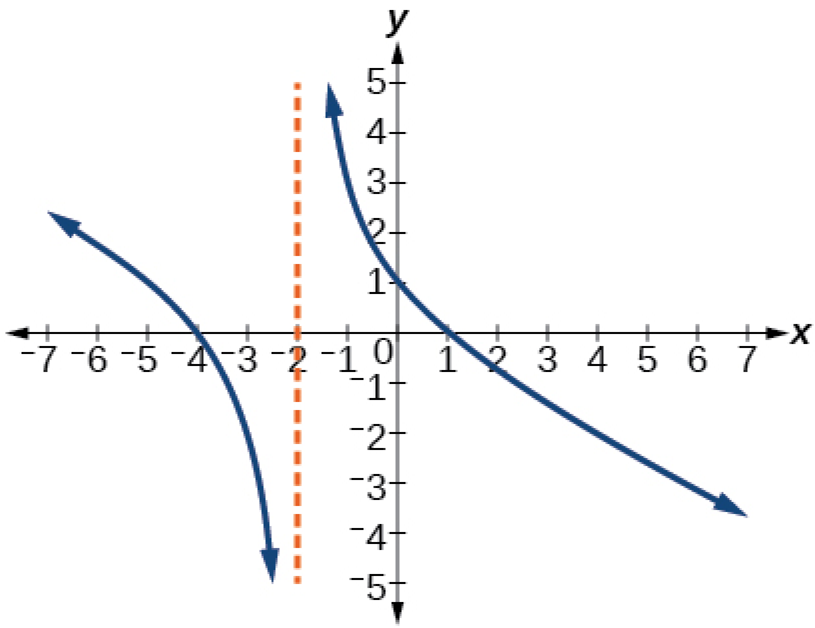


Solution:

$y = \frac{1}{3} \frac{x^2+x-6}{x-1}$

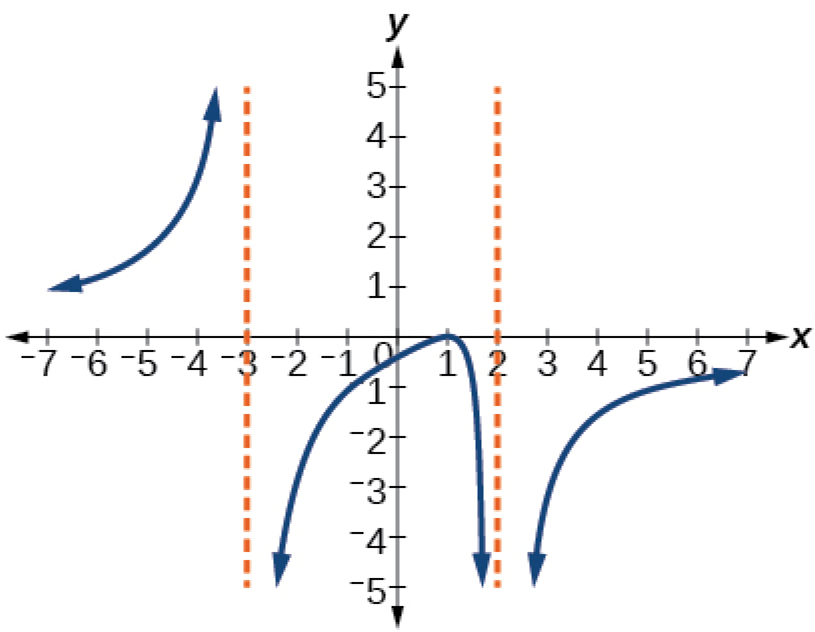
Exercise:

Problem:



Exercise:

Problem:

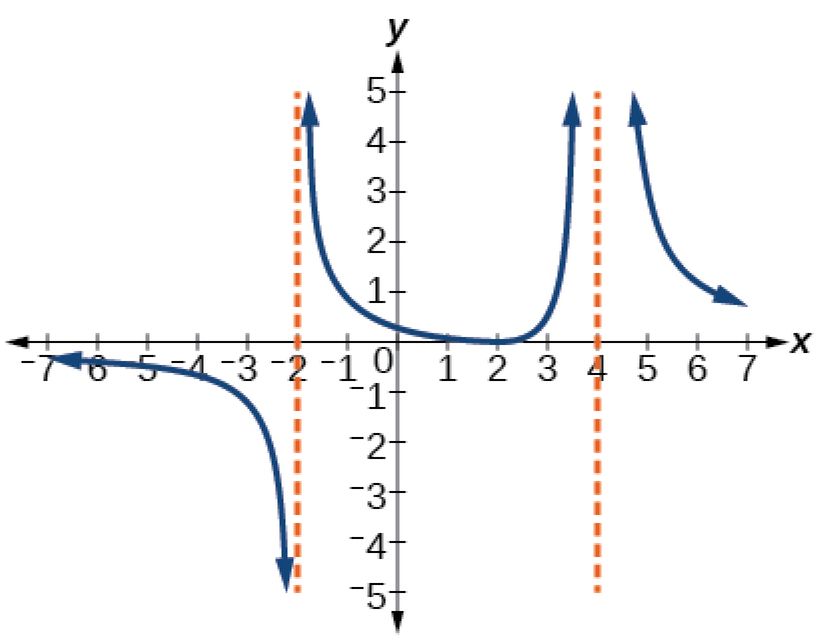


Solution:

$$y = -6\frac{(x-1)^2}{(x+3)(x-2)^2}$$

Exercise:

Problem:



Numeric

For the following exercises, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote

Exercise:

Problem: $f(x) = \frac{1}{x-2}$

Solution:

x	2.01	2.001	2.0001	1.99	1.999
y	100	1,000	10,000	–100	–1,000

x	**10**	**100**	**1,000**	**10,000**	**100,000**
y	.125	.0102	.001	.0001	.00001

Vertical asymptote $x = 2,$ Horizontal asymptote $y = 0$

Exercise:

Problem: $f(x) = \frac{x}{x-3}$

Exercise:

Problem: $f(x) = \frac{2x}{x+4}$

Solution:

x	–4.1	–4.01	–4.001	–3.99	–3.999
y	82	802	8,002	–798	–7998

x	10	100	1,000	10,000	100,000
y	1.4286	1.9331	1.992	1.9992	1.999992

Vertical asymptote $x = -4,$ Horizontal asymptote $y = 2$

Exercise:

Problem: $f(x) = \frac{2x}{(x-3)^2}$

Exercise:

Problem: $f(x) = \frac{x^2}{x^2+2x+1}$

Solution:

x	–.9	–.99	–.999	–1.1	–1.01
y	81	9,801	998,001	121	10,201

x	10	100	1,000	10,000	100,000
y	.82645	.9803	.998	.9998	

Vertical asymptote $x = -1,$ Horizontal asymptote $y = 1$

Technology

For the following exercises, use a calculator to graph $f\left(x\right).$ Use the graph to solve $f\left(x\right) > 0.$

Exercise:

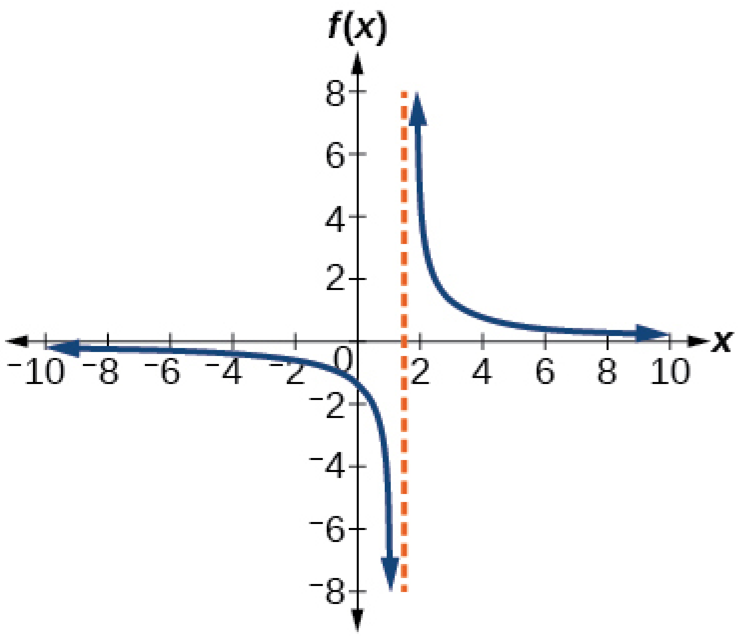
Problem: $f(x) = \frac{2}{x+1}$

Exercise:

Problem: $f(x) = \frac{4}{2x-3}$

Solution:

$\left(\frac{3}{2}, \infty\right)$



Exercise:

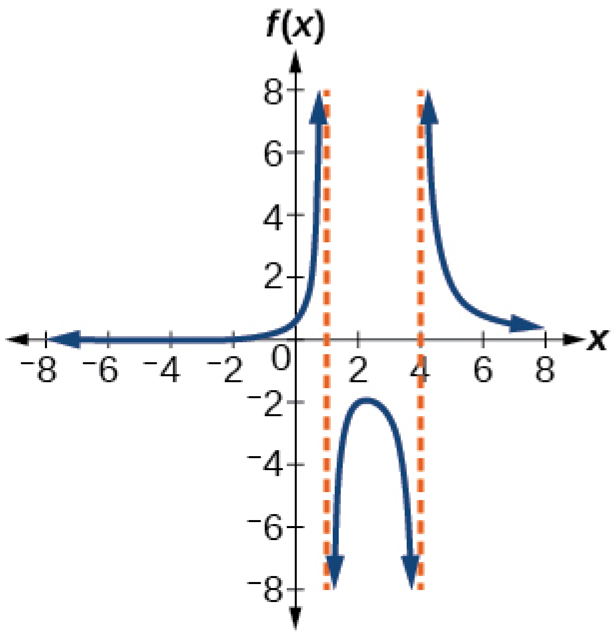
Problem: $f(x) = \frac{2}{(x-1)(x+2)}$

Exercise:

Problem: $f(x) = \frac{x+2}{(x-1)(x-4)}$

Solution:

$(-2, 1) \cup (4, \infty)$



Exercise:

Problem: $f(x) = \frac{(x+3)^2}{(x-1)^2(x+1)}$

Extensions

For the following exercises, identify the removable discontinuity.

Exercise:

Problem: $f(x) = \frac{x^2-4}{x-2}$

Solution:

$(2, 4)$

Exercise:

Problem: $f(x) = \frac{x^3+1}{x+1}$

Exercise:

Problem: $f(x) = \frac{x^2+x-6}{x-2}$

Solution:

$(2, 5)$

Exercise:

Problem: $f(x) = \frac{2x^2+5x-3}{x+3}$

Exercise:

Problem: $f(x) = \frac{x^3+x^2}{x+1}$

Solution:

$(-1, 1)$

Real-World Applications

For the following exercises, express a rational function that describes the situation.

Exercise:

Problem:

A large mixing tank currently contains 200 gallons of water, into which 10 pounds of sugar have been mixed. A tap will open, pouring 10 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 3 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

Exercise:

Problem:

A large mixing tank currently contains 300 gallons of water, into which 8 pounds of sugar have been mixed. A tap will open, pouring 20 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 2 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

Solution:

$C(t) = \frac{8+2t}{300+20t}$

For the following exercises, use the given rational function to answer the question.

Exercise:

Problem:

The concentration C of a drug in a patient's bloodstream t hours after injection in given by $C(t) = \frac{2t}{3+t^2}$. What happens to the concentration of the drug as t increases?

Exercise:

Problem:

The concentration C of a drug in a patient's bloodstream t hours after injection is given by $C(t) = \frac{100t}{2t^2+75}$. Use a calculator to approximate the time when the concentration is highest.

Solution:

After about 6.12 hours.

For the following exercises, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.

Exercise:

Problem:

An open box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that will have minimum surface area. Let x = length of the side of the base.

Exercise:

Problem:

A rectangular box with a square base is to have a volume of 20 cubic feet. The material for the base costs 30 cents/ square foot. The material for the sides costs 10 cents/square foot. The material for the top costs 20 cents/square foot. Determine the dimensions that will yield minimum cost. Let x = length of the side of the base.

Solution:

$A(x) = 50x^2 + \frac{800}{x}$. 2 by 2 by 5 feet.

Exercise:

Problem:

A right circular cylinder has volume of 100 cubic inches. Find the radius and height that will yield minimum surface area. Let x = radius.

Exercise:

Problem:

A right circular cylinder with no top has a volume of 50 cubic meters. Find the radius that will yield minimum surface area. Let x = radius.

Solution:

$A(x) = \pi x^2 + \frac{100}{x}$. Radius = 2.52 meters.

Exercise:

Problem:

A right circular cylinder is to have a volume of 40 cubic inches. It costs 4 cents/square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let x = radius.

Glossary

arrow notation
: a way to symbolically represent the local and end behavior of a function by using arrows to indicate that an input or output approaches a value

horizontal asymptote
: a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound.

rational function
: a function that can be written as the ratio of two polynomials

removable discontinuity
: a single point at which a function is undefined that, if filled in, would make the function continuous; it appears as a hole on the graph of a function

vertical asymptote
: a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a

Graphs of Polynomial Functions
In this section, you will:

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Use the Intermediate Value Theorem.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in [link].

Year	2006	2007	2008	2009	2010	2011	2012	2013
Revenues	52.4	52.8	51.2	49.5	48.6	48.6	48.7	47.1

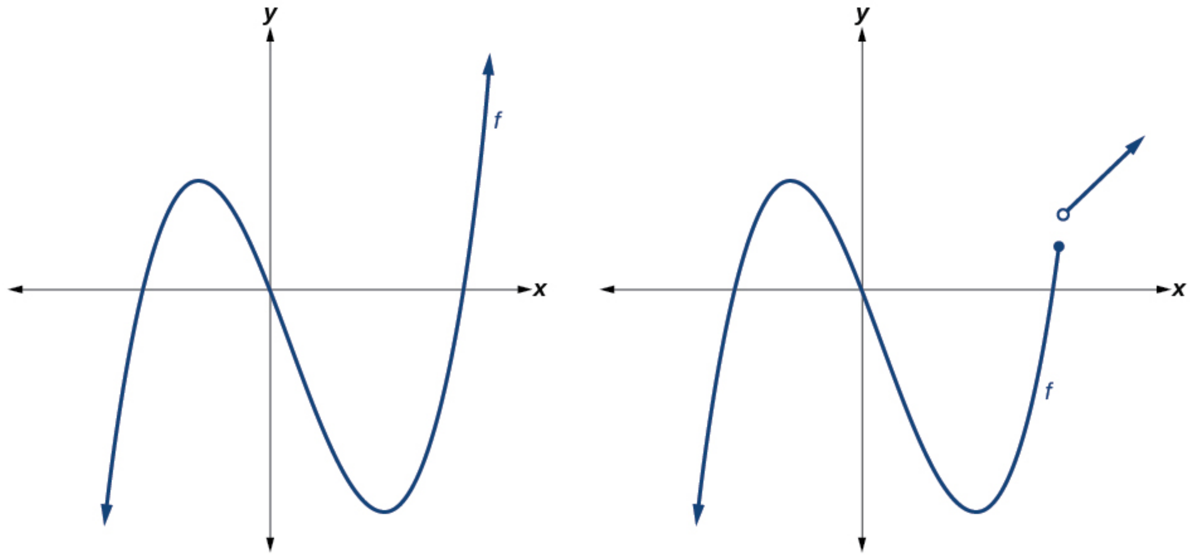
The revenue can be modeled by the polynomial function
Equation:

$$R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332$$

where R represents the revenue in millions of dollars and t represents the year, with $t = 6$ corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing? These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

Recognizing Characteristics of Graphs of Polynomial Functions

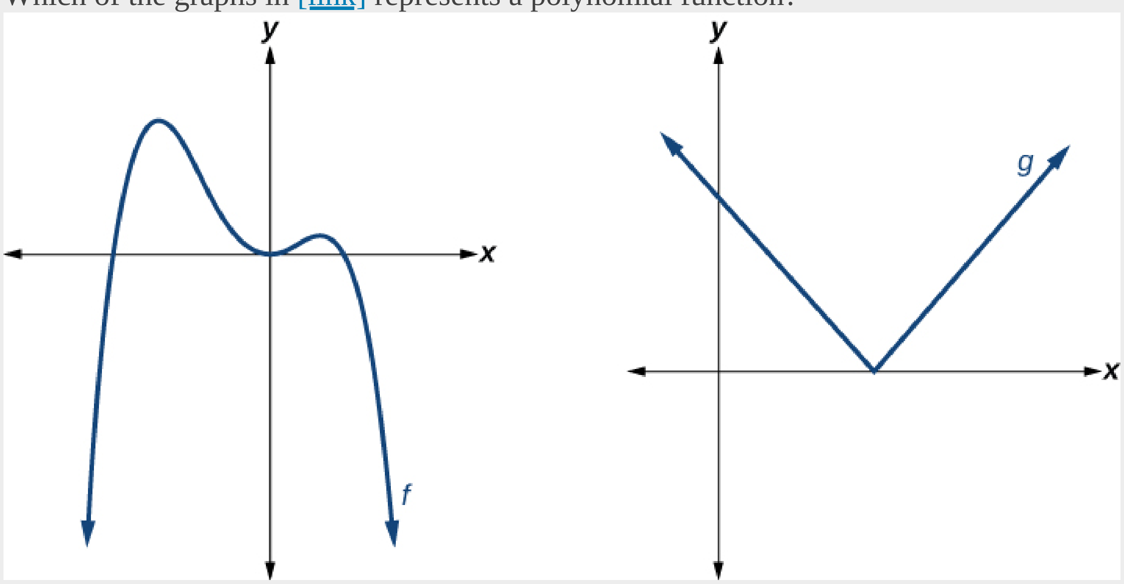
Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. [link] shows a graph that represents a polynomial function and a graph that represents a function that is not a polynomial.

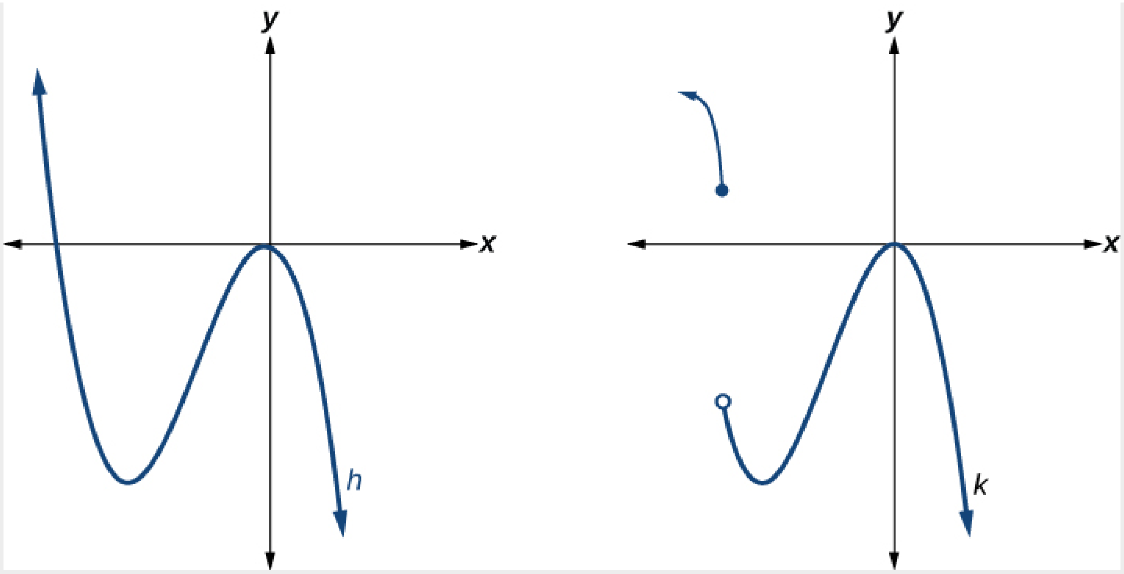


Example:
Exercise:

Problem:
Recognizing Polynomial Functions

Which of the graphs in [link] represents a polynomial function?





Solution:

The graphs of f and h are graphs of polynomial functions. They are smooth and continuous.

The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Note:
Do all polynomial functions have as their domain all real numbers?
Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f(x) = 0$ are called zeros of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros**.**

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.

3. Technology is used to determine the intercepts.

Note:
Given a polynomial function f, find the *x*-intercepts by factoring.

1. Set $f(x) = 0$.
2. If the polynomial function is not given in factored form:
    a. Factor out any common monomial factors.
    b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x-intercepts.

Example:
Exercise:

Problem:
Finding the *x*-Intercepts of a Polynomial Function by Factoring

Find the *x*-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution:

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

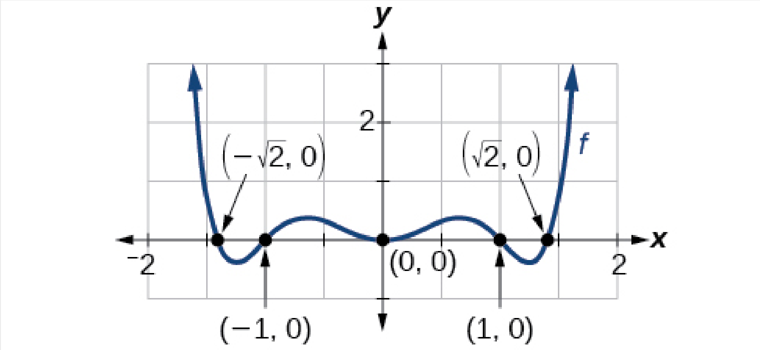
Equation:

$$\begin{aligned} x^6 - 3x^4 + 2x^2 &= 0 && \text{Factor out the greatest common factor.} \\ x^2(x^4 - 3x^2 + 2) &= 0 && \text{Factor the trinomial.} \\ x^2(x^2 - 1)(x^2 - 2) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

Equation:

$$\begin{array}{ccccc} & & (x^2 - 1) = 0 & & (x^2 - 2) = 0 \\ x^2 = 0 & \text{or} & x^2 = 1 & \text{or} & x^2 = 2 \\ x = 0 & & x = \pm 1 & & x = \pm\sqrt{2} \end{array}$$

This gives us five x-intercepts: $(0,0),(1,0),(-1,0),(\sqrt{2},0),$ and $(-\sqrt{2},0)$. See [link]. We can see that this is an even function.



Example:
Exercise:

Problem:
Finding the *x*-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x)=x^3-5x^2-x+5$.

Solution:

Find solutions for $f(x)=0$ by factoring.

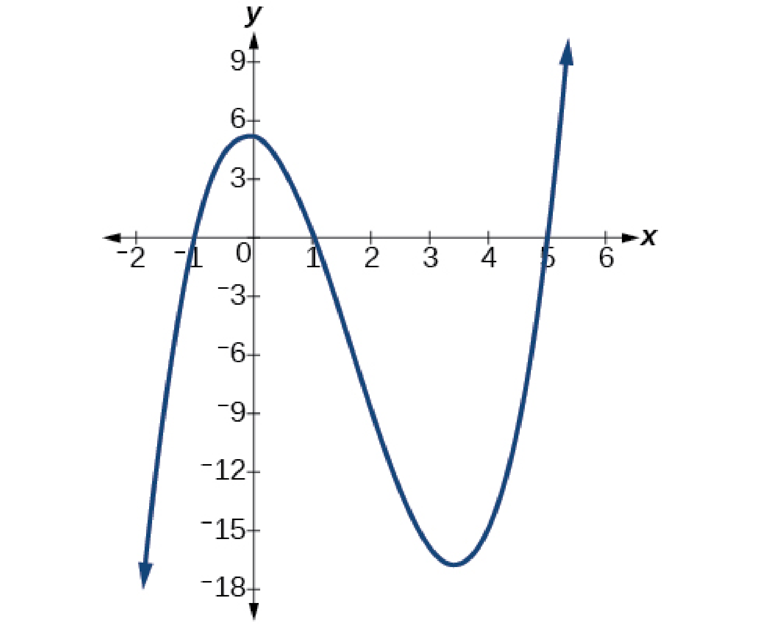
Equation:

$$\begin{array}{rl} x^3-5x^2-x+5=0 & \text{Factor by grouping.} \\ x^2(x-5)-(x-5)=0 & \text{Factor out the common factor.} \\ (x^2-1)(x-5)=0 & \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5)=0 & \text{Set each factor equal to zero.} \end{array}$$

Equation:

$$\begin{array}{ccccc} x+1=0 & \text{or} & x-1=0 & \text{or} & x-5=0 \\ x=-1 & & x=1 & & x=5 \end{array}$$

There are three x-intercepts: $(-1,0),(1,0),$ and $(5,0)$. See [link].



Example:
Exercise:

Problem:
Finding the *y*- and *x*-Intercepts of a Polynomial in Factored Form

Find the *y*- and *x*-intercepts of $g(x) = (x-2)^2(2x+3)$.

Solution:

The *y*-intercept can be found by evaluating $g(0)$.
Equation:

$$\begin{aligned} g(0) &= (0-2)^2(2(0)+3) \\ &= 12 \end{aligned}$$

So the *y*-intercept is $(0, 12)$.

The *x*-intercepts can be found by solving $g(x) = 0$.
Equation:

$$(x-2)^2(2x+3) = 0$$

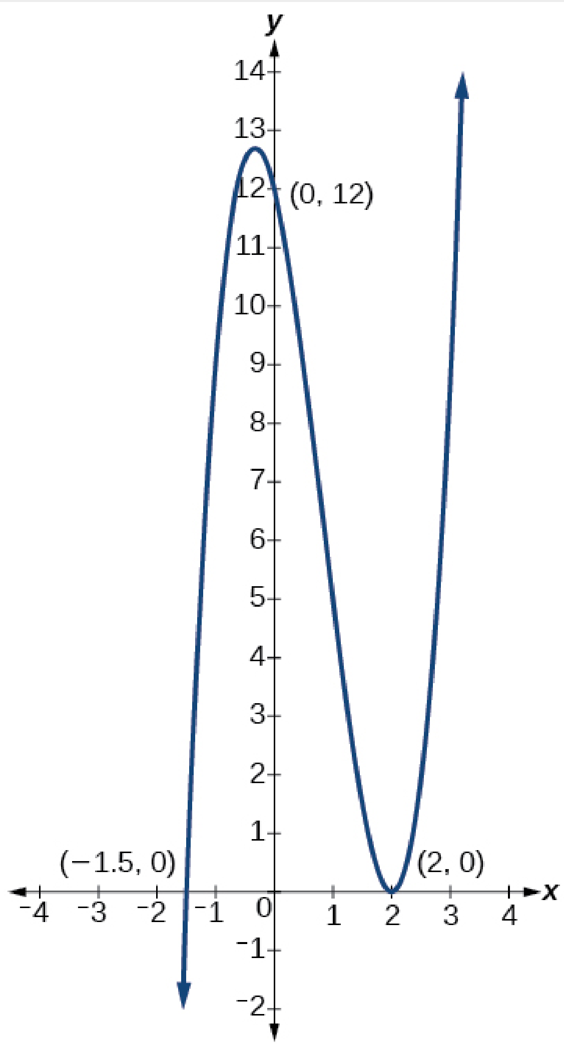
Equation:

$$\begin{array}{rcl} (x-2)^2 = 0 & & (2x+3) = 0 \\ x - 2 = 0 & \text{or} & x = -\frac{3}{2} \\ x = 2 & & \end{array}$$

So the x-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

Analysis

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in [link].



Example:
Exercise:

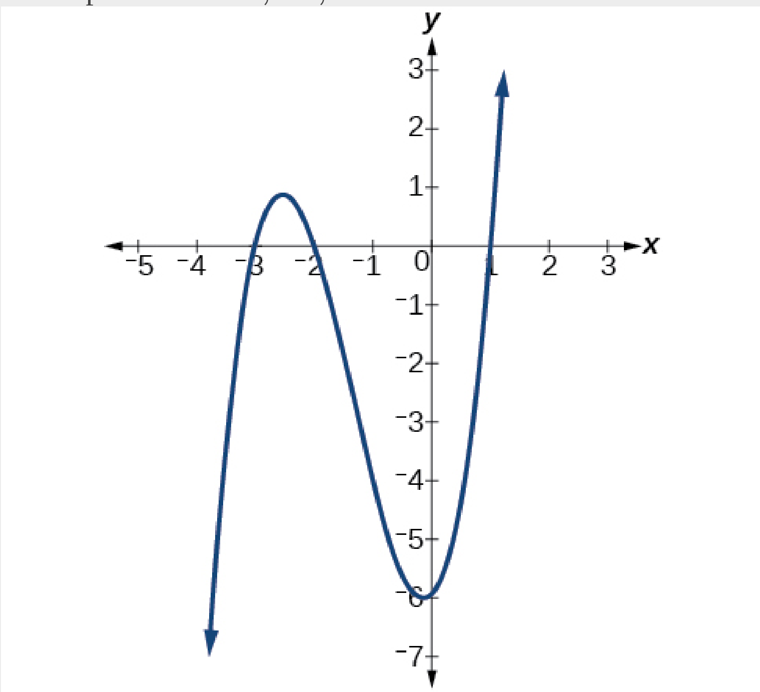
Problem:
Finding the x-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6.$

Solution:

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in [link], it appears that there are *x*-intercepts at $x = -3, -2,$ and $1.$



We can check whether these are correct by substituting these values for x and verifying that
Equation:

$$h(-3) = h(-2) = h(1) = 0.$$

Since $h(x) = x^3 + 4x^2 + x - 6,$ we have:
Equation:

$$\begin{aligned} h(-3) &= (-3)^3 + 4(-3)^2 + (-3) - 6 = -27 + 36 - 3 - 6 = 0 \\ h(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0 \\ h(1) &= (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0 \end{aligned}$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.
Equation:

$$\begin{aligned} h(x) &= x^3 + 4x^2 + x - 6 \\ &= (x+3)(x+2)(x-1) \end{aligned}$$

Note:
Exercise:

Problem: Find the y- and x-intercepts of the function $f(x) = x^4 - 19x^2 + 30x$.

Solution:

y-intercept $(0,0)$; x-intercepts $(0,0)$, $(-5,0)$, $(2,0)$, and $(3,0)$

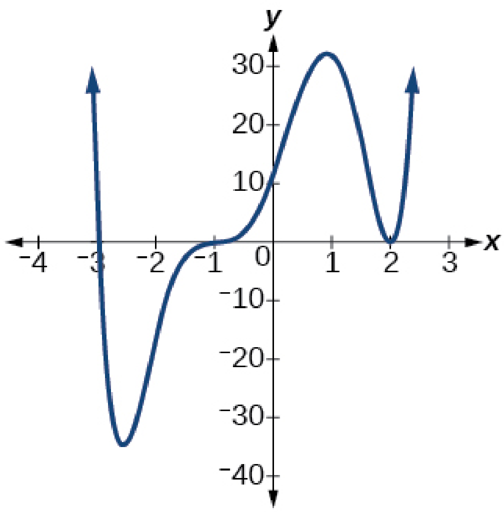
Identifying Zeros and Their Multiplicities

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off.

Suppose, for example, we graph the function
Equation:

$$f(x) = (x+3)(x-2)^2(x+1)^3.$$

Notice in [link] that the behavior of the function at each of the x-intercepts is different.



Identifying the behavior of the graph at an *x*-intercept by examining the multiplicity of the zero.

The x-intercept -3 is the solution of equation $(x+3)=0$. The graph passes directly through the x-intercept at $x=-3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

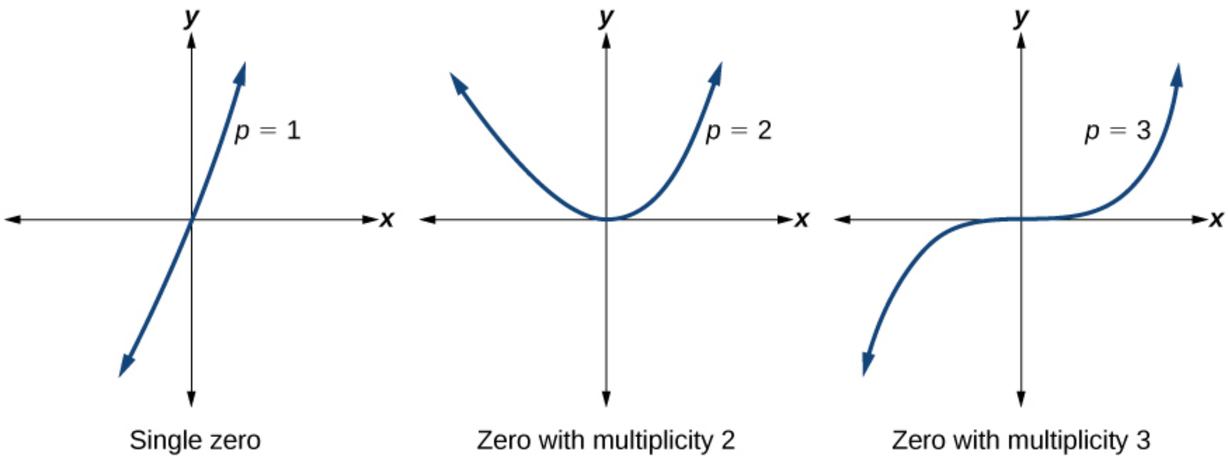
The x-intercept 2 is the repeated solution of equation $(x-2)^2=0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.
Equation:

$$(x-2)^2=(x-2)(x-2)$$

The factor is repeated, that is, the factor $(x-2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x=2,$ has multiplicity 2 because the factor $(x-2)$ occurs twice.

The x-intercept -1 is the repeated solution of factor $(x+1)^3=0$. The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near the intercept as the toolkit function $f(x)=x^3$. We call this a triple zero, or a zero with multiplicity 3.

For zeros with even multiplicities, the graphs *touch* or are tangent to the x-axis. For zeros with odd multiplicities, the graphs *cross* or intersect the x-axis. See [link] for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.



For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

Note:
Graphical Behavior of Polynomials at x-Intercepts
If a polynomial contains a factor of the form $(x - h)^p$, the behavior near the x-intercept h is determined by the power p. We say that $x = h$ is a zero of **multiplicity** p.
The graph of a polynomial function will touch the x-axis at zeros with even multiplicities. The graph will cross the *x*-axis at zeros with odd multiplicities.
The sum of the multiplicities is the degree of the polynomial function.

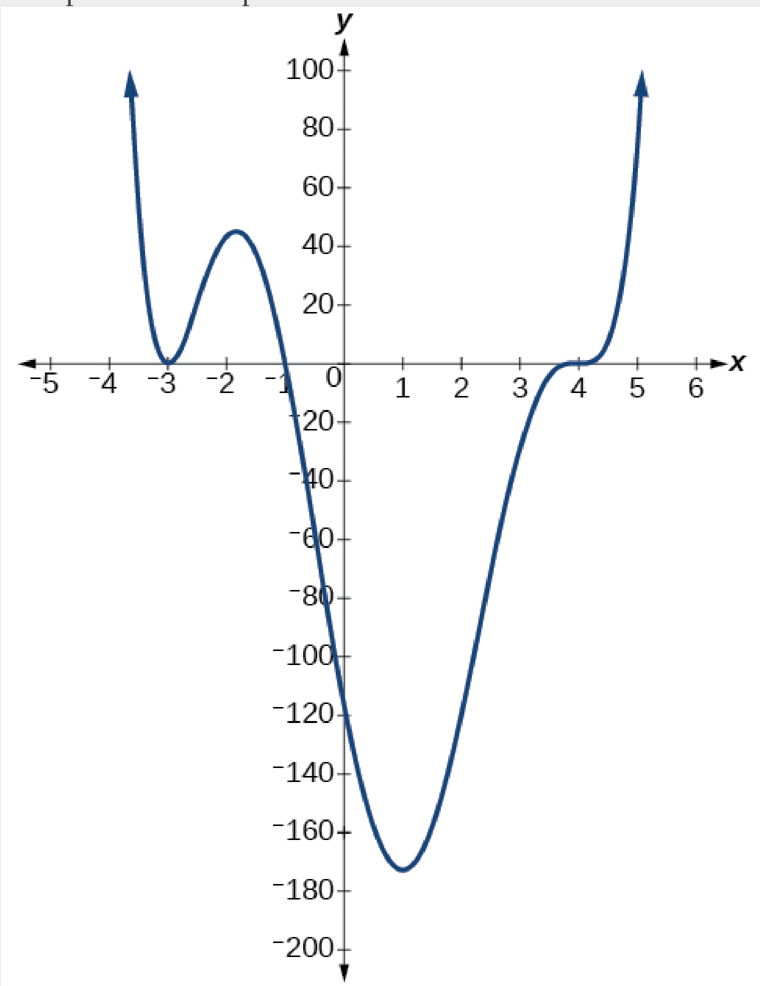
Note:
Given a graph of a polynomial function of degree n, identify the zeros and their multiplicities.

1. If the graph crosses the *x*-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the *x*-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the *x*-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is n.

Example:
Exercise:

Problem:
Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in [link] to identify the zeros of the function and their possible multiplicities.



Solution:

The polynomial function is of degree n. The sum of the multiplicities must be n.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the *x*-axis, so the multiplicity of the zero must be even. The zero of -3 has multiplicity 2.

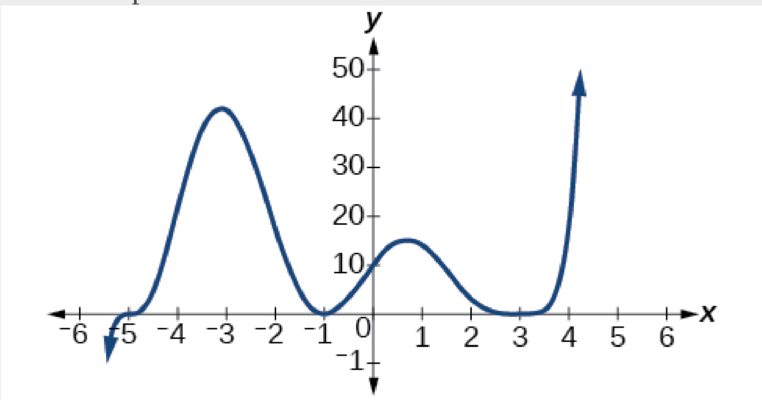
The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x = 4$. The graph crosses the *x*-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is likely 3 and that the sum of the multiplicities is likely 6.

Note:
Exercise:

Problem:

Use the graph of the function of degree 5 in [link] to identify the zeros of the function and their multiplicities.



Solution:

The graph has a zero of –5 with multiplicity 3, a zero of –1 with multiplicity 2, and a zero of 3 with multiplicity 4.

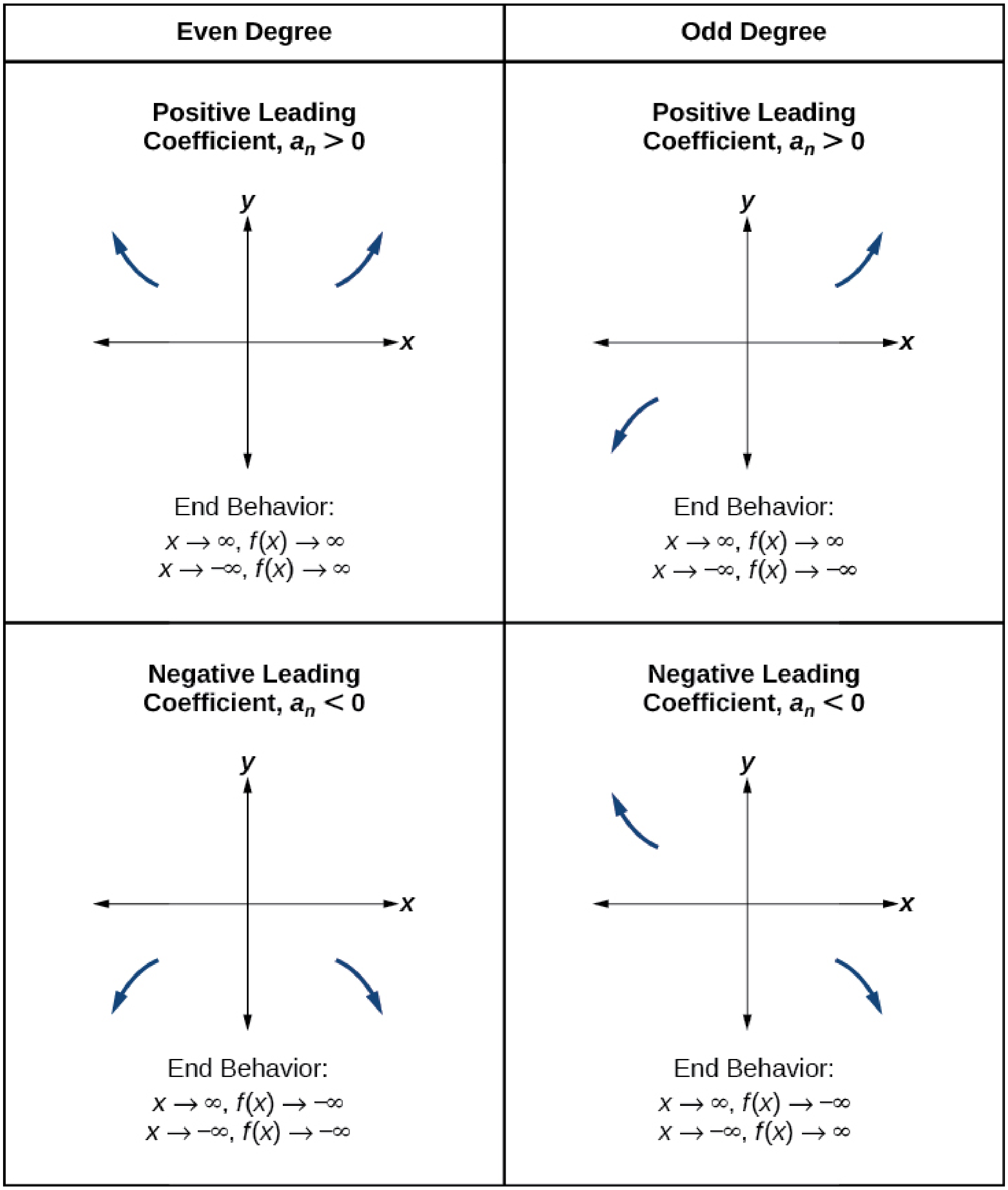
Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form
Equation:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

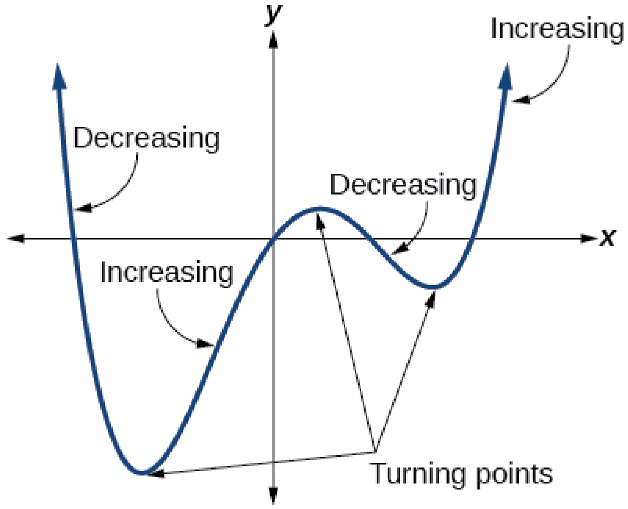
will either ultimately rise or fall as x increases without bound and will either rise or fall as x decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the *end behavior* of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. [link] summarizes all four cases.



Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function’s local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in [link]. The graph has three turning points.



This function f is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

Note:
Interpreting Turning Points
A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising).
A polynomial of degree n will have at most $n-1$ turning points.

Example:
Exercise:

Problem:
Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

a. $f(x)=-x^3+4x^5-3x^2+1$
b. $f(x)=-(x-1)^2\left(1+2x^2\right)$

Solution:

a. $f(x)=-x^{\ 3}+4x^5-3x^2+1$

First, rewrite the polynomial function in descending order:
$f(x)=4x^5-x^3-3x^2+1$

Identify the degree of the polynomial function. This polynomial function is of degree 5.

The maximum number of turning points is $5 - 1 = 4$.

b. $f(x) = -(x-1)^2\left(1+2x^2\right)$

First, identify the leading term of the polynomial function if the function were expanded.

$$f(x) = -(x-1)^2\ (1+2x^2)$$
$$a_n = -(x^2)\ (2x^2)\ -2x^4$$

Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum number of turning points is $4 - 1 = 3$.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

Note:
Given a polynomial function, sketch the graph.

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

Example:
Exercise:

Problem:
Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution:

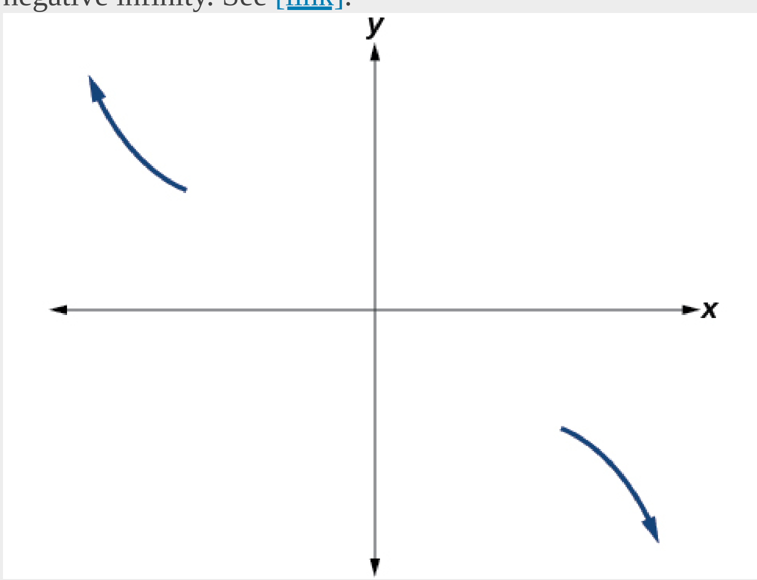
This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$, the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The y-intercept is found by evaluating $f(0)$.
Equation:

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The y-intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See [link].

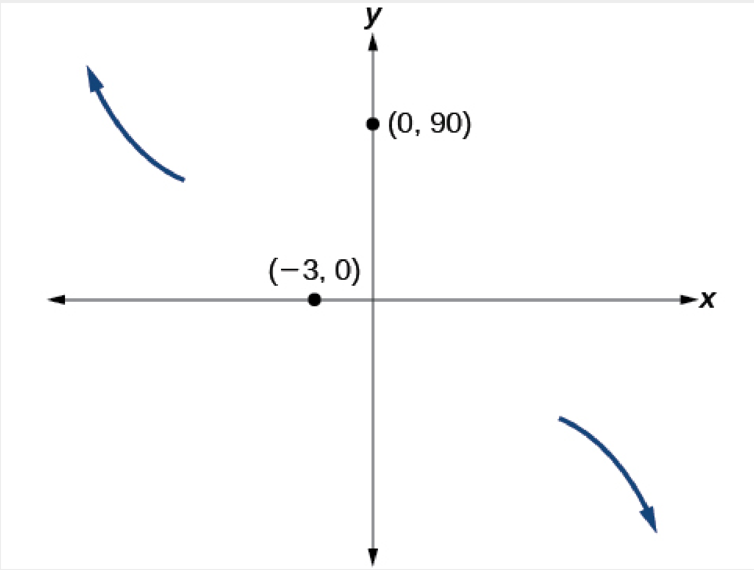


To sketch this, we consider that:

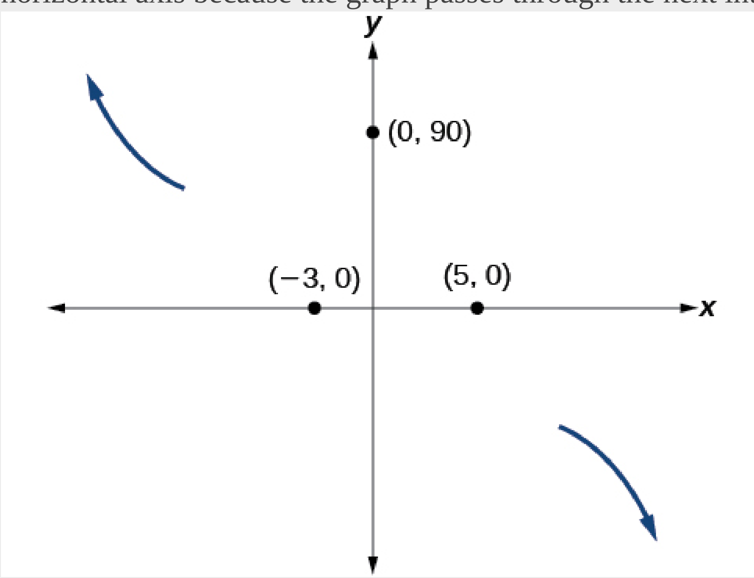
- As $x \to -\infty$ the function $f(x) \to \infty$, so we know the graph starts in the second quadrant and is decreasing toward the x-axis.

- Since $f(-x)=-2(-x+3)^2(-x-5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3,0)$, the graph bounces off of the x-axis, so the function must start increasing.

  At $(0,90)$, the graph crosses the y-axis at the y-intercept. See [link].

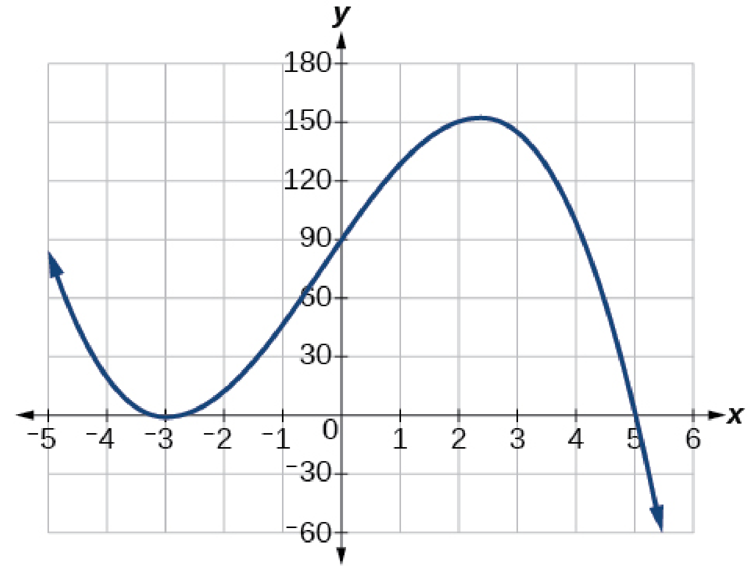


Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5,0)$. See [link].



As $x\to\infty$ the function $f(x)\to-\infty$, so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in [link], and verify that the resulting graph looks like our sketch in [link].

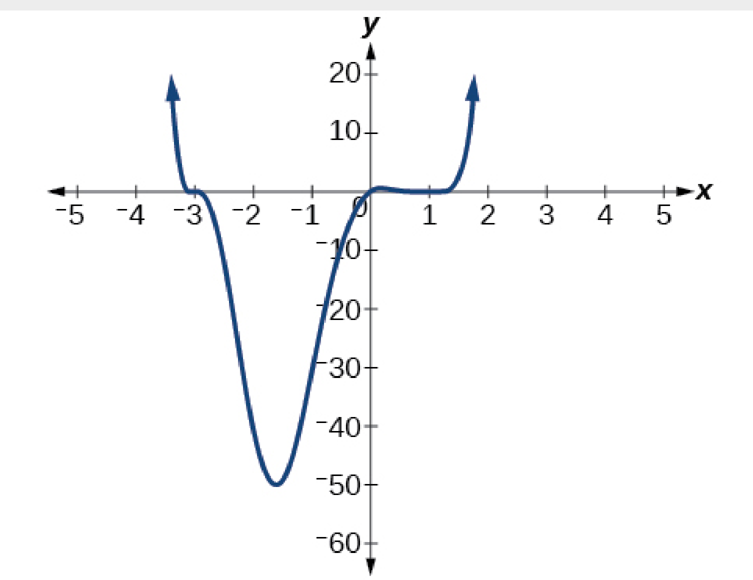


The complete graph of the polynomial function $f(x) = -2(x+3)^2(x-5)$

Note:
Exercise:

Problem: Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.

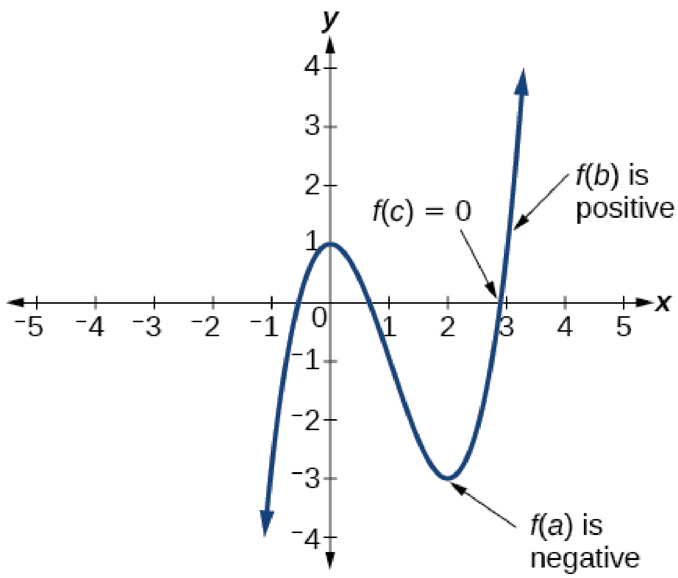
Solution:



Using the Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the x-axis, we can confirm that there is a zero between them. Consider a polynomial function f whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function f at $x = a$ lies above the x-axis and another point at $x = b$ lies below the x-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x-axis. Call this point $(c,\ f(c))$. This means that we are assured there is a solution c where $f(c) = 0$.

In other words, the Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis. [link] shows that there is a zero between a and b.



Using the Intermediate Value Theorem to show there exists a zero.

Note:
Intermediate Value Theorem

Let f be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

Example:
Exercise:

Problem:
Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Solution:

As a start, evaluate $f(x)$ at the integer values $x = 1, 2, 3, \text{and} 4$. See [link].

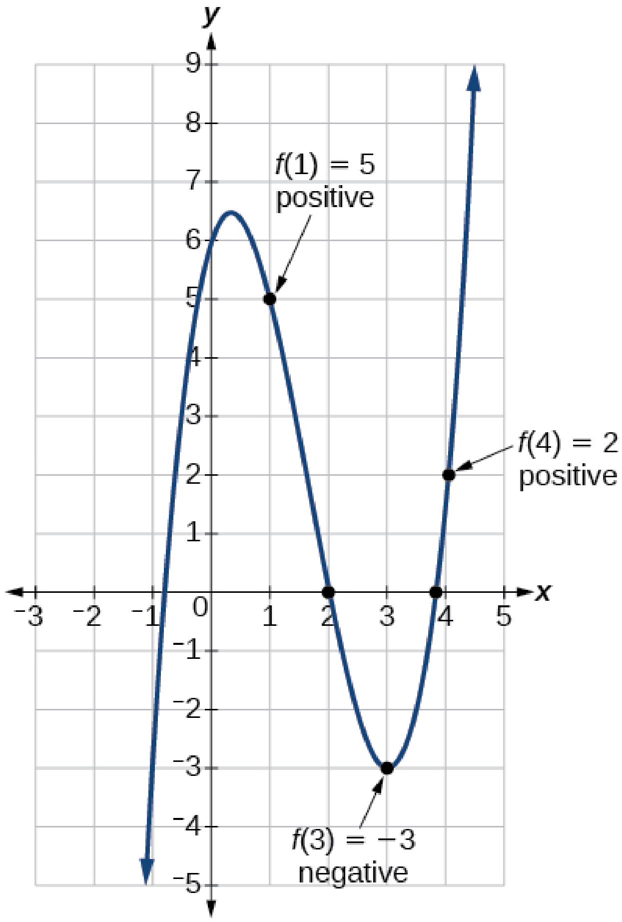
x	1	2	3	4
$f(x)$	5	0	–3	2

We see that one zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4.

We have shown that there are at least two real zeros between $x = 1$ and $x = 4$.

Analysis

We can also see on the graph of the function in [link] that there are two real zeros between $x = 1$ and $x = 4$.



Note:
Exercise:

Problem:

Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

Solution:

Because f is a polynomial function and since $f(1)$ is negative and $f(2)$ is positive, there is at least one real zero between $x = 1$ and $x = 2$.

Writing Formulas for Polynomial Functions

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a polynomial function written in factored form will have an x-intercept

where each factor is equal to zero, we can form a function that will pass through a set of x-intercepts by introducing a corresponding set of factors.

Note:
Factored Form of Polynomials
If a polynomial of lowest degree p has horizontal intercepts at $x = x_1, x_2, \ldots, x_n,$ then the polynomial can be written in the factored form: $f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$ where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the *x*-intercept.

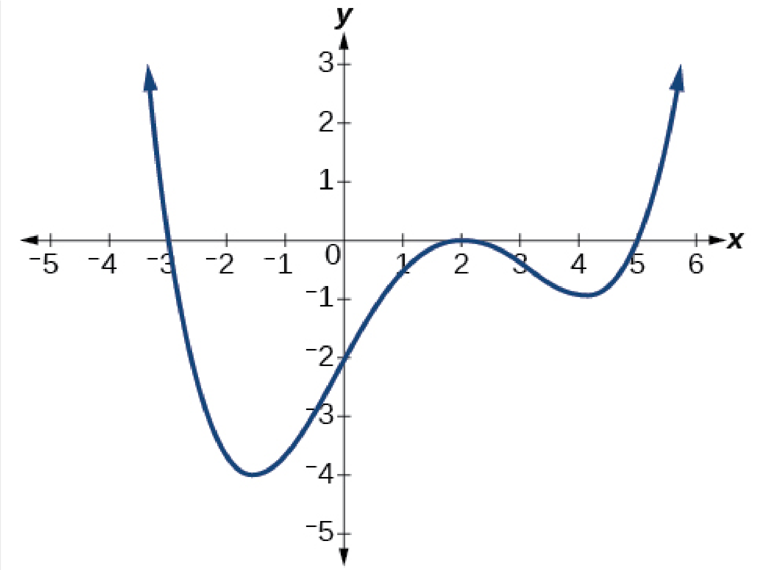
Note:
Given a graph of a polynomial function, write a formula for the function.

1. Identify the *x*-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the *x*-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the *y*-intercept may be easiest) to determine the stretch factor.

Example:
Exercise:

Problem:
Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in [link].



Solution:

This graph has three x-intercepts: $x = -3, 2,$ and 5. The y-intercept is located at $(0, -2)$. At $x = -3$ and $x = 5,$ the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2,$ the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

Equation:

$$f(x) = a(x + 3)(x - 2)^2(x - 5)$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, -2),$ to solve for a.

Equation:

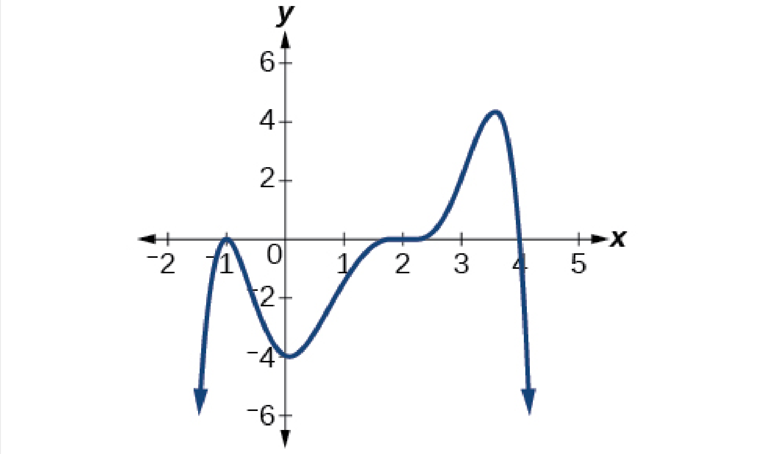
$$\begin{aligned} f(0) &= a(0 + 3)(0 - 2)^2(0 - 5) \\ -2 &= a(0 + 3)(0 - 2)^2(0 - 5) \\ -2 &= -60a \\ a &= \tfrac{1}{30} \end{aligned}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x + 3)(x - 2)^2(x - 5).$

Note:

Exercise:

Problem: Given the graph shown in [link], write a formula for the function shown.



Solution:

$f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

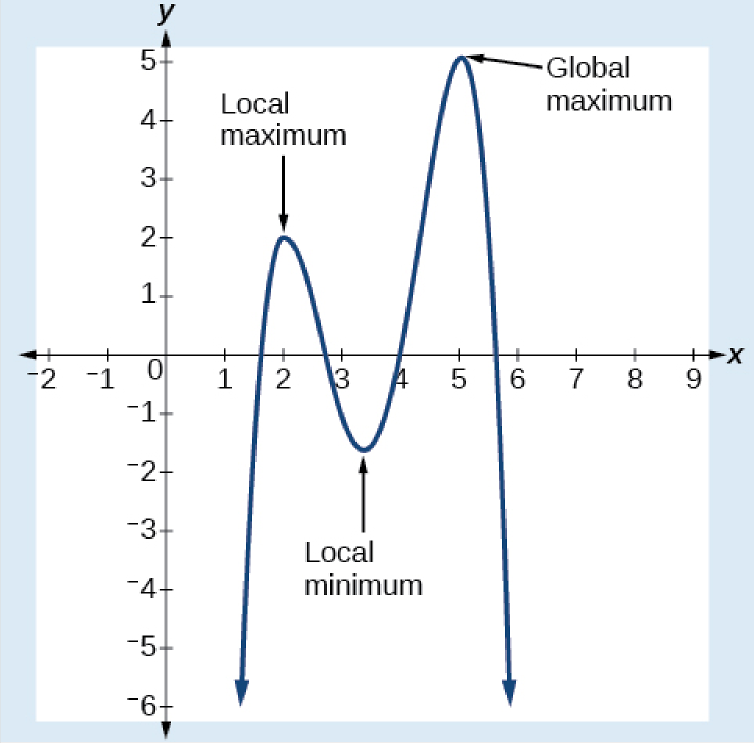
Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a **global maximum** or a **global minimum**. These are also referred to as the absolute maximum and absolute minimum values of the function.

Note:
Local and Global Extrema
A local maximum or local minimum at $x = a$ (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around $x = a$. If a function has a local maximum at a, then $f(a) \geq f(x)$ for all x in an open interval around $x = a$. If a function has a local minimum at a, then $f(a) \leq f(x)$ for all x in an open interval around $x = a$.
A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at a, then $f(a) \geq f(x)$ for all x. If a function has a global minimum at a, then $f(a) \leq f(x)$ for all x.

We can see the difference between local and global extrema in [link].



Note:
Do all polynomial functions have a global minimum or maximum?
No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ *has neither a global maximum nor a global minimum.*

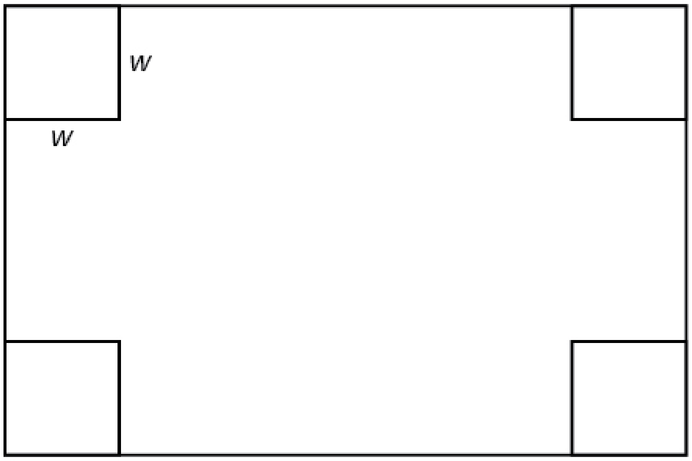
Example:
Exercise:

Problem:
Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

Solution:

We will start this problem by drawing a picture like that in [link], labeling the width of the cut-out squares with a variable, w.

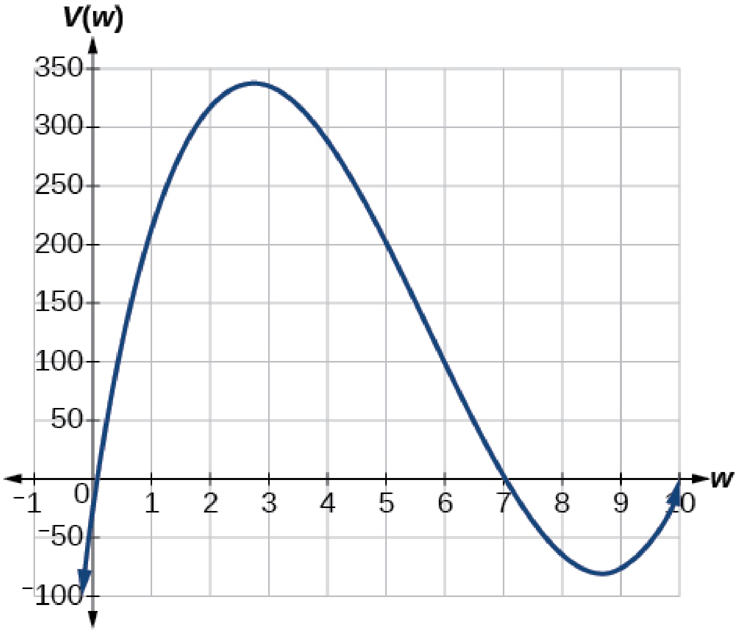


Notice that after a square is cut out from each end, it leaves a $(14-2w)$ cm by $(20-2w)$ cm rectangle for the base of the box, and the box will be w cm tall. This gives the volume

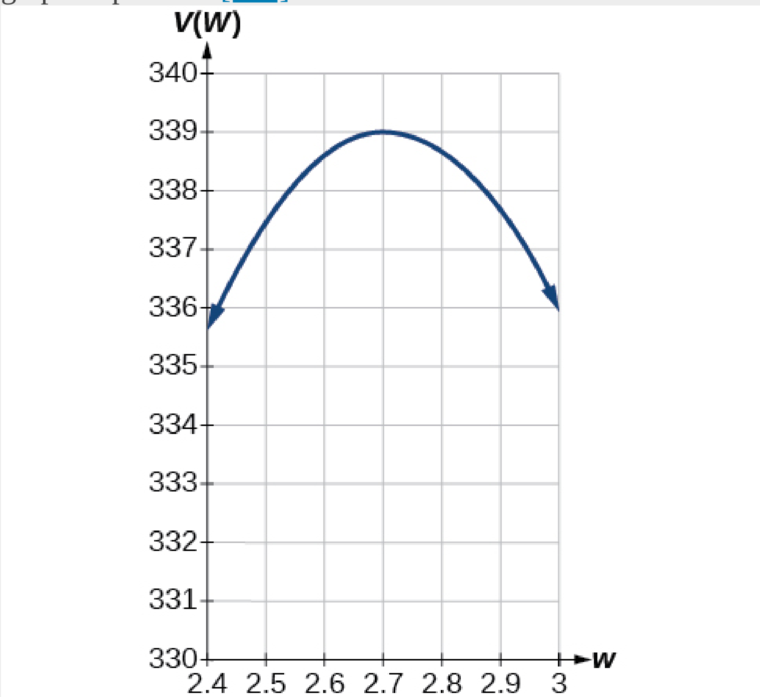
Equation:

$$\begin{aligned} V(w) &= (20-2w)(14-2w)w \\ &= 280w-68w^2+4w^3 \end{aligned}$$

Notice, since the factors are $w,\ 20–2w$ and $14–2w,$ the three zeros are 10, 7, and 0, respectively. Because a height of 0 cm is not reasonable, we consider the only the zeros 10 and 7. The shortest side is 14 and we are cutting off two squares, so values w may take on are greater than zero or less than 7. This means we will restrict the domain of this function to $0<w<7.$ Using technology to sketch the graph of $V\left(w\right)$ on this reasonable domain, we get a graph like that in [link]. We can use this graph to estimate the maximum value for the volume, restricted to values for w that are reasonable for this problem—values from 0 to 7.



From this graph, we turn our focus to only the portion on the reasonable domain, $[0,\ 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce [link].



From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm, when the squares measure approximately 2.7 cm on each side.

Note:
Exercise:

Problem:

Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

Solution:

The minimum occurs at approximately the point $(0, -6.5)$, and the maximum occurs at approximately the point $(3.5, 7)$.

Note:

Access the following online resource for additional instruction and practice with graphing polynomial functions.

- Intermediate Value Theorem

Key Concepts

- Polynomial functions of degree 2 or more are smooth, continuous functions. See [link].
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero. See [link], [link], and [link].
- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis. See [link].
- The multiplicity of a zero determines how the graph behaves at the x-intercepts. See [link].
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree n has at most $n-1$ turning points. See [link].
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points. See [link] and [link].
- Graphing a polynomial function helps to estimate local and global extremas. See [link].
- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c)=0.$ See [link].

Section Exercises

Verbal

Exercise:

Problem:

What is the difference between an x-intercept and a zero of a polynomial function f?

Solution:

The x-intercept is where the graph of the function crosses the x-axis, and the zero of the function is the input value for which $f(x)=0.$

Exercise:

Problem:

If a polynomial function of degree n has n distinct zeros, what do you know about the graph of the function?

Exercise:

Problem:

Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.

Solution:

If we evaluate the function at a and at b and the sign of the function value changes, then we know a zero exists between a and b.

Exercise:

Problem: Explain how the factored form of the polynomial helps us in graphing it.

Exercise:

Problem:

If the graph of a polynomial just touches the x-axis and then changes direction, what can we conclude about the factored form of the polynomial?

Solution:

There will be a factor raised to an even power.

Algebraic

For the following exercises, find the x- or t-intercepts of the polynomial functions.

Exercise:

Problem: $C\left(t\right)=2\left(t-4\right)\left(t+1\right)(t-6)$

Exercise:

Problem: $C\left(t\right)=3\left(t+2\right)\left(t-3\right)(t+5)$

Solution:

$(-2,0),(3,0),(-5,0)$

Exercise:

Problem: $C\left(t\right)=4t{(t-2)}^{2}(t+1)$

Exercise:

Problem: $C\left(t\right)=2t\left(t-3\right){(t+1)}^{2}$

Solution:

$(3,0),(-1,0),(0,0)$

Exercise:

Problem: $C\left(t\right)=2t^{4}-8t^{3}+6t^{2}$

Exercise:

Problem: $C\left(t\right)=4t^{4}+12t^{3}-40t^{2}$

Solution:

$(0,0),\;\;(-5,0),\;\;(2,0)$

Exercise:

Problem: $f(x)=x^{4}-x^{2}$

Exercise:

Problem: $f(x)=x^{3}+x^{2}-20x$

Solution:

$(0,0),\;\;(-5,0),\;\;(4,0)$

Exercise:

Problem: $f(x)=x^{3}+6x^{2}-7x$

Exercise:

Problem: $f(x)=x^{3}+x^{2}-4x-4$

Solution:

$(2,0),\;\;(-2,0),\;\;(-1,0)$

Exercise:

Problem: $f(x)=x^{3}+2x^{2}-9x-18$

Exercise:

Problem: $f(x)=2x^{3}-x^{2}-8x+4$

Solution:

$(-2,0),\ (2,0),\ \left(\frac{1}{2},0\right)$

Exercise:

Problem: $f(x)=x^6-7x^3-8$

Exercise:

Problem: $f(x)=2x^4+6x^2-8$

Solution:

$(1,0),\ \ (-1,0)$

Exercise:

Problem: $f(x)=x^3-3x^2-x+3$

Exercise:

Problem: $f(x)=x^6-2x^4-3x^2$

Solution:

$(0,0),\ (\sqrt{3},0),\ (-\sqrt{3},0)$

Exercise:

Problem: $f(x)=x^6-3x^4-4x^2$

Exercise:

Problem: $f(x)=x^5-5x^3+4x$

Solution:

$(0,0),\ \ (1,0),\ \ (-1,0),\ \ (2,0),\ \ (-2,0)$

For the following exercises, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.

Exercise:

Problem: $f(x)=x^3-9x,$ between $x=-4$ and $x=-2.$

Exercise:

Problem: $f(x)=x^3-9x,$ between $x=2$ and $x=4.$

Solution:

$f(2)=-10$ and $f(4)=28$. Sign change confirms.

Exercise:

Problem: $f(x)=x^5-2x,$ between $x=1$ and $x=2$.

Exercise:

Problem: $f(x)=-x^4+4,$ between $x=1$ and $x=3$.

Solution:

$f(1)=3$ and $f(3)=-77.$ Sign change confirms.

Exercise:

Problem: $f(x)=-2x^3-x,$ between $x=-1$ and $x=1$.

Exercise:

Problem: $f(x)=x^3-100x+2,$ between $x=0.01$ and $x=0.1$

Solution:

$f(0.01)=1.000001$ and $f(0.1)=-7.999.$ Sign change confirms.

For the following exercises, find the zeros and give the multiplicity of each.

Exercise:

Problem: $f(x)=(x+2)^3(x-3)^2$

Exercise:

Problem: $f(x)=x^2(2x+3)^5(x-4)^2$

Solution:

0 with multiplicity 2, $-\frac{3}{2}$ with multiplicity 5, 4 with multiplicity 2

Exercise:

Problem: $f(x)=x^3(x-1)^3\,(x+2)$

Exercise:

Problem: $f(x) = x^2 \left(x^2 + 4x + 4\right)$

Solution:

0 with multiplicity 2, –2 with multiplicity 2

Exercise:

Problem: $f(x) = (2x + 1)^3 \left(9x^2 - 6x + 1\right)$

Exercise:

Problem: $f(x) = (3x + 2)^5 \left(x^2 - 10x + 25\right)$

Solution:

$-\frac{2}{3}$ with multiplicity 5, 5 with multiplicity 2

Exercise:

Problem: $f(x) = x \left(4x^2 - 12x + 9\right) \left(x^2 + 8x + 16\right)$

Exercise:

Problem: $f(x) = x^6 - x^5 - 2x^4$

Solution:

0 with multiplicity 4, 2 with multiplicity 1, -1 with multiplicity 1

Exercise:

Problem: $f(x) = 3x^4 + 6x^3 + 3x^2$

Exercise:

Problem: $f(x) = 4x^5 - 12x^4 + 9x^3$

Solution:

$\frac{3}{2}$ with multiplicity 2, 0 with multiplicity 3

Exercise:

Problem: $f(x) = 2x^4 \left(x^3 - 4x^2 + 4x\right)$

Exercise:

Problem: $f(x) = 4x^4\left(9x^4 - 12x^3 + 4x^2\right)$

Solution:

0 with multiplicity 6, $\frac{2}{3}$ with multiplicity 2

Graphical

For the following exercises, graph the polynomial functions. Note x- and y-intercepts, multiplicity, and end behavior.

Exercise:

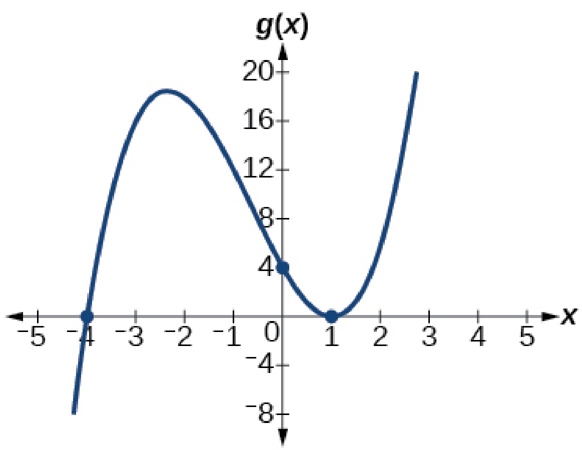
Problem: $f\left(x\right) = (x + 3)^2(x - 2)$

Exercise:

Problem: $g\left(x\right) = (x + 4)(x - 1)^2$

Solution:

x-intercepts, $(1, 0)$ with multiplicity 2, $(-4, 0)$ with multiplicity 1, y- intercept $(0, 4)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.



Exercise:

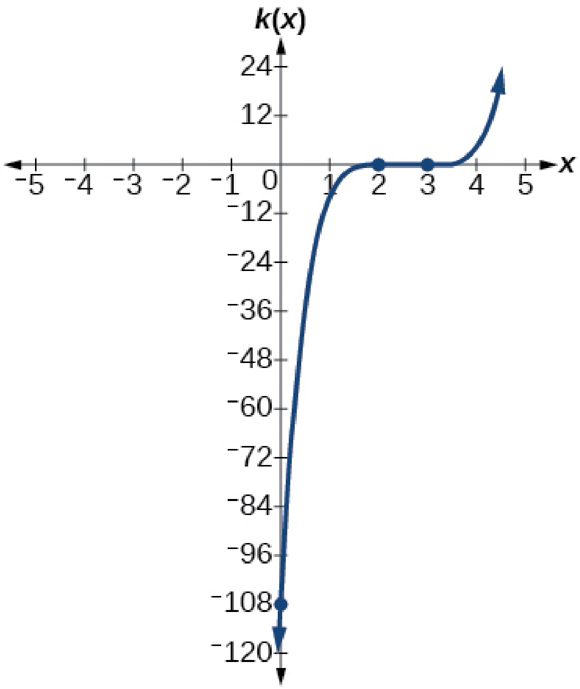
Problem: $h\left(x\right) = (x - 1)^3(x + 3)^2$

Exercise:

Problem: $k\left(x\right) = (x - 3)^3(x - 2)^2$

Solution:

x-intercepts $(3, 0)$ with multiplicity 3, $(2, 0)$ with multiplicity 2, y- intercept $(0, –108)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.



Exercise:

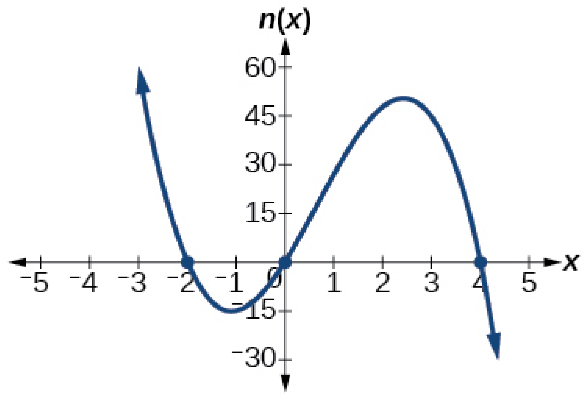
Problem:$m\left(x\right) = -2x\left(x - 1\right)(x + 3)$

Exercise:

Problem:$n\left(x\right) = -3x\left(x + 2\right)(x - 4)$

Solution:

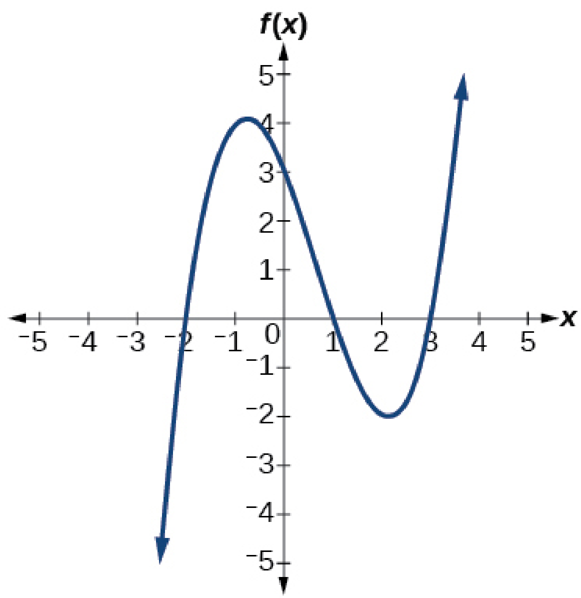
x-intercepts $(0,\,0),\ (-2,\,0),\ (4,\,0)$ with multiplicity 1, y-intercept $(0,\,0)$. As $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$.



For the following exercises, use the graphs to write the formula for a polynomial function of least degree.

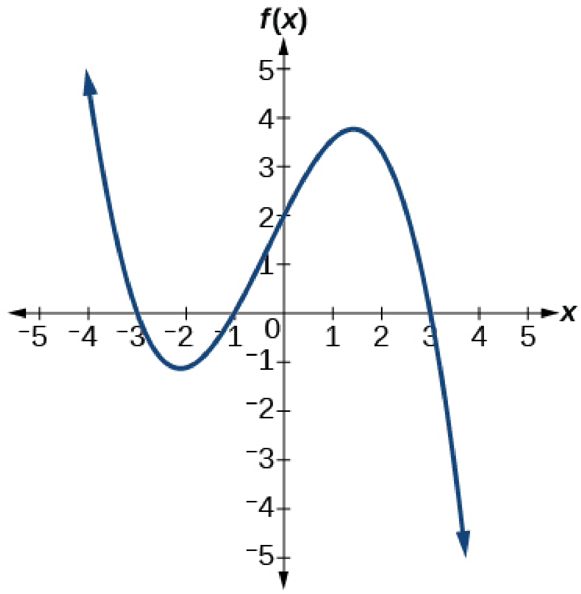
Exercise:

Problem:



Exercise:

Problem:

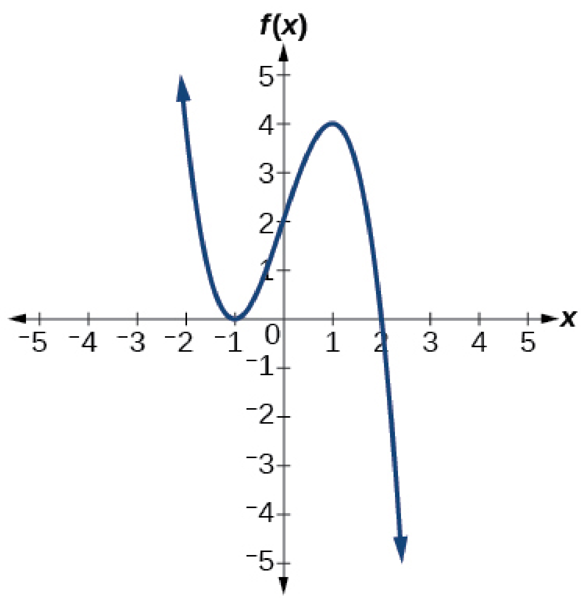


Solution:

$f(x) = -\frac{2}{9}(x-3)(x+1)(x+3)$

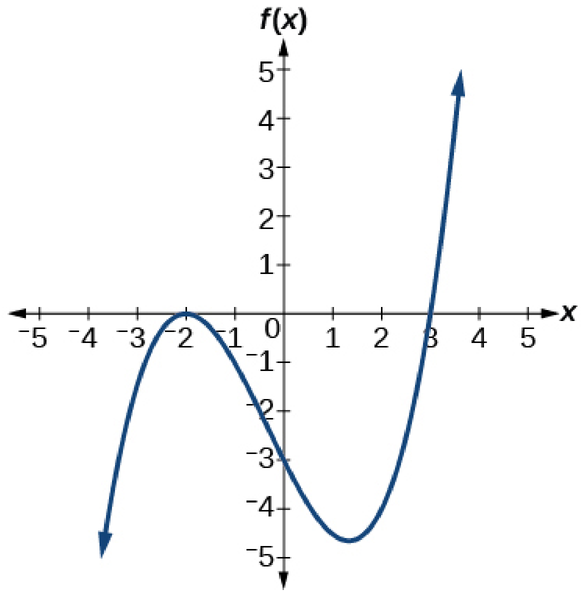
Exercise:

Problem:



Exercise:

Problem:

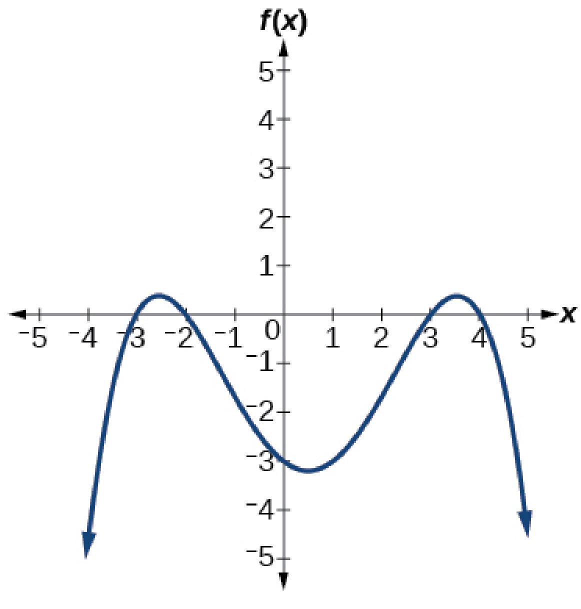


Solution:

$f(x) = \frac{1}{4}(x+2)^2(x-3)$

Exercise:

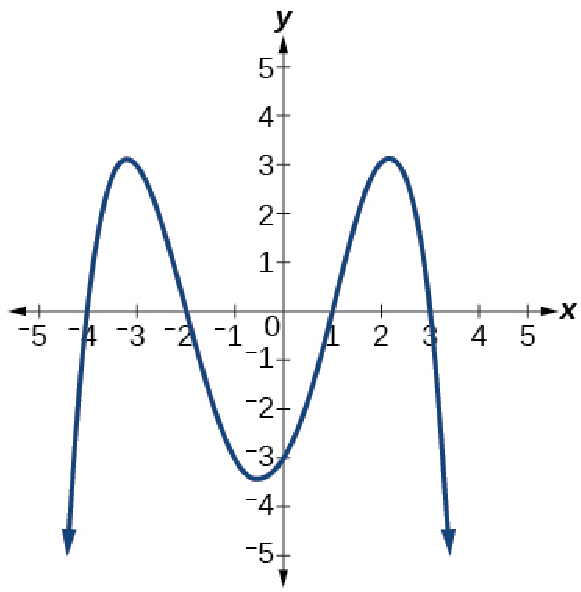
Problem:



For the following exercises, use the graph to identify zeros and multiplicity.

Exercise:

Problem:

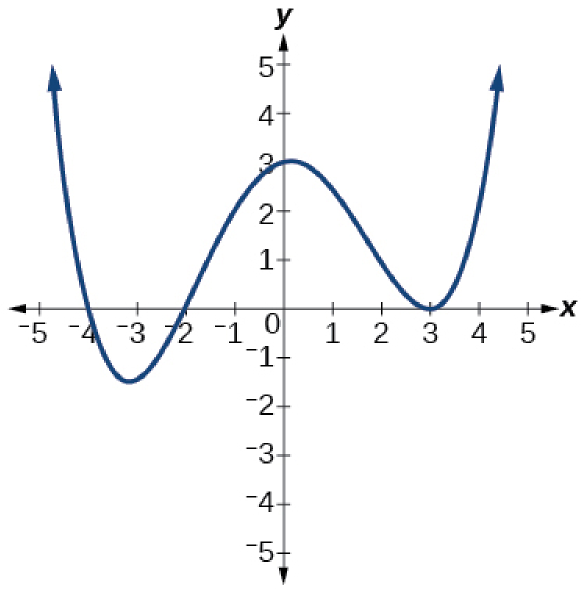


Solution:

–4, –2, 1, 3 with multiplicity 1

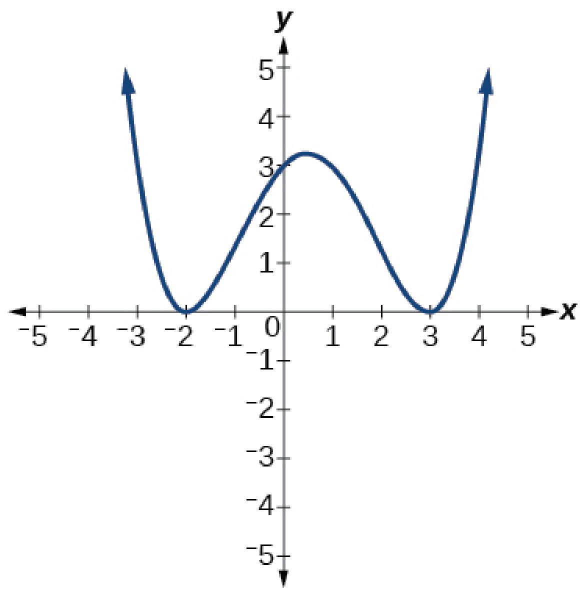
Exercise:

Problem:



Exercise:

Problem:

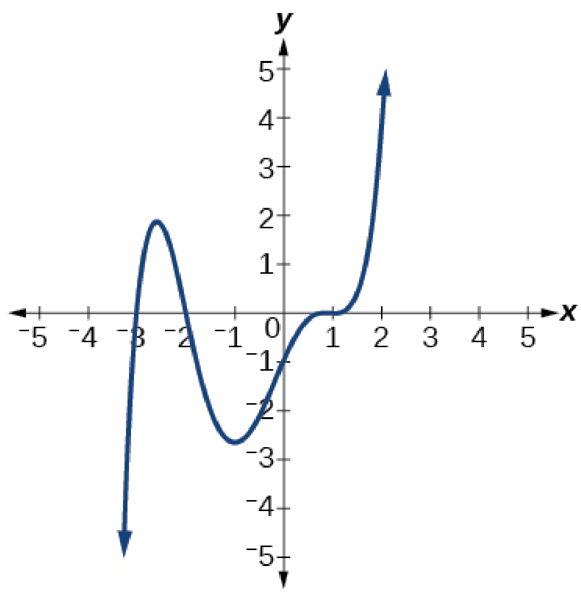


Solution:

–2, 3 each with multiplicity 2

Exercise:

Problem:



For the following exercises, use the given information about the polynomial graph to write the equation.

Exercise:

Problem: Degree 3. Zeros at $x = -2,\ x = 1$, and $x = 3$. y-intercept at $(0, -4)$.

Solution:

$f(x) = -\frac{2}{3}(x+2)(x-1)(x-3)$

Exercise:

Problem: Degree 3. Zeros at $x = -5,\ x = -2$, and $x = 1$. y-intercept at $(0, 6)$

Exercise:

Problem:

Degree 5. Roots of multiplicity 2 at $x = 3$ and $x = 1$, and a root of multiplicity 1 at $x = -3$. y-intercept at $(0, 9)$

Solution:

$f(x) = \frac{1}{3}(x-3)^2(x-1)^2(x+3)$

Exercise:

Problem:

Degree 4. Root of multiplicity 2 at $x = 4$, and a roots of multiplicity 1 at $x = 1$ and $x = -2$. y-intercept at $(0, -3)$.

Exercise:

Problem:

Degree 5. Double zero at $x = 1$, and triple zero at $x = 3$. Passes through the point $(2, 15)$.

Solution:

$f(x) = -15(x-1)^2(x-3)^3$

Exercise:

Problem: Degree 3. Zeros at $x = 4$, $x = 3$, and $x = 2$. y-intercept at $(0, -24)$.

Exercise:

Problem: Degree 3. Zeros at $x = -3,\ x = -2$ and $x = 1$. y-intercept at $(0, 12)$.

Solution:

$f(x) = -2\,(x+3)\,(x+2)\,(x-1)$

Exercise:

Problem:

Degree 5. Roots of multiplicity 2 at $x=-3$ and $x=2$ and a root of multiplicity 1 at $x=-2.$

y-intercept at $(0,\ 4).$

Exercise:

Problem:

Degree 4. Roots of multiplicity 2 at $x=\frac{1}{2}$ and roots of multiplicity 1 at $x=6$ and $x=-2.$

y-intercept at $(0{,}18).$

Solution:

$f(x)=-\frac{3}{2}(2x-1)^2\,(x-6)\,(x+2)$

Exercise:

Problem: Double zero at $x=-3$ and triple zero at $x=0.$ Passes through the point $(1,32).$

Technology

For the following exercises, use a calculator to approximate local minima and maxima or the global minimum and maximum.

Exercise:

Problem:$f(x)=x^3-x-1$

Solution:

local max $(–.58, –.62),$ local min $(.58, –1.38)$

Exercise:

Problem:$f(x)=2x^3-3x-1$

Exercise:

Problem:$f(x)=x^4+x$

Solution:

global min $(–.63, –.47)$

Exercise:

Problem: $f(x) = -x^4 + 3x - 2$

Exercise:

Problem: $f(x) = x^4 - x^3 + 1$

Solution:

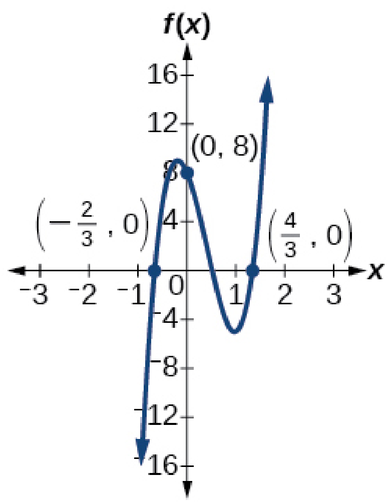
global min $(.75, .89)$

Extensions

For the following exercises, use the graphs to write a polynomial function of least degree.

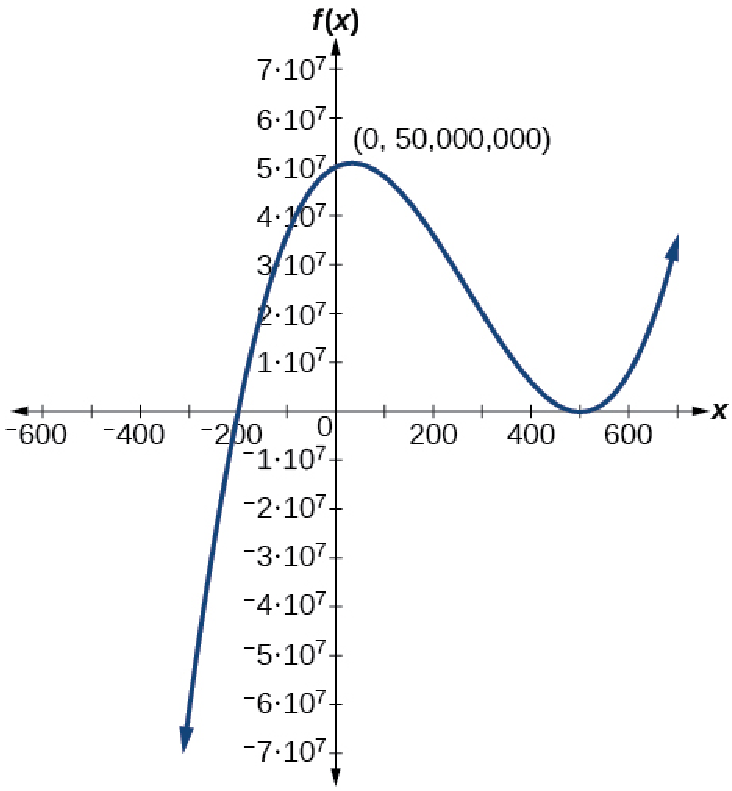
Exercise:

Problem:



Exercise:

Problem:

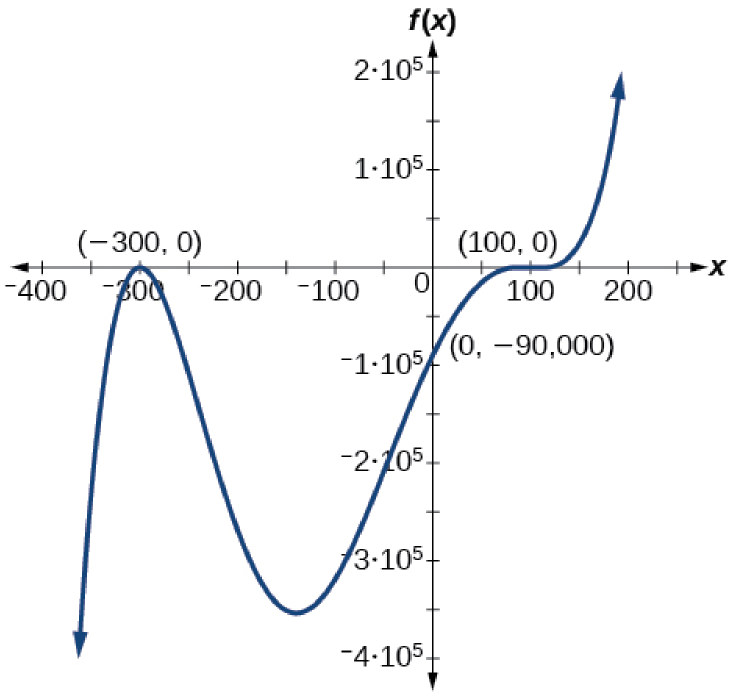


Solution:

$f(x) = (x - 500)^2(x + 200)$

Exercise:

Problem:



Real-World Applications

For the following exercises, write the polynomial function that models the given situation.

Exercise:

Problem:

A rectangle has a length of 10 units and a width of 8 units. Squares of x by x units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of x.

Solution:

$f(x) = 4x^3 - 36x^2 + 80x$

Exercise:

Problem:

Consider the same rectangle of the preceding problem. Squares of $2x$ by $2x$ units are cut out of each corner. Express the volume of the box as a polynomial in terms of x.

Exercise:

Problem:

A square has sides of 12 units. Squares $x + 1$ by $x + 1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of x.

Solution:

$f(x) = 4x^3 - 36x^2 + 60x + 100$

Exercise:

Problem:

A cylinder has a radius of $x + 2$ units and a height of 3 units greater. Express the volume of the cylinder as a polynomial function.

Exercise:

Problem:

A right circular cone has a radius of $3x + 6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ for radius r and height h.

Solution:

$f(x) = \pi(9x^3 + 45x^2 + 72x + 36)$

Glossary

global maximum
: highest turning point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all x.

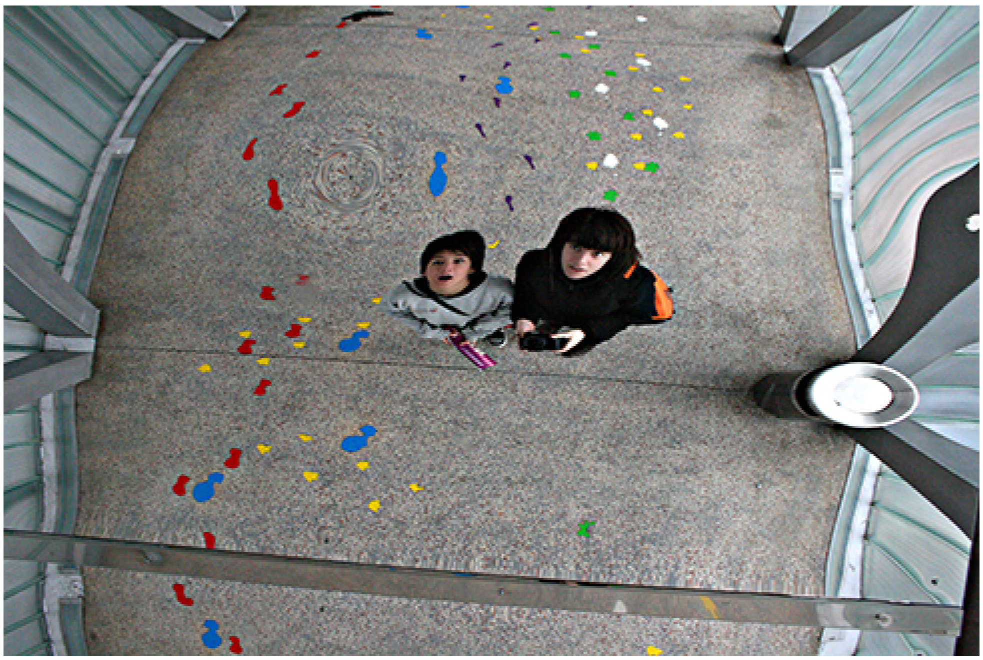
global minimum
: lowest turning point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all x.

Intermediate Value Theorem
: for two numbers a and b in the domain of $f,$ if $a < b$ and $f(a) \neq f(b),$ then the function f takes on every value between $f(a)$ and $f(b);$ specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis

multiplicity
: the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x - h)^p,$ $x = h$ is a zero of multiplicity p.

Transformation of Functions
In this section, you will:

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.



(credit: "Misko"/Flickr)

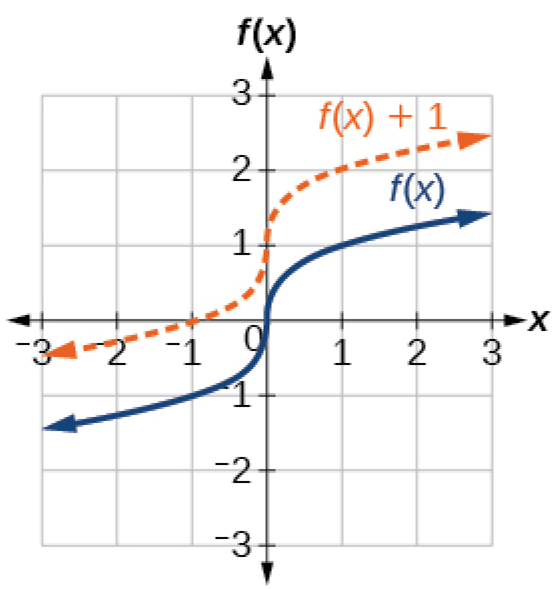
We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

Graphing Functions Using Vertical and Horizontal Shifts

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f\left(x\right)$ is shifted vertically k units. See [link] for an example.



Vertical shift by $k = 1$ of the cube root function

$$f(x) = \sqrt[3]{x}.$$

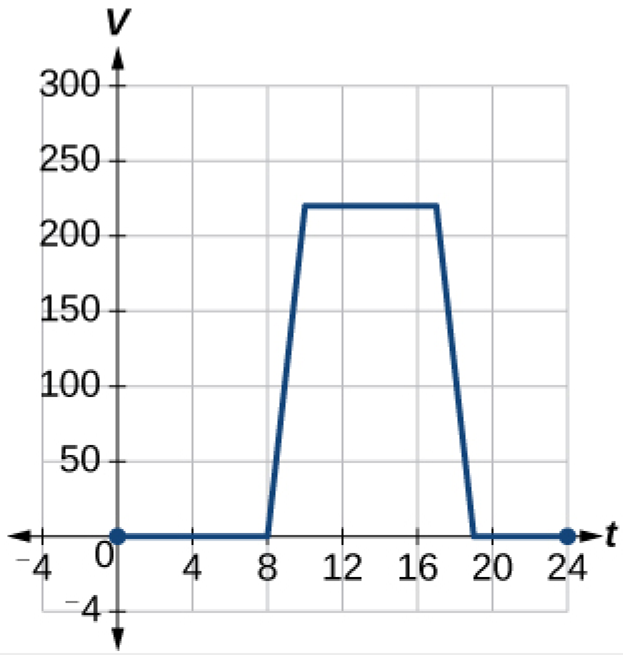
To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Note:
Vertical Shift
Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example:
Exercise:

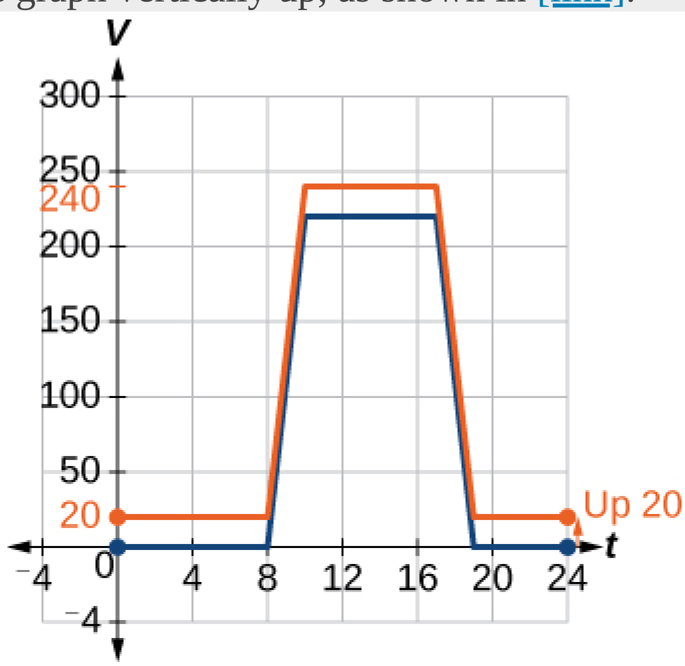
Problem:
Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. [link] shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.



Solution:

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in [link].



Notice that in [link], for each input value, the output value has increased by 20, so if we call the new function $S(t)$, we could write

Equation:

$$S(t) = V(t) + 20$$

This notation tells us that, for any value of t, $S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See [link].

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

Note:

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example:
Exercise:

Problem:
Shifting a Tabular Function Vertically

A function $f\left(x\right)$ is given in [link]. Create a table for the function $g(x)=f(x)-3.$

x	2	4	6	8
$f(x)$	1	3	7	11

Solution:

The formula $g(x)=f(x)-3$ tells us that we can find the output values of g by subtracting 3 from the output values of $f.$ For example:
Equation:

$$\begin{array}{lll}f(2)=1 & & \text{Given}\\ g(x)=f(x)-3 & & \text{Given transformation}\\ g(2)=f(2)-3 & & \\ =1-3 & & \\ =-2 & & \end{array}$$

Subtracting 3 from each $f\left(x\right)$ value, we can complete a table of values for $g\left(x\right)$ as shown in [link].

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	−2	0	4	8

Analysis

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Note:
Exercise:

Problem:

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t),$ and then find a formula for $b(t)$.

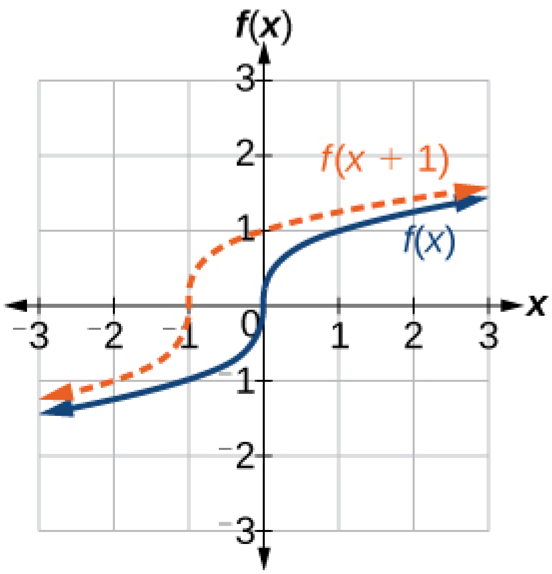
Solution:
Equation:

$$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$$

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a

movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in [link].



Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = +1$ shifts the graph to the left, that is, towards *negative* values of x.

For example, if $f(x) = x^2$, then $g(x) = (x - 2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Note:
Horizontal Shift
Given a function f, a new function $g(x) = f(x - h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example:
Exercise:

Problem:
Adding a Constant to an Input

Returning to our building airflow example from [link], suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

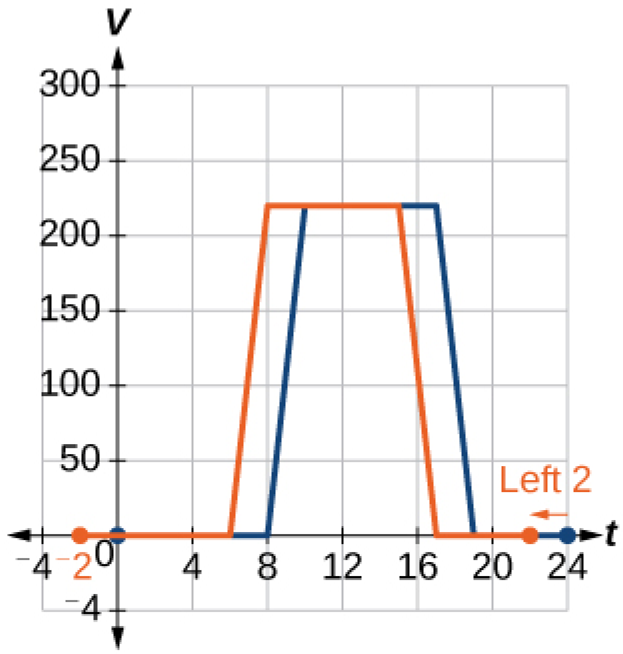
Solution:

We can set $V\left(t\right)$ to be the original program and $F\left(t\right)$ to be the revised program.
Equation:

$$\begin{array}{l}V\left(t\right)=\text{ the original venting plan}\\ \text{F}\left(t\right)=\text{starting 2 hrs sooner}\end{array}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function $F,$ the airflow starts to change at 6 a.m. The comparable function values are $V(8)=F(6).$ See [link]. Notice also that the vents first opened to 220 ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220 ft^2 at 8 a.m., so $V(10)=F(8).$

In both cases, we see that, because $F\left(t\right)$ starts 2 hours sooner, $h=-2.$ That means that the same output values are reached when $F(t)=V(t-(-2))=V\left(t+2\right).$



Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the *left*.

Horizontal changes or “inside changes” affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F : F(t) = V(t+2)$.

Note:
Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Example:
Exercise:

Problem:
Shifting a Tabular Function Horizontally

A function $f(x)$ is given in [link]. Create a table for the function $g(x) = f(x - 3)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Solution:

The formula $g(x) = f(x - 3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 *larger*. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

Equation:

$$\begin{aligned} g(5) &= f(5 - 3) \\ &= f(2) \\ &= 1 \end{aligned}$$

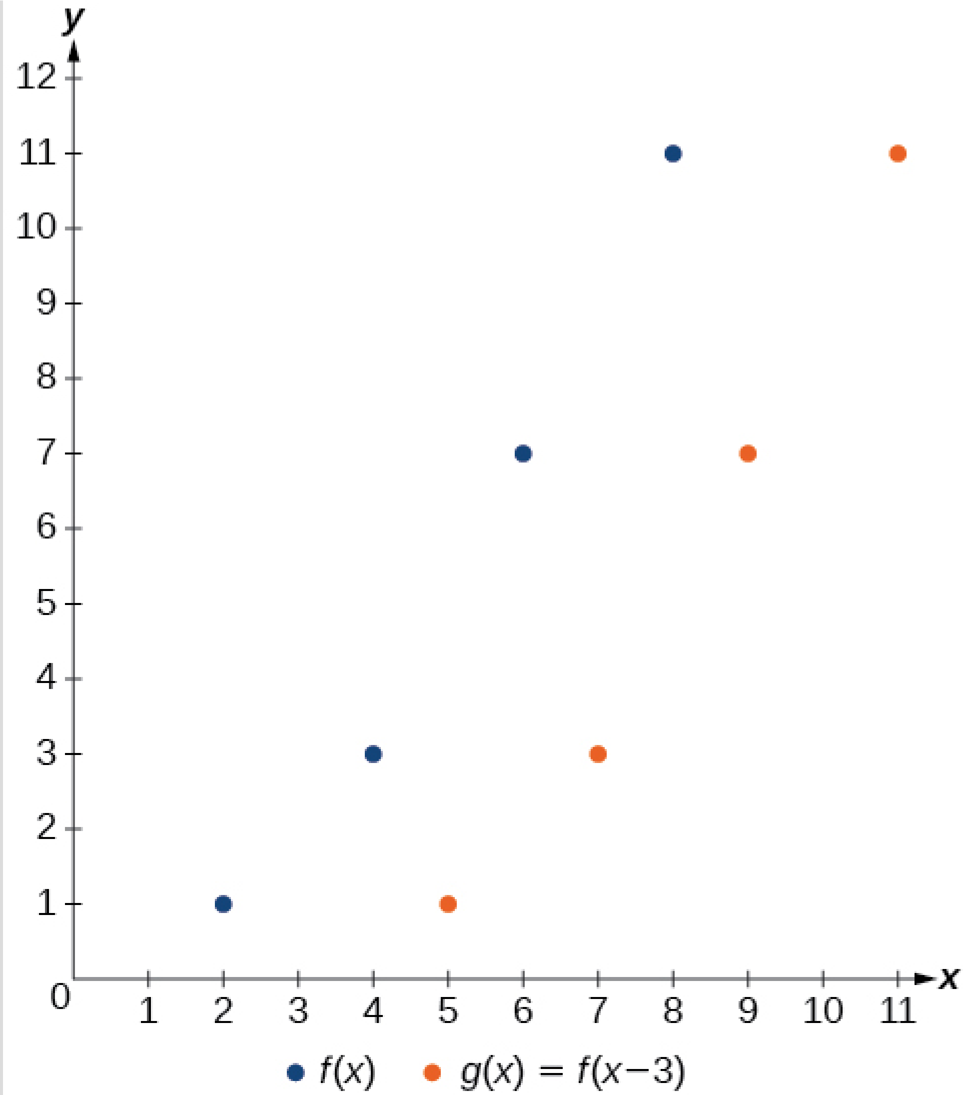
We continue with the other values to create [link].

x	5	7	9	11
$x-3$	2	4	6	8
$f(x-3)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x),$ but the corresponding input values, $x,$ have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis

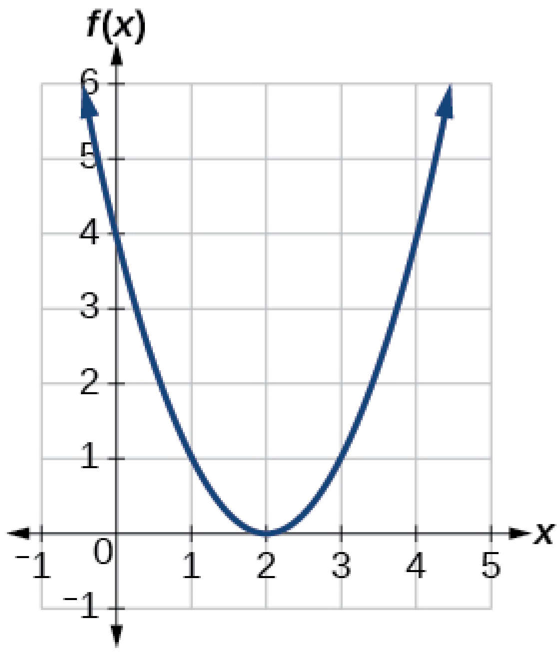
[link] represents both of the functions. We can see the horizontal shift in each point.



Example:
Exercise:

Problem:
Identifying a Horizontal Shift of a Toolkit Function

[link] represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.



Solution:

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the *x*-values are shifted to the right 2 units. The vertex used to be at (0,0), but now the vertex is at (2,0). The graph is the basic quadratic function shifted 2 units to the right, so

Equation:

$$g(x) = f(x - 2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0;$ the *x*-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x - 2)$.

Equation:

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x - 2) \\ g(x) &= f(x - 2) = (x - 2)^2 \end{aligned}$$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0)=0.$ In our shifted function, $g(2)=0.$ To obtain the output value of 0 from the function $f,$ we need to decide whether a plus or a minus sign will work to satisfy $g(2)=f(x-2)=f(0)=0.$ For this to work, we will need to *subtract* 2 units from our input values.

Example:
Exercise:

Problem:
Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)+10$ and $G(m+10).$

Solution:

$G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

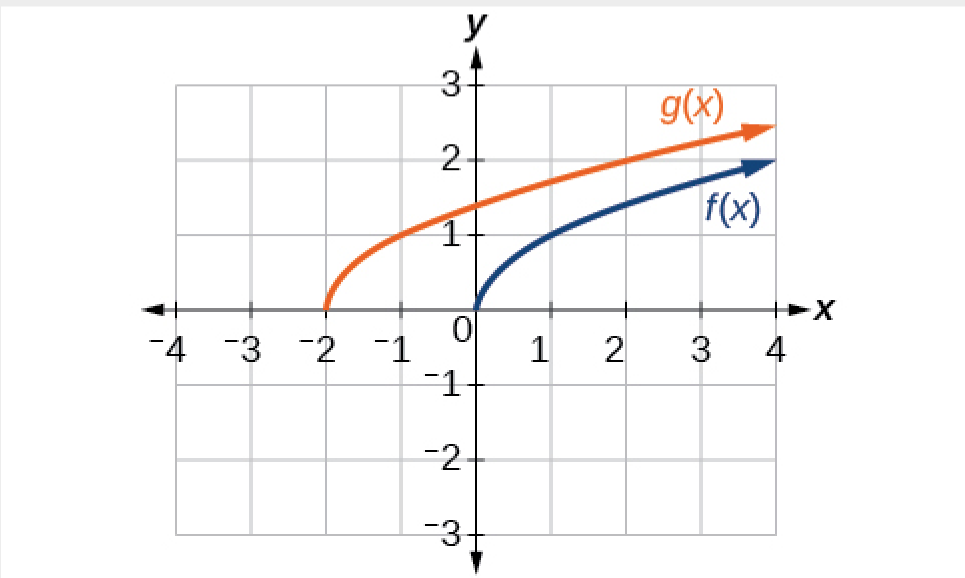
Note:
Exercise:

Problem:

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Solution:

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.



Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output (y-) axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input (x-) axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down *and* right or left.

Note:

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Example:

Exercise:

Problem:

Graphing Combined Vertical and Horizontal Shifts

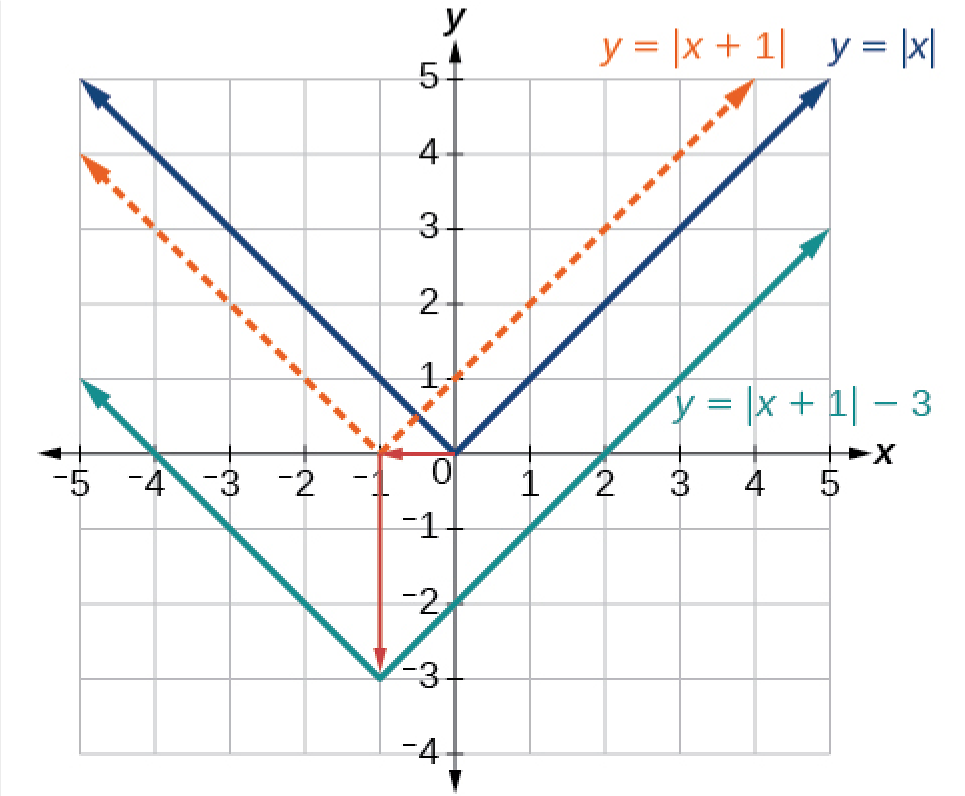
Given $f(x) = |x|$, sketch a graph of $h(x) = f(x+1) - 3$.

Solution:

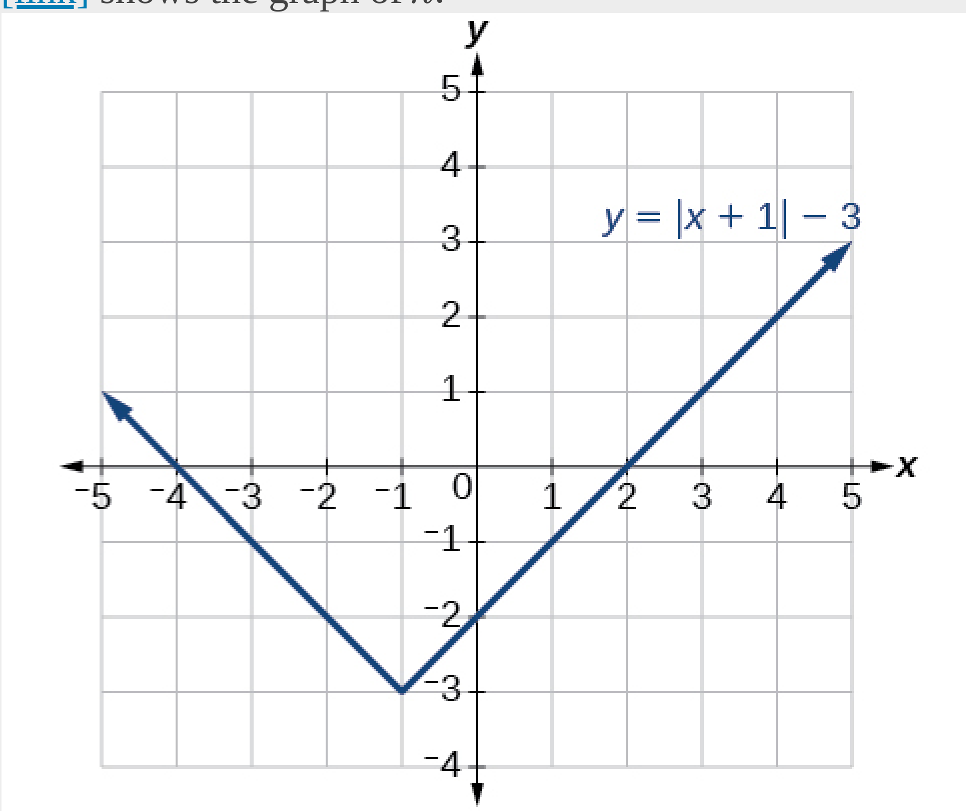
The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in [link].

Let us follow one point of the graph of $f(x) = |x|$.

- The point$(0, 0)$is transformed first by shifting left 1 unit: $(0, 0) \to (-1, 0)$
- The point$(-1, 0)$is transformed next by shifting down 3 units: $(-1, 0) \to (-1, -3)$



[link] shows the graph of h.

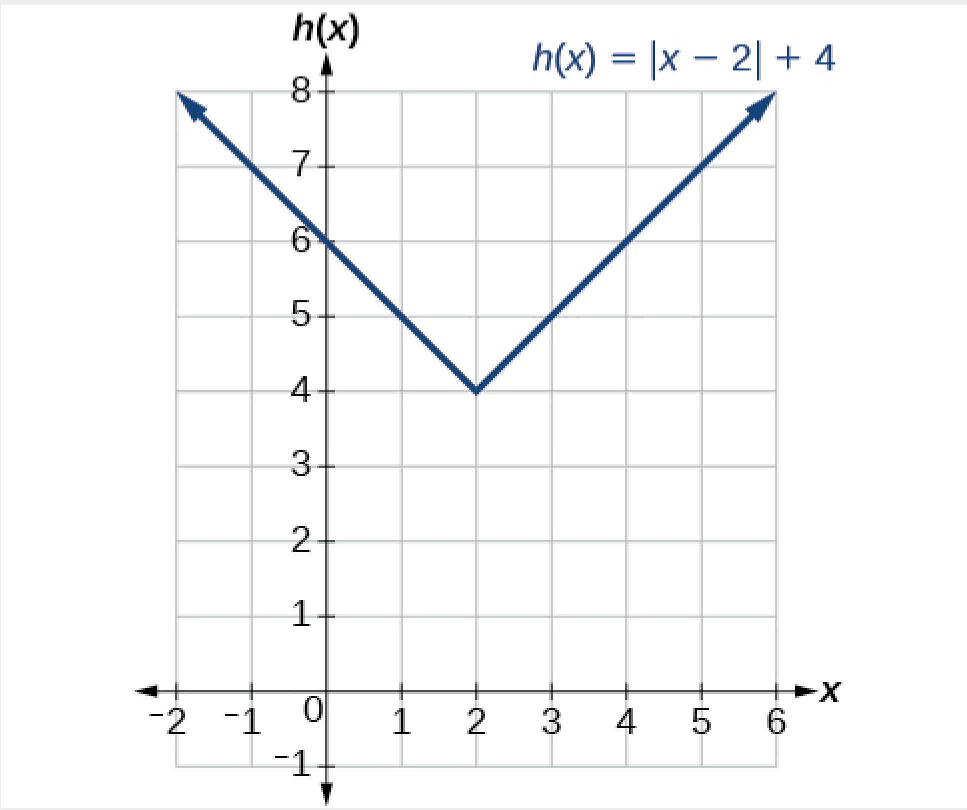


Note:
Exercise:

Problem:

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x-2)+4$.

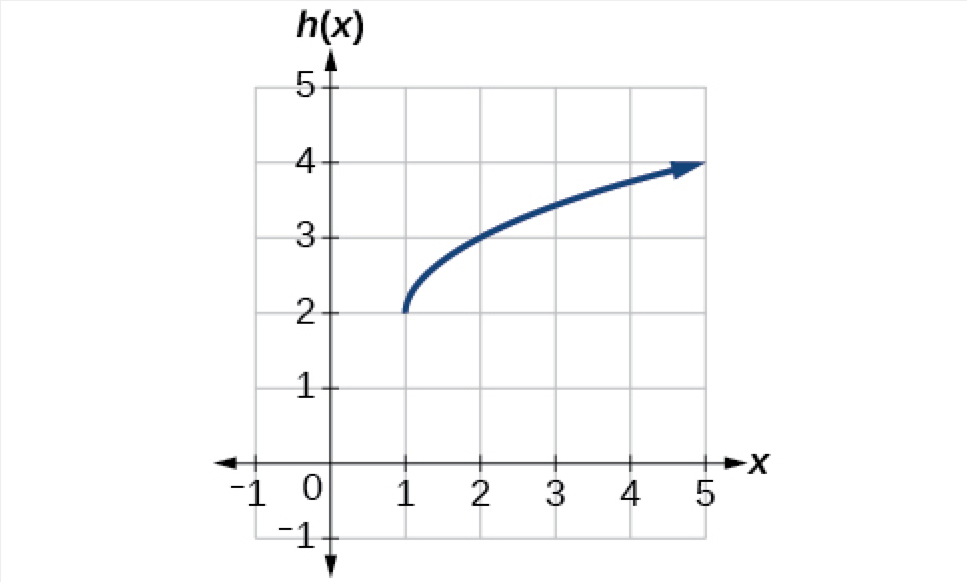
Solution:



Example:
Exercise:

Problem:
Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in [link], which is a transformation of the toolkit square root function.



Solution:

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

Equation:

$$h(x) = f(x - 1) + 2$$

Using the formula for the square root function, we can write

Equation:

$$h(x) = \sqrt{x - 1} + 2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Note:
Exercise:

Problem:

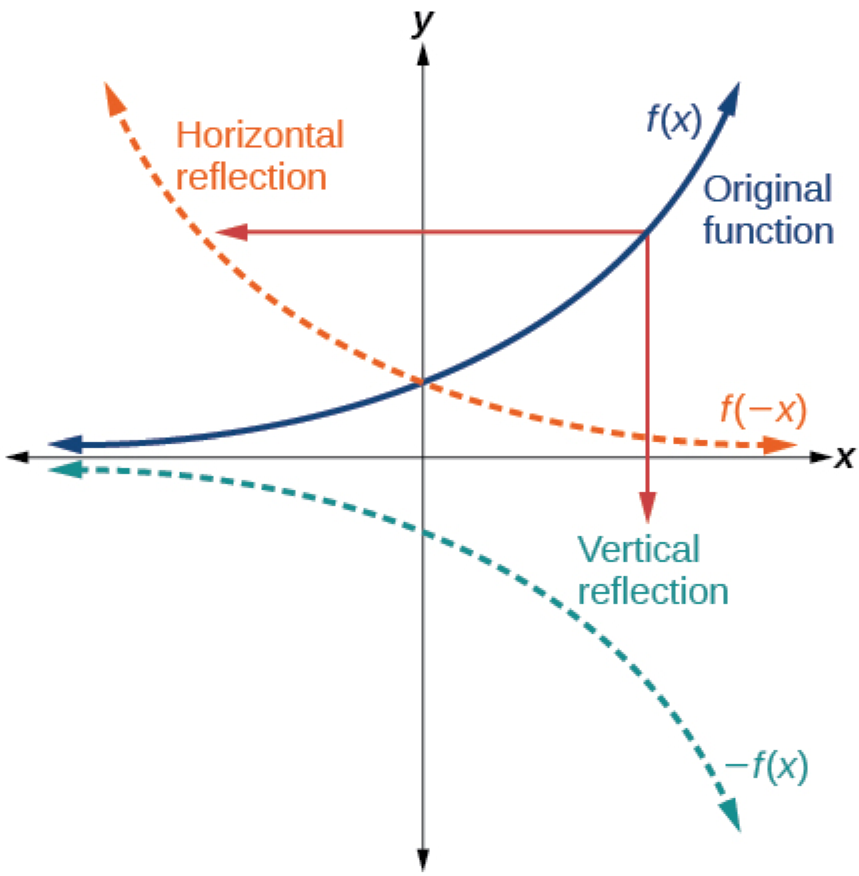
Write a formula for a transformation of the toolkit reciprocal function $f\left(x\right)=\frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Solution:

$g\left(x\right)=\frac{1}{x-1}+1$

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the *x*- or *y*-axis. A **vertical reflection** reflects a graph vertically across the *x*-axis, while a **horizontal reflection** reflects a graph horizontally across the *y*-axis. The reflections are shown in [link].



Vertical and horizontal reflections of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the *x*-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the *y*-axis.

Note:
Reflections
Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the *x*-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the *y*-axis.

Note:
Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the *x*-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the *y*-axis.

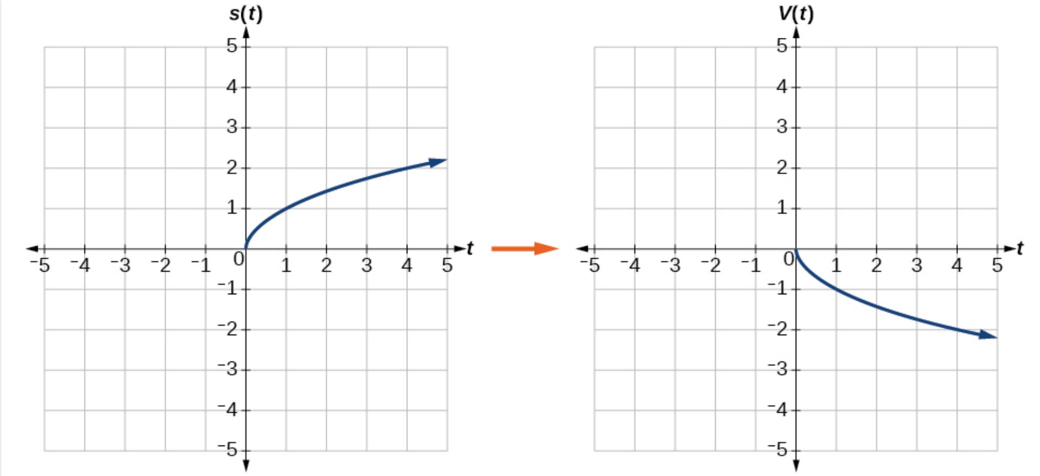
Example:
Exercise:

Problem:
Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution:

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal *t*-axis as shown in [link].



Vertical reflection of the square root function

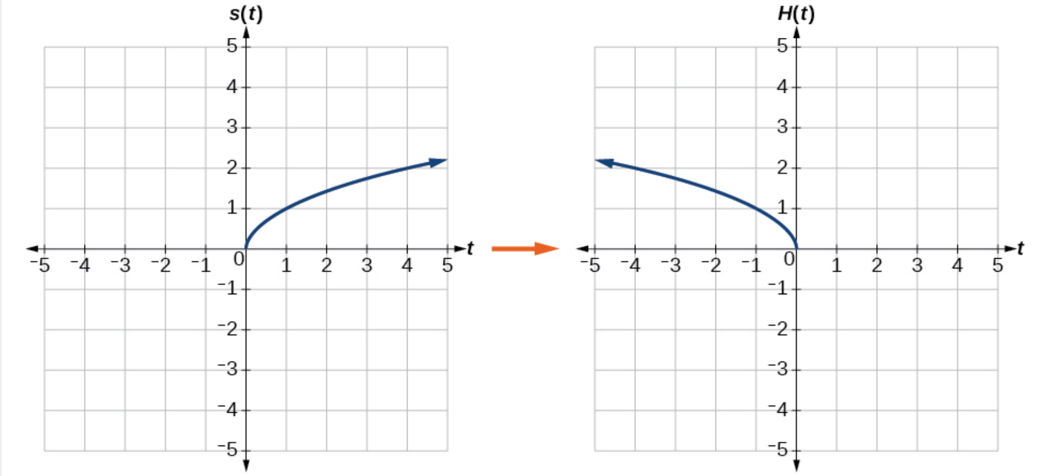
Because each output value is the opposite of the original output value, we can write

Equation:

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in [link].



Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

Equation:

$$H(t)=s(-t)\text{ or }H(t)=\sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

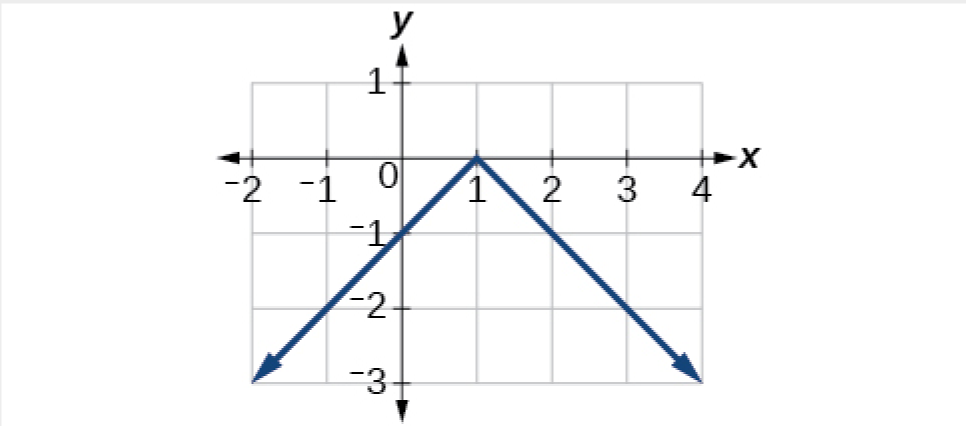
Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0,\infty)$ and range $[0,\infty),$ the vertical reflection gives the $V(t)$ function the range $(-\infty,\ 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty,\ 0]$.

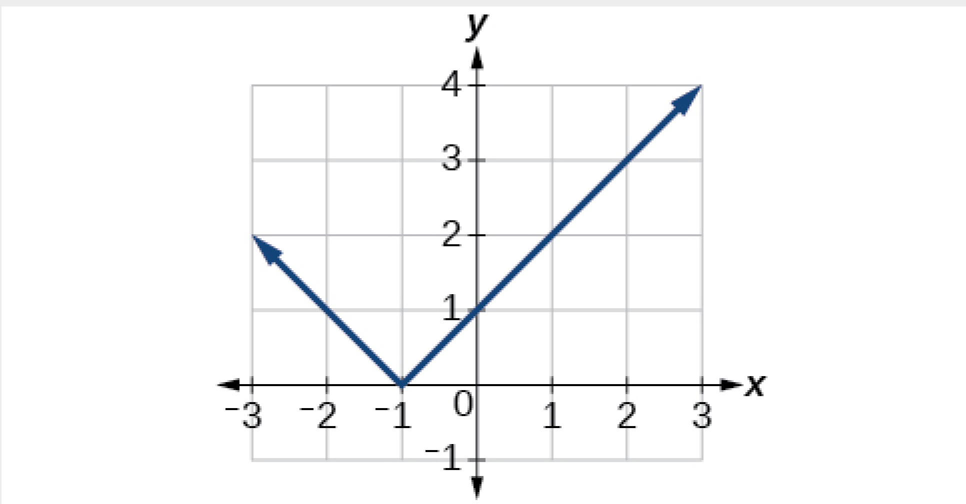
Note:
Exercise:

Problem:

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

Solution:

a. 

b. 

Example:
Exercise:

Problem:
Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as [link]. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

x	2	4	6	8
$f(x)$	1	3	7	11

Solution:

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the *x*-values stay the same and each output value will be the opposite of the original output value. See [link].

x	2	4	6	8
$g(x)$	–1	–3	–7	–11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See [link].

x	−2	−4	−6	−8
$h(x)$	1	3	7	11

Note:
Exercise:

Problem:

A function $f(x)$ is given as [link]. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

x	−2	0	2	4
$f(x)$	5	10	15	20

Solution:

a. $g(x) = -f(x)$

x	-2	0	2	4
$g(x)$	-5	-10	-15	-20

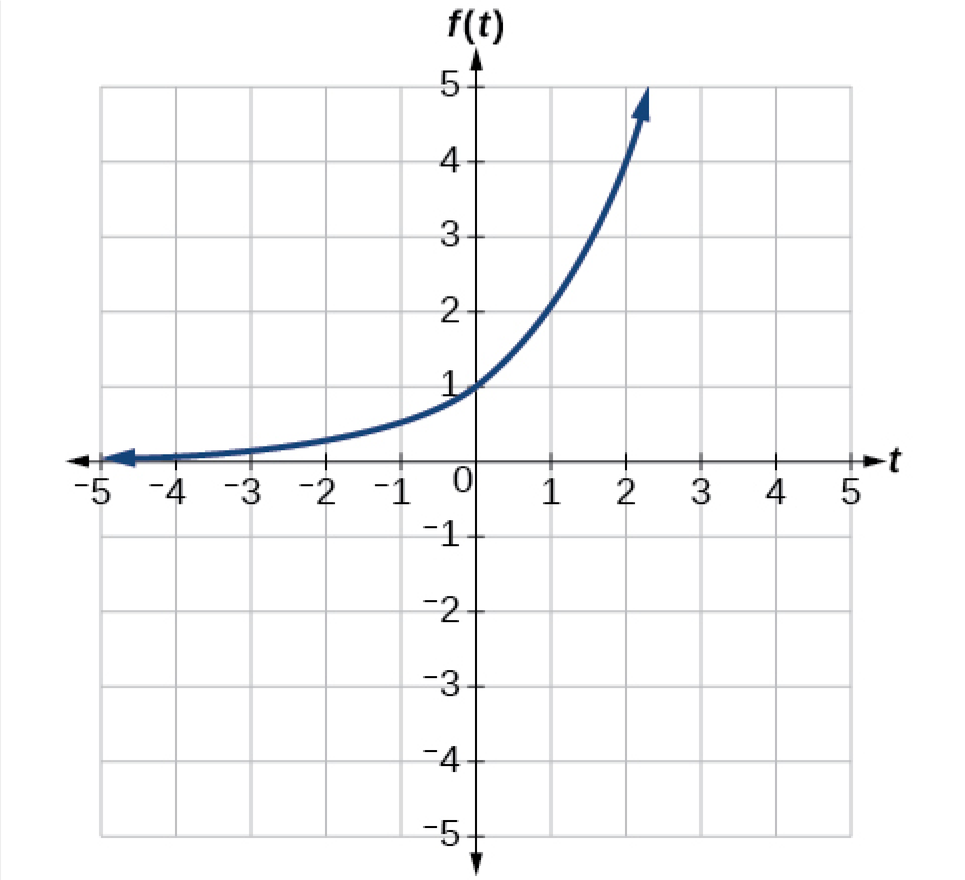
b. $h(x) = f(-x)$

x	-2	0	2	4
$h(x)$	15	10	5	unknown

Example:
Exercise:

Problem:
Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1,$ where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in [link]. Sketch a graph of $k(t)$.



Solution:

This equation combines three transformations into one equation.

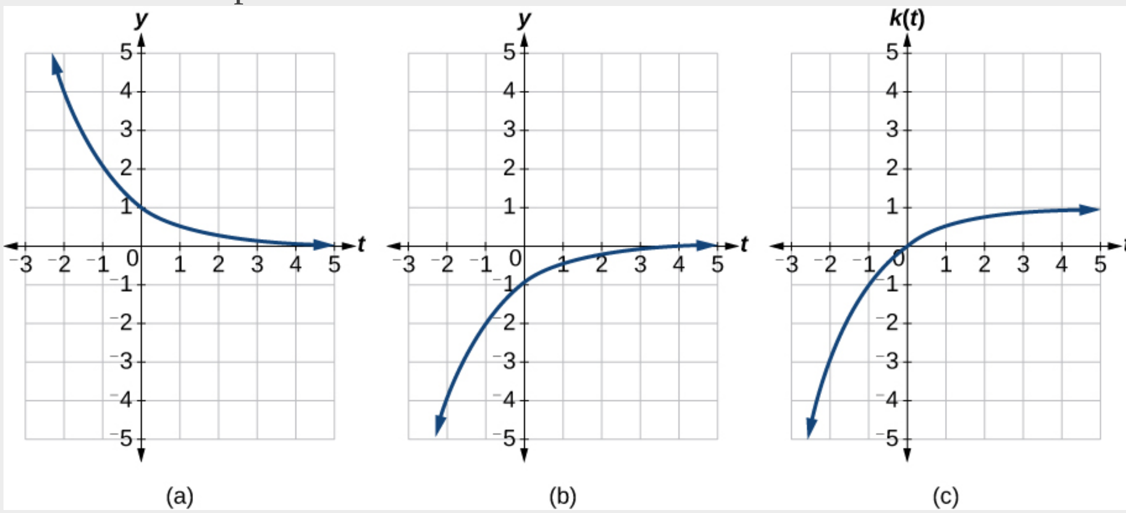
- A horizontal reflection: $f(-t) = 2^{-t}$
- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t) + 1 = -2^{-t} + 1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points (0, 1) and (1, 2).

1. First, we apply a horizontal reflection: (0, 1) (–1, 2).
2. Then, we apply a vertical reflection: (0, −1) (-1, –2).
3. Finally, we apply a vertical shift: (0, 0) (-1, -1).

This means that the original points, (0,1) and (1,2) become (0,0) and (-1,-1) after we apply the transformations.

In [link], the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.



Analysis

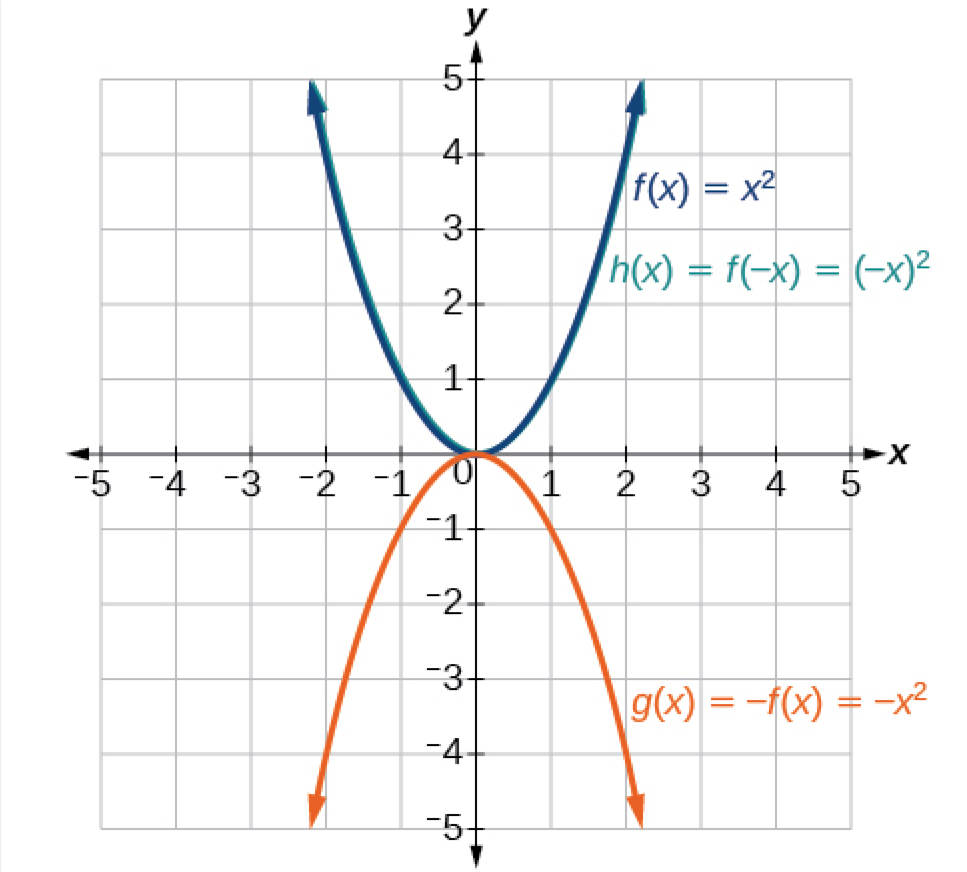
As a model for learning, this function would be limited to a domain of $t \geq 0,$ with corresponding range $[0, 1).$

Note:
Exercise:

Problem:

Given the toolkit function $f(x) = x^2,$ graph $g(x) = -f(x)$ and $h(x) = f(-x).$ Take note of any surprising behavior for these functions.

Solution:

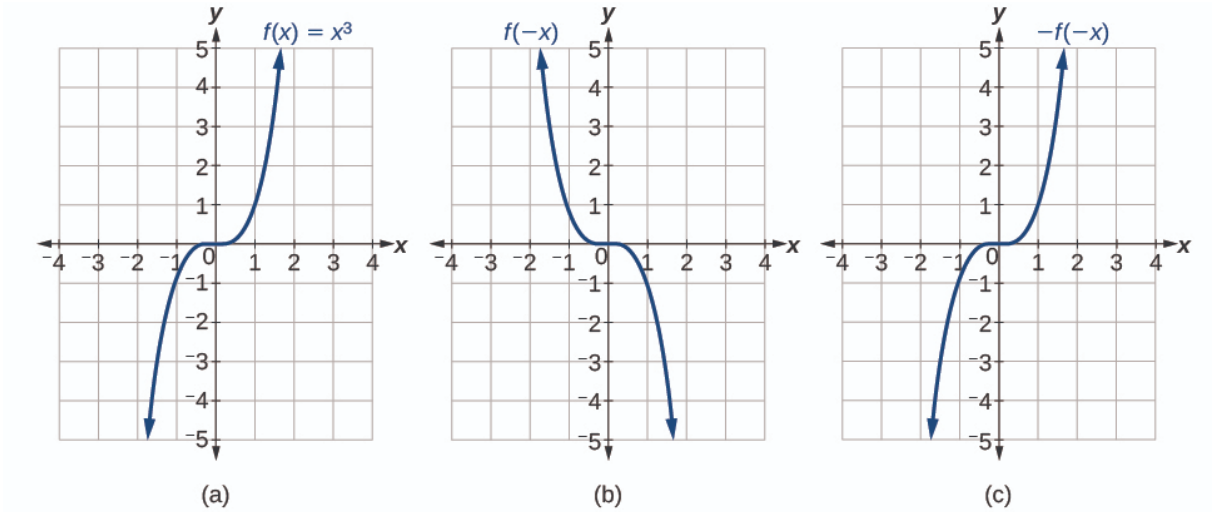


Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the *y*-axis. Functions whose graphs are symmetric about the *y*-axis are called **even functions.**

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over *both* axes, the result would be the original graph, as shown in [link].



(a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Note:
Even and Odd Functions
A function is called an **even function** if for every input x
Equation:

$$f(x) = f(-x)$$

The graph of an even function is symmetric about the y-axis.
A function is called an **odd function** if for every input x
Equation:

$$f(x) = -f(-x)$$

The graph of an odd function is symmetric about the origin.

Note:
Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Example:
Exercise:

Problem:
Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution:

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let’s begin with the rule for even functions.
Equation:

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

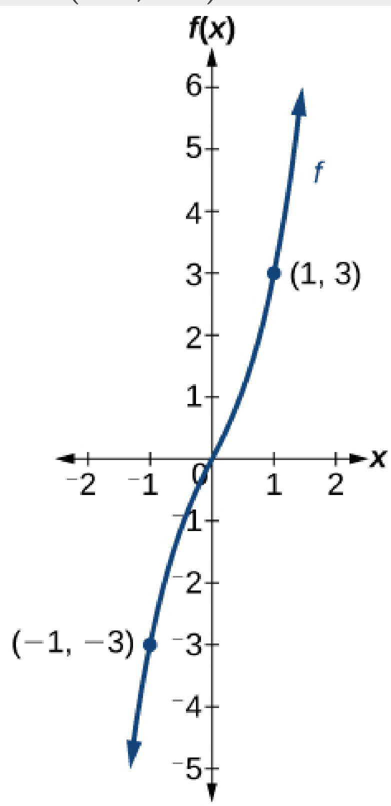
Equation:

$$-f(-x) = -\left(-x^3 - 2x\right) = x^3 + 2x$$

Because $-f(-x) = f(x),$ this is an odd function.

Analysis

Consider the graph of f in [link]. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, (1, 3) is on the graph of f, and the corresponding point$(-1, -3)$is also on the graph.



Note:
Exercise:

Problem: Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Solution:

even

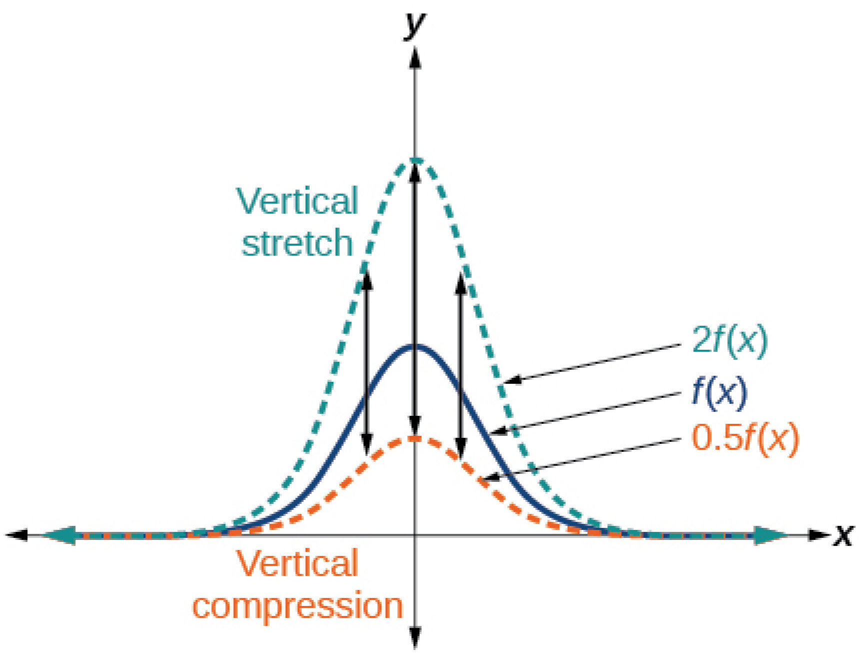
Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. [link] shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.



Vertical stretch and compression

Note:
Vertical Stretches and Compressions
Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

Note:
Given a function, graph its vertical stretch.

1. Identify the value of a.
2. Multiply all range values by a.
3. If $a > 1,$ the graph is stretched by a factor of a.

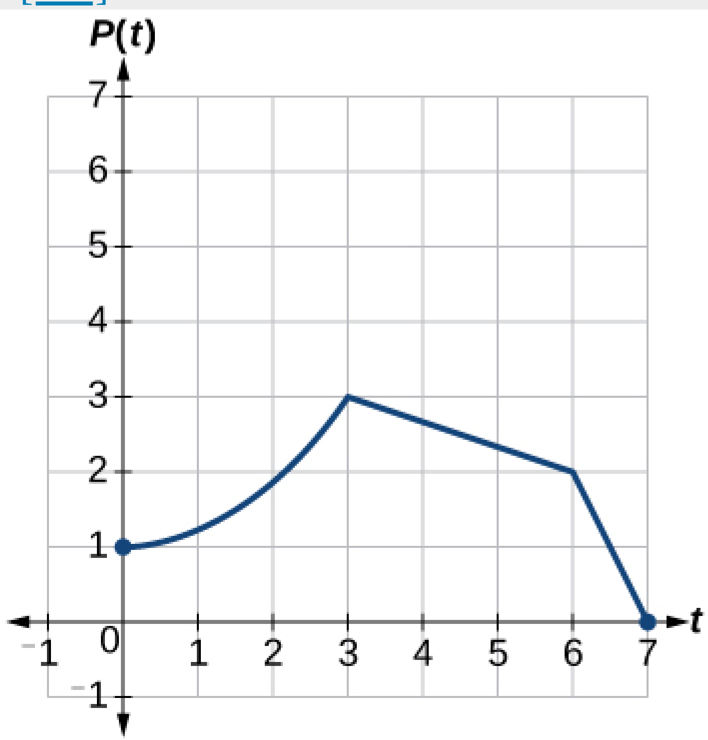
   If $0 < a < 1,$ the graph is compressed by a factor of a.

   If $a < 0,$ the graph is either stretched or compressed and also reflected about the *x*-axis.

Example:
Exercise:

Problem:
Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in [link].



A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

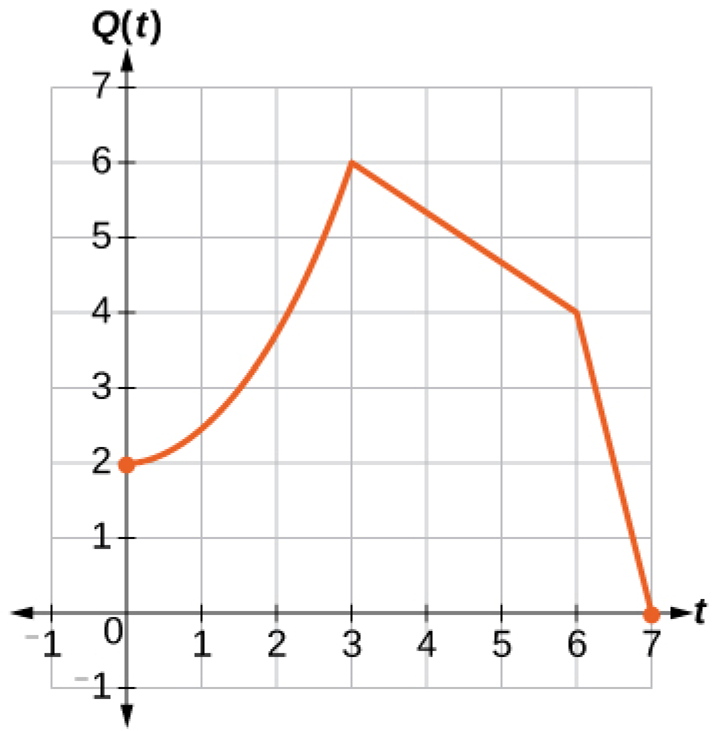
Solution:

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in [link].

If we choose four reference points, (0, 1), (3, 3), (6, 2) and (7, 0) we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.
Equation:

$$\begin{array}{c}(0,\ 1)\to (0,\ 2)\\ (3,\ 3)\to (3,\ 6)\\ (6,\ 2)\to (6,\ 4)\\ (7,\ 0)\to (7,\ 0)\end{array}$$



Symbolically, the relationship is written as

Equation:

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

Note:

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.

2. Multiply all of the output values by a.

Example:
Exercise:

Problem:
Finding a Vertical Compression of a Tabular Function

A function f is given as [link]. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Solution:

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

Equation:

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce [link].

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Note:
Exercise:

Problem:

A function f is given as [link]. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

x	2	4	6	8
$f(x)$	12	16	20	0

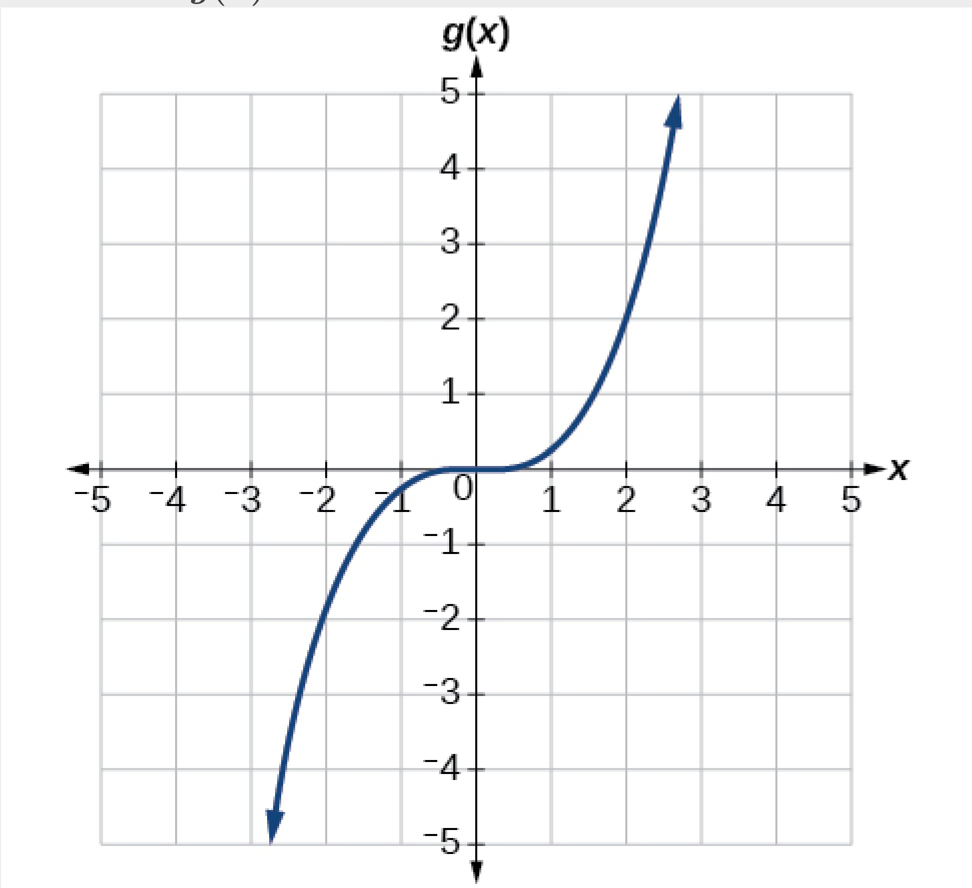
Solution:

x	2	4	6	8
$g(x)$	9	12	15	0

Example:
Exercise:

Problem:
Recognizing a Vertical Stretch

The graph in [link] is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.



Solution:

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8.$ Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2).$ From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x).$

We can write a formula for g by using the definition of the function f.
Equation:

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3$$

Note:
Exercise:

Problem:

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

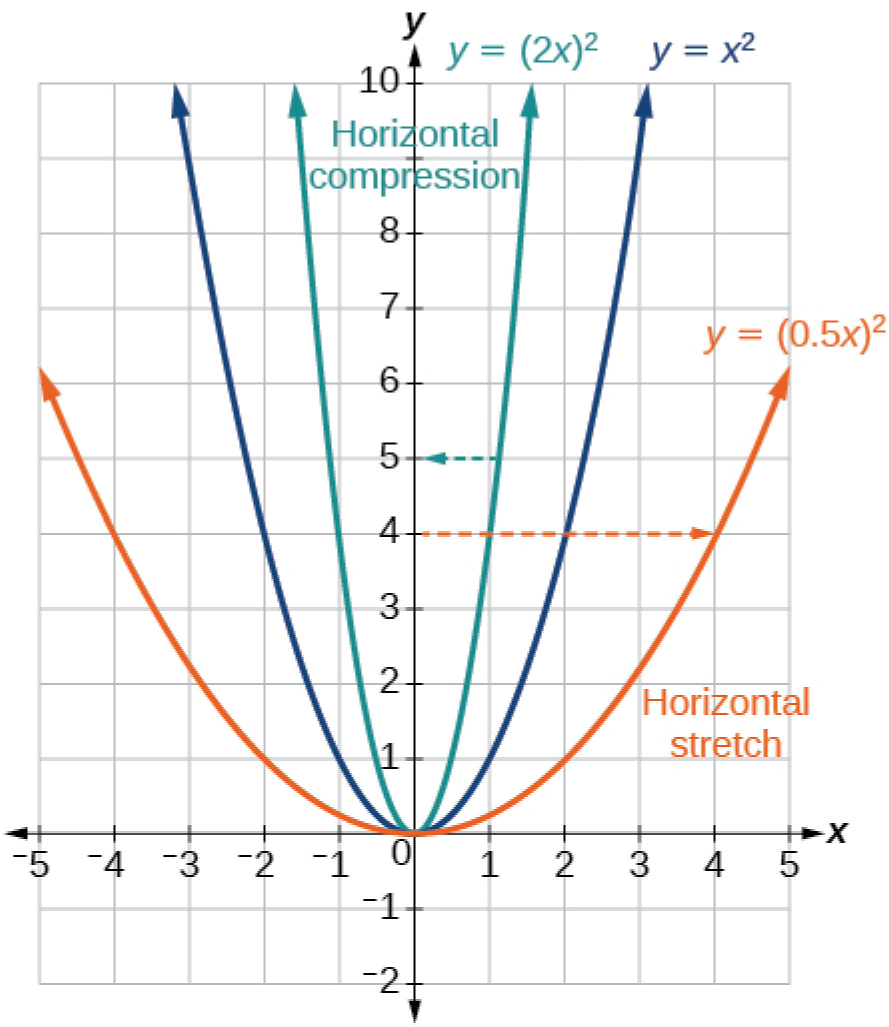
Solution:

$g(x) = 3x - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is

stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.



Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe [link]. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

Note:
Horizontal Stretches and Compressions
Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

Note:
Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example:
Exercise:

Problem:
Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

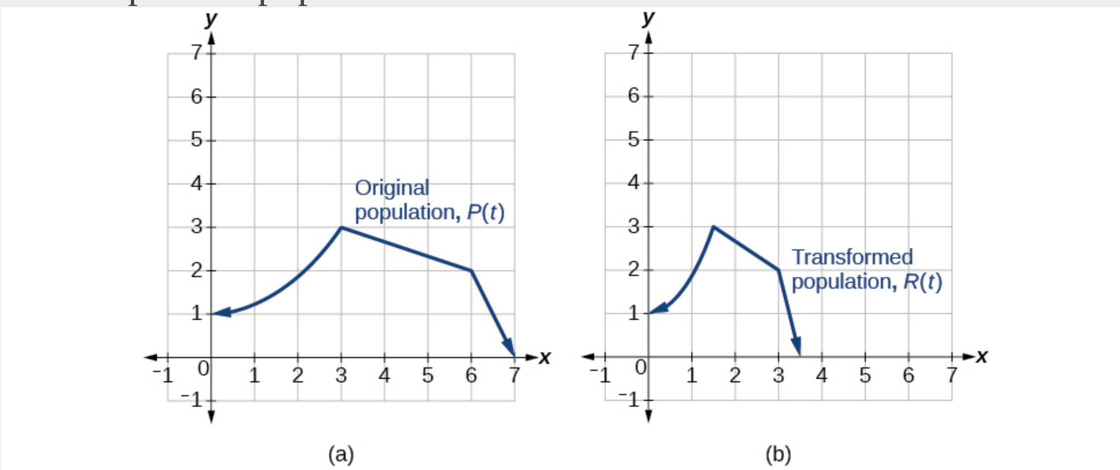
Solution:

Symbolically, we could write

Equation:

$$\begin{aligned} R(1) &= P(2), \\ R(2) &= P(4), \text{ and in general,} \\ R(t) &= P(2t). \end{aligned}$$

See [link] for a graphical comparison of the original population and the compressed population.



(a) Original population graph (b) Compressed population graph

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example:

Exercise:

Problem:

Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as [link]. Create a table for the function $g(x)=f\left(\frac{1}{2}x\right)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Solution:

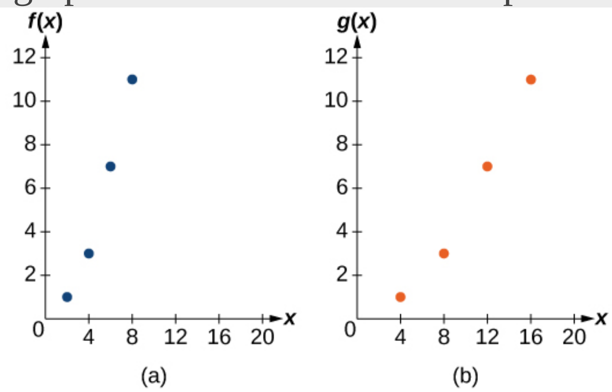
The formula $g(x)=f\left(\frac{1}{2}x\right)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2)=f\left(\frac{1}{2}\cdot 2\right)=f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.
Equation:

$$g(4)=f\left(\frac{1}{2}\cdot 4\right)=f(2)=1$$

We do the same for the other values to produce [link].

x	4	8	12	16
$g(x)$	1	3	7	11

[link] shows the graphs of both of these sets of points.



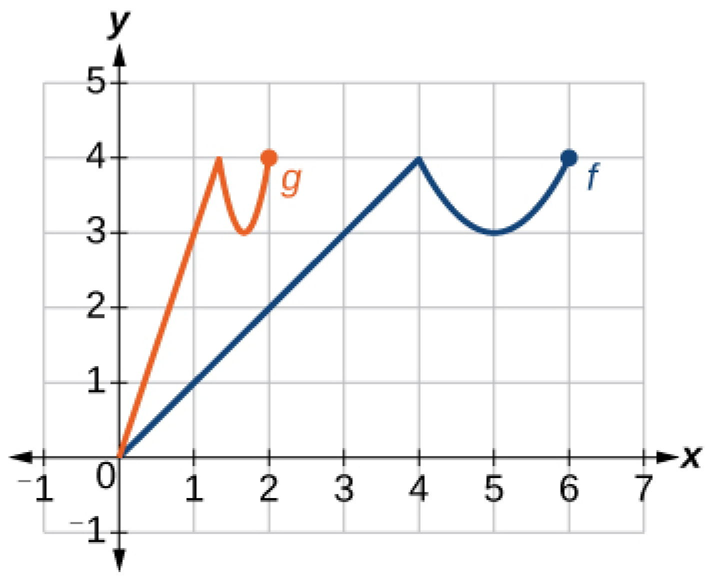
Analysis

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example:
Exercise:

Problem:
Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in [link].



Solution:

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at $(6, 4)$ and $g(x)$ ends at $(2, 4),$ we can see that the x-values have been compressed by $\frac{1}{3}$, because $6\left(\frac{1}{3}\right) = 2.$ We might also notice that $g(2) = f\left(6\right)$ and $g(1) = f\left(3\right).$ Either way, we can describe this relationship as $g(x) = f\left(3x\right).$ This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f\left(\frac{1}{4}x\right).$ This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Note:
Exercise:

Problem:

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Solution:

$g(x) = f\left(\frac{1}{3}x\right)$ so using the square root function we get
$g(x) = \sqrt{\frac{1}{3}x}$

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x) + 3,$ which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x),$ we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x + 3),$ for example, we have to think about how the inputs to the function g relate to the inputs to the function $f.$ Suppose we know $f(7) = 12.$ What input to g would produce that output? In other words, what value of x will allow $g(x) = f(2x + 3) = 12?$ We would need $2x + 3 = 7.$ To solve for $x,$ we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

Equation:

$$f(bx+p)=f\left(b\left(x+\frac{p}{b}\right)\right)$$

Let’s work through an example.

Equation:

$$f\left(x\right)=\left(2x+4\right)^2$$

We can factor out a 2.

Equation:

$$f\left(x\right)=\left(2\left(x+2\right)\right)^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Note:

Combining Transformations

When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.

When combining horizontal transformations written in the form $f(bx-h)$, first horizontally shift by h and then horizontally stretch by $\frac{1}{b}$.

When combining horizontal transformations written in the form $f(b(x-h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h.

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example:
Exercise:

Problem:
Finding a Triple Transformation of a Tabular Function

Given [link] for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x) + 1$.

x	6	12	18	24
$f(x)$	10	14	15	17

Solution:

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$.See [link].

x	2	4	6	8
$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to

the previous transformation. See [link].

x	2	4	6	8
$2f(3x)$	20	28	30	34

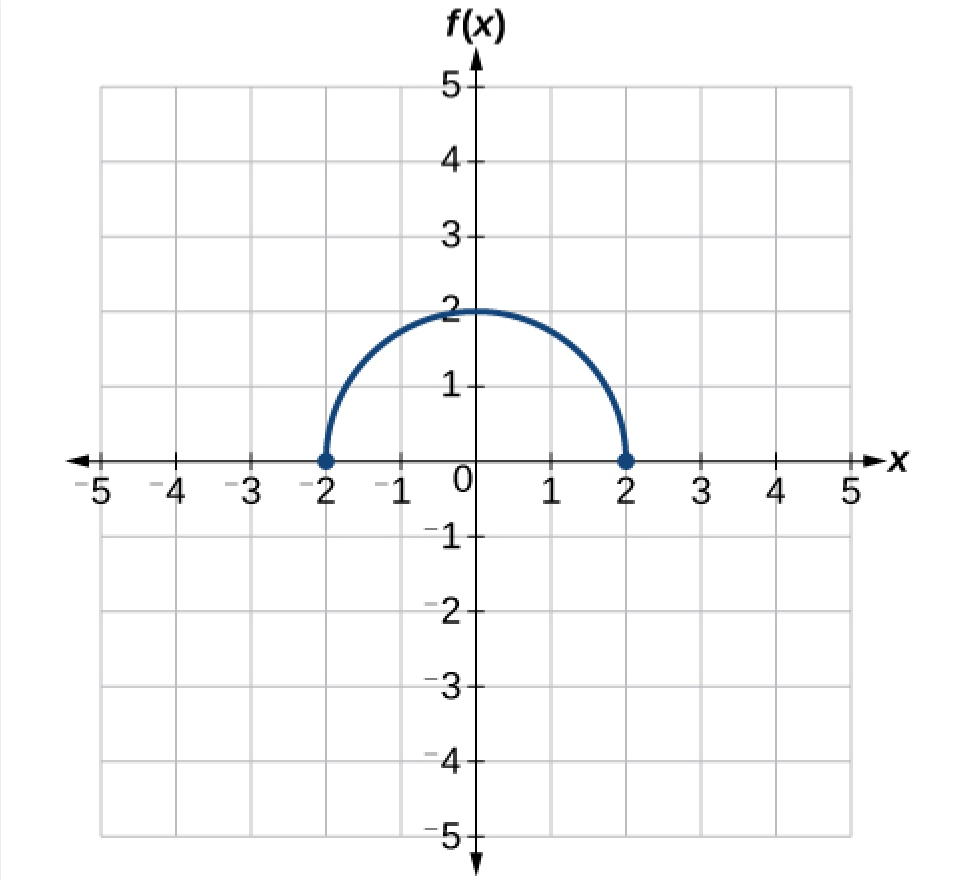
Finally, we can apply the vertical shift, which will add 1 to all the output values. See [link].

x	2	4	6	8
$g(x) = 2f(3x) + 1$	21	29	31	35

Example:
Exercise:

Problem:
Finding a Triple Transformation of a Graph

Use the graph of $f\,(x)$ in [link] to sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3.$

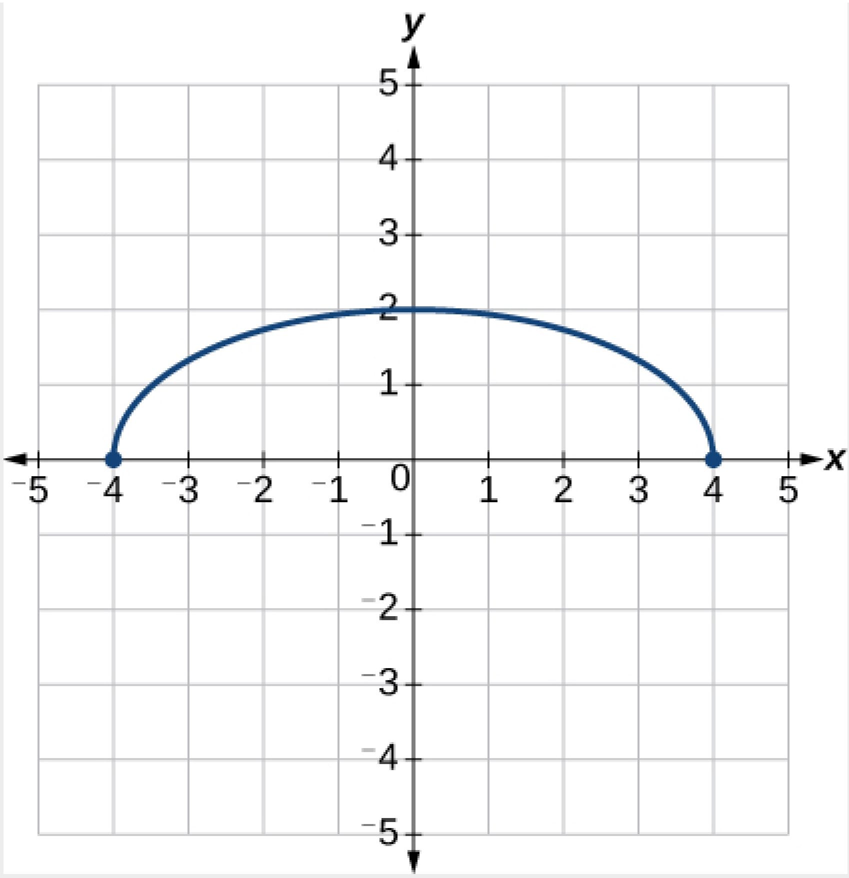


Solution:

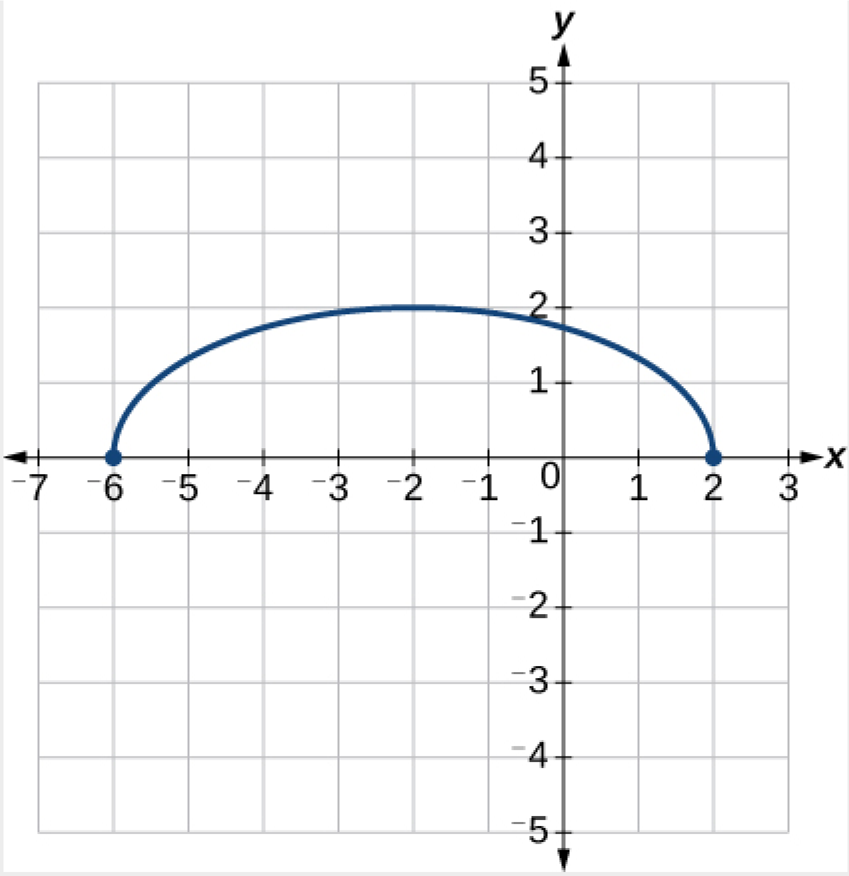
To simplify, let’s start by factoring out the inside of the function.
Equation:

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3$$

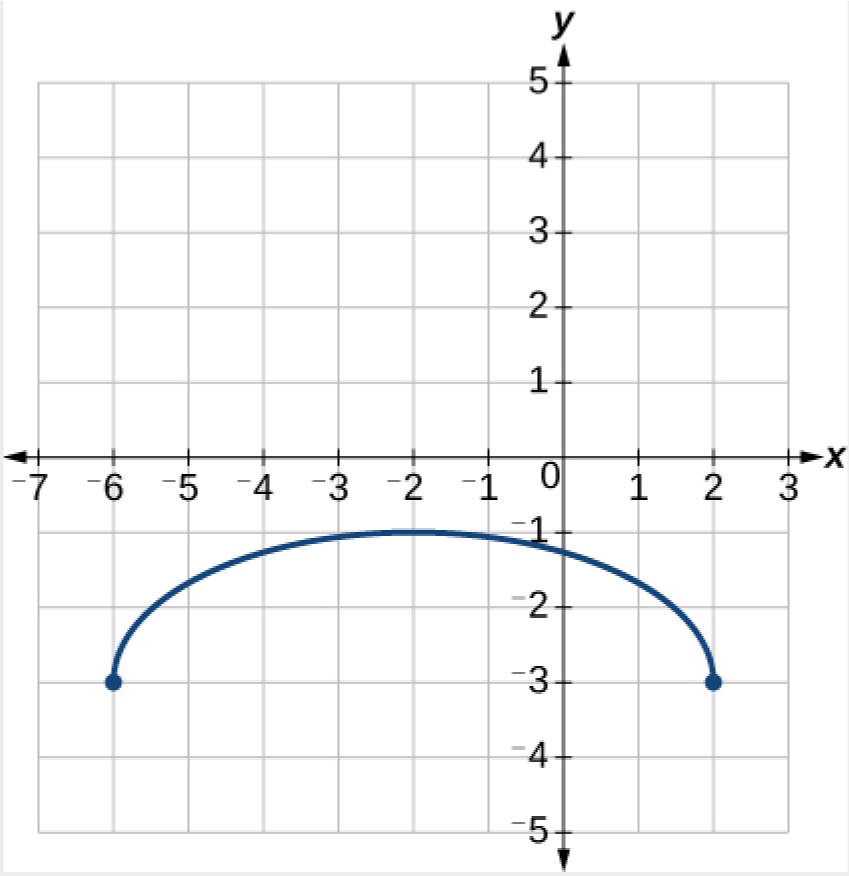
By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point (0,2) remains at (0,2) while the point (2,0) will stretch to (4,0). See [link].



Next, we horizontally shift left by 2 units, as indicated by $x + 2$. See [link].



Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See [link].



Note:
Access this online resource for additional instruction and practice with transformation of functions.

- Function Transformations

Key Equations

Vertical shift	$g(x) = f(x) + k$ (up for $k > 0$)
Horizontal shift	$g(x) = f(x - h)$(right for $h > 0$)
Vertical reflection	$g(x) = -f(x)$
Horizontal reflection	$g(x) = f(-x)$
Vertical stretch	$g(x) = af(x)$ ($a > 0$)
Vertical compression	$g(x) = af(x)$ $(0 < a < 1)$
Horizontal stretch	$g(x) = f(bx)(0 < b < 1)$
Horizontal compression	$g(x) = f(bx)$ $(b > 1)$

Key Concepts

- A function can be shifted vertically by adding a constant to the output. See [link] and [link].
- A function can be shifted horizontally by adding a constant to the input. See [link], [link], and [link].
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts. See [link].
- Vertical and horizontal shifts are often combined. See [link] and [link].
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph. See [link].
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly. See [link].

- A function presented as an equation can be reflected by applying transformations one at a time. See [link].
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither. See [link].
- A function can be compressed or stretched vertically by multiplying the output by a constant. See [link], [link], and [link].
- A function can be compressed or stretched horizontally by multiplying the input by a constant. See [link], [link], and [link].
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?

Solution:

A horizontal shift results when a constant is added to or subtracted from the input. A vertical shifts results when a constant is added to or subtracted from the output.

Exercise:

Problem:

When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?

Exercise:

Problem:

When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?

Solution:

A horizontal compression results when a constant greater than 1 is multiplied by the input. A vertical compression results when a constant between 0 and 1 is multiplied by the output.

Exercise:

Problem:

When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the *x*-axis from a reflection with respect to the *y*-axis?

Exercise:

Problem:

How can you determine whether a function is odd or even from the formula of the function?

Solution:

For a function $f,$ substitute $(-x)$ for (x) in $f(x).$ Simplify. If the resulting function is the same as the original function, $f(-x)=f(x),$ then the function is even. If the resulting function is the opposite of the original function, $f(-x)=-f(x),$ then the original function is odd.

If the function is not the same or the opposite, then the function is neither odd nor even.

Algebraic

Exercise:

Problem:

Write a formula for the function obtained when the graph of $f(x) = \sqrt{x}$ is shifted up 1 unit and to the left 2 units.

Exercise:

Problem:

Write a formula for the function obtained when the graph of $f(x) = |x|$ is shifted down 3 units and to the right 1 unit.

Solution:

$g(x) = |x - 1| - 3$

Exercise:

Problem:

Write a formula for the function obtained when the graph of $f(x) = \frac{1}{x}$ is shifted down 4 units and to the right 3 units.

Exercise:

Problem:

Write a formula for the function obtained when the graph of $f(x) = \frac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

Solution:

$$g(x) = \frac{1}{(x+4)^2} + 2$$

For the following exercises, describe how the graph of the function is a transformation of the graph of the original function f.

Exercise:

Problem: $y = f(x - 49)$

Exercise:

Problem: $y = f(x + 43)$

Solution:

The graph of $f(x + 43)$ is a horizontal shift to the left 43 units of the graph of f.

Exercise:

Problem: $y = f(x + 3)$

Exercise:

Problem: $y = f(x - 4)$

Solution:

The graph of $f(x - 4)$ is a horizontal shift to the right 4 units of the graph of f.

Exercise:

Problem: $y = f(x) + 5$

Exercise:

Problem: $y = f(x) + 8$

Solution:

The graph of $f(x) + 8$ is a vertical shift up 8 units of the graph of f.

Exercise:

Problem: $y = f(x) - 2$

Exercise:

Problem: $y = f(x) - 7$

Solution:

The graph of $f(x) - 7$ is a vertical shift down 7 units of the graph of f.

Exercise:

Problem: $y = f(x - 2) + 3$

Exercise:

Problem: $y = f(x + 4) - 1$

Solution:

The graph of $f(x + 4) - 1$ is a horizontal shift to the left 4 units and a vertical shift down 1 unit of the graph of f.

For the following exercises, determine the interval(s) on which the function is increasing and decreasing.

Exercise:

Problem: $f(x) = 4(x + 1)^2 - 5$

Exercise:

Problem: $g(x) = 5(x+3)^2 - 2$

Solution:

decreasing on $(-\infty, -3)$ and increasing on $(-3, \infty)$

Exercise:

Problem: $a(x) = \sqrt{-x+4}$

Exercise:

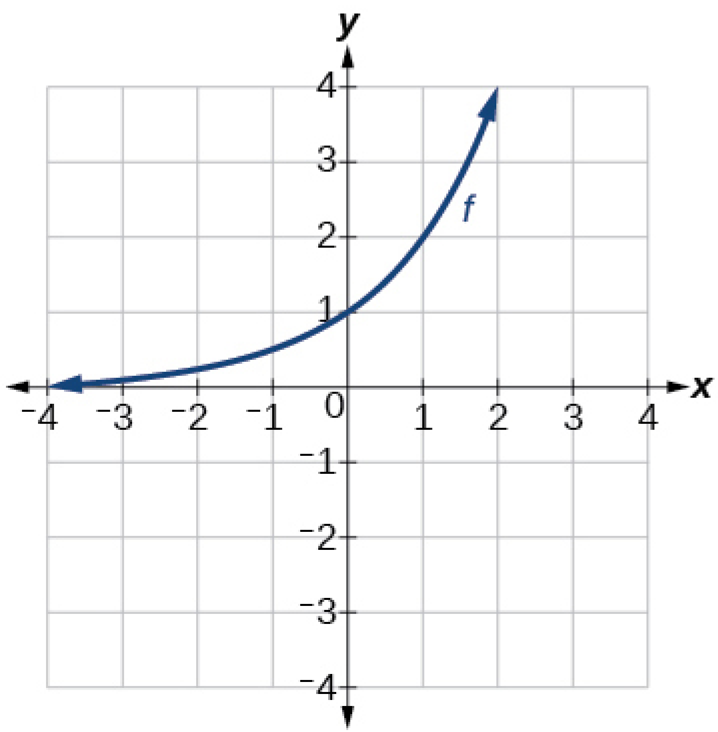
Problem: $k(x) = -3\sqrt{x} - 1$

Solution:

decreasing on $(0,\ \infty)$

Graphical

For the following exercises, use the graph of $f(x) = 2^x$ shown in [link] to sketch a graph of each transformation of $f(x)$.



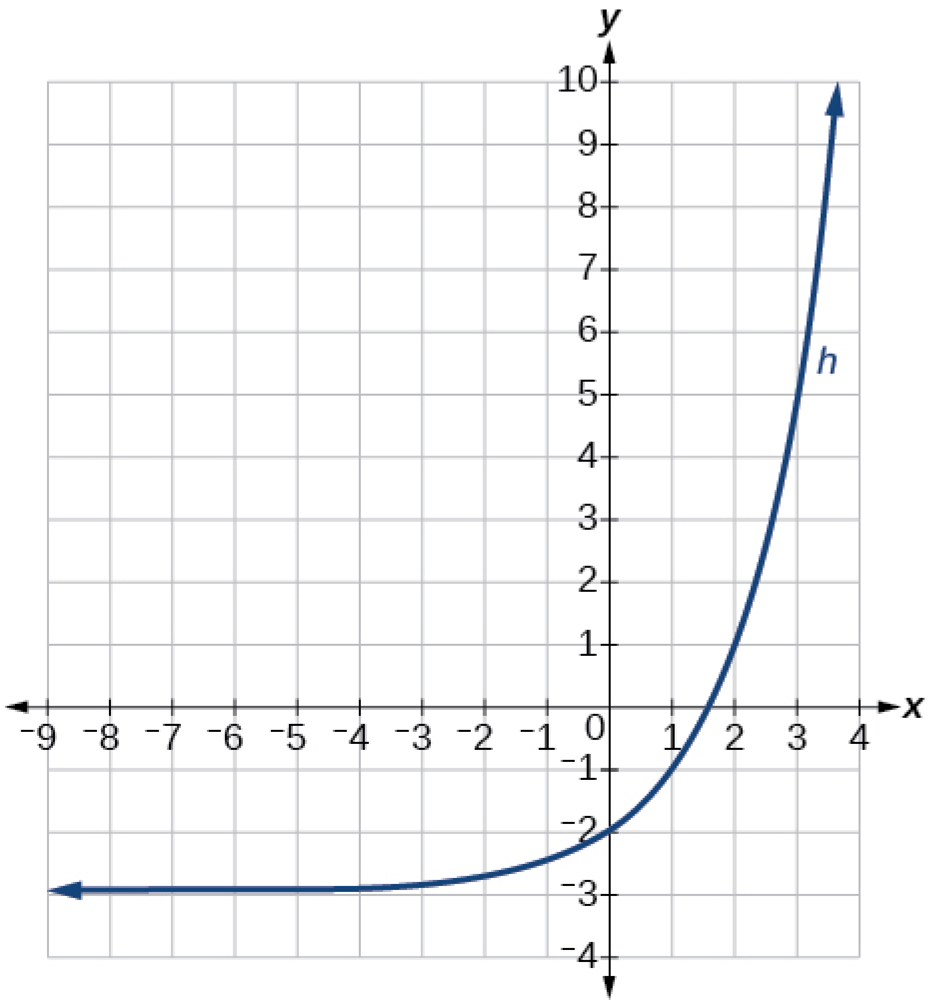
Exercise:

Problem: $g(x) = 2^x + 1$

Exercise:

Problem: $h(x) = 2^x - 3$

Solution:



Exercise:

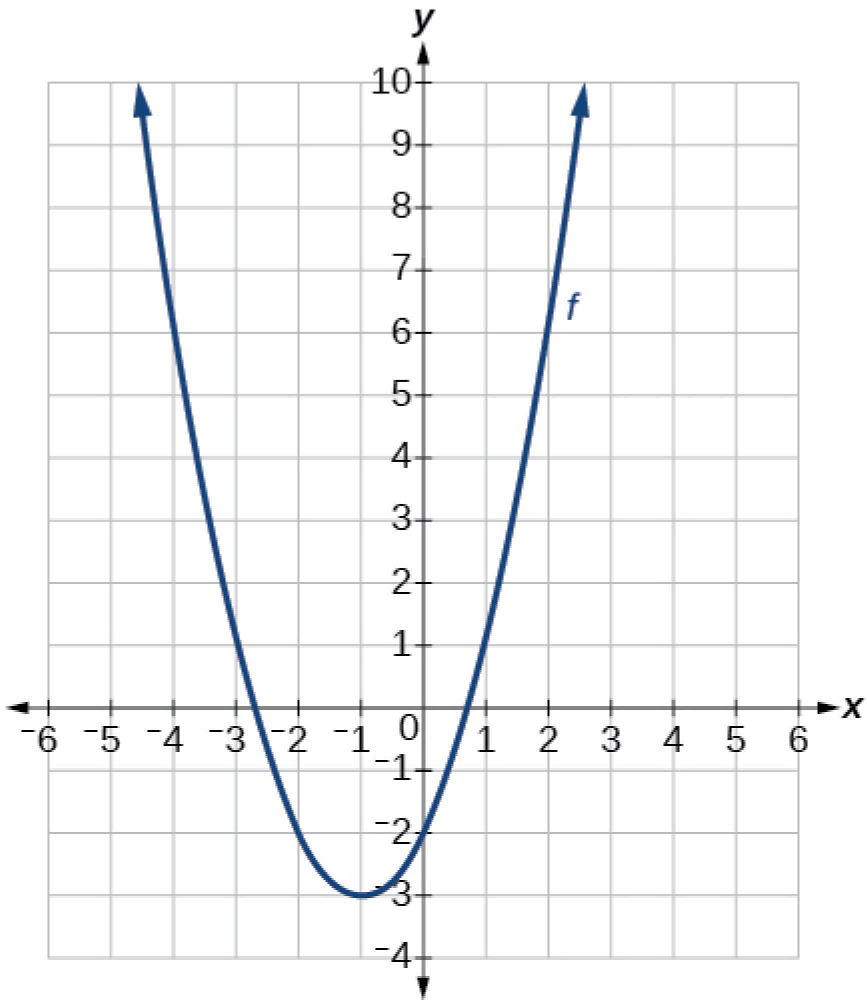
Problem: $w(x) = 2^{x-1}$

For the following exercises, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

Exercise:

Problem: $f(t) = (t+1)^2 - 3$

Solution:



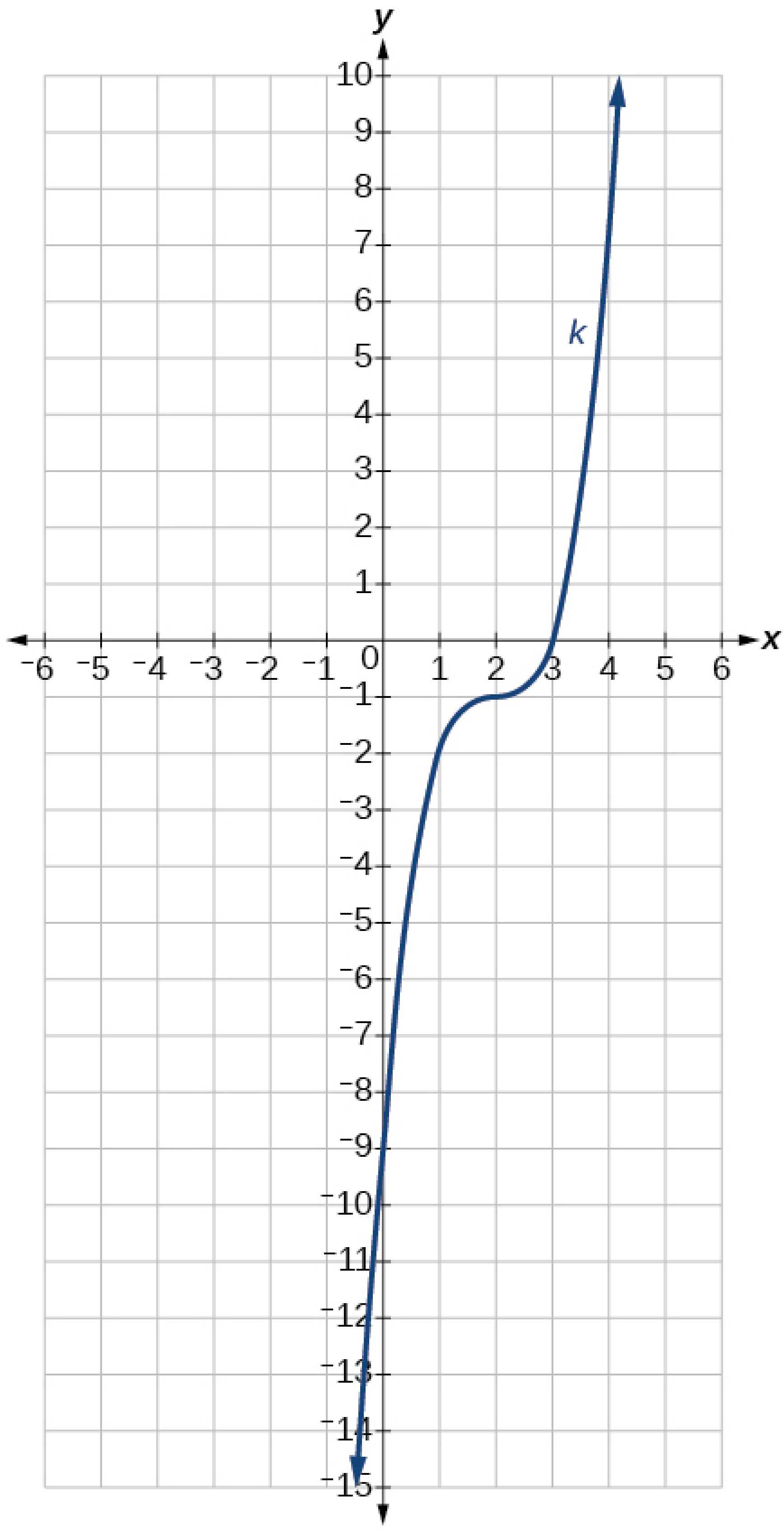
Exercise:

Problem: $h(x) = |x - 1| + 4$

Exercise:

Problem: $k(x) = (x - 2)^3 - 1$

Solution:



y
x
k

Exercise:

Problem: $m(t) = 3 + \sqrt{t+2}$

Numeric

Exercise:

Problem:

Tabular representations for the functions $f,$ $g,$ and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	−2	−1	0	1	2
$f(x)$	−2	−1	−3	1	2

x	−1	0	1	2	3
$g(x)$	−2	−1	−3	1	2

x	−2	−1	0	1	2
$h(x)$	−1	0	−2	2	3

Solution:

$g(x) = f(x-1),\ h(x) = f(x) + 1$

Exercise:

Problem:

Tabular representations for the functions $f,\ g,$ and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	−2	−1	0	1	2
$f(x)$	−1	−3	4	2	1

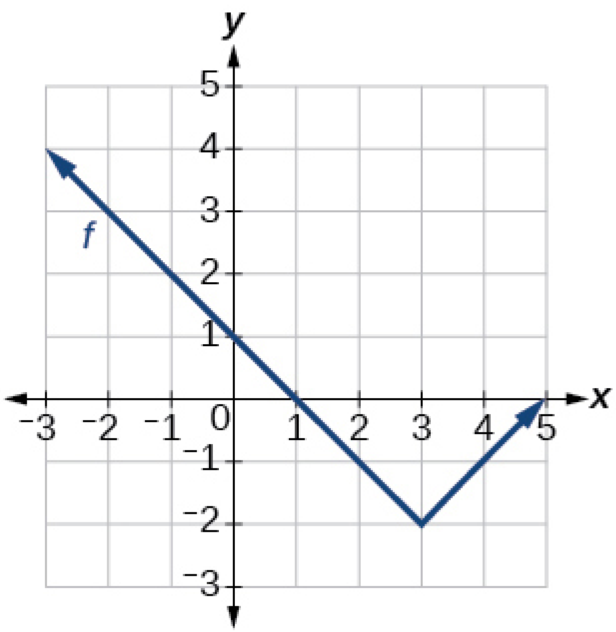
x	−3	−2	−1	0	1
$g(x)$	−1	−3	4	2	1

x	−2	−1	0	1	2
$h(x)$	−2	−4	3	1	0

For the following exercises, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.

Exercise:

Problem:

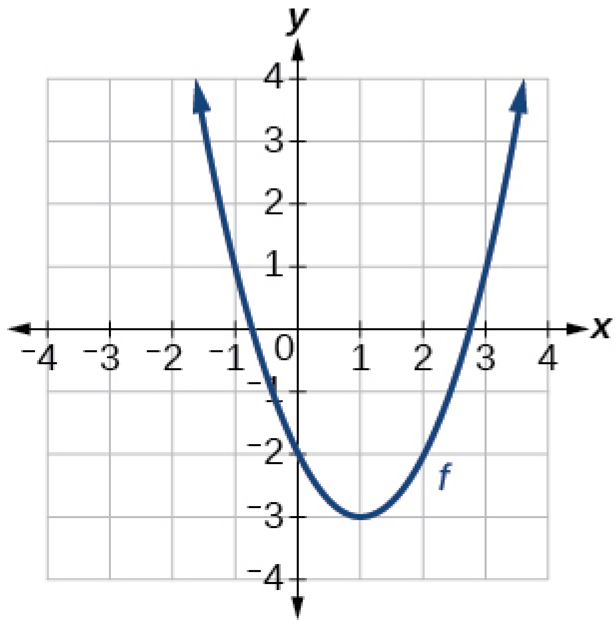


Solution:

$f(x) = |x - 3| - 2$

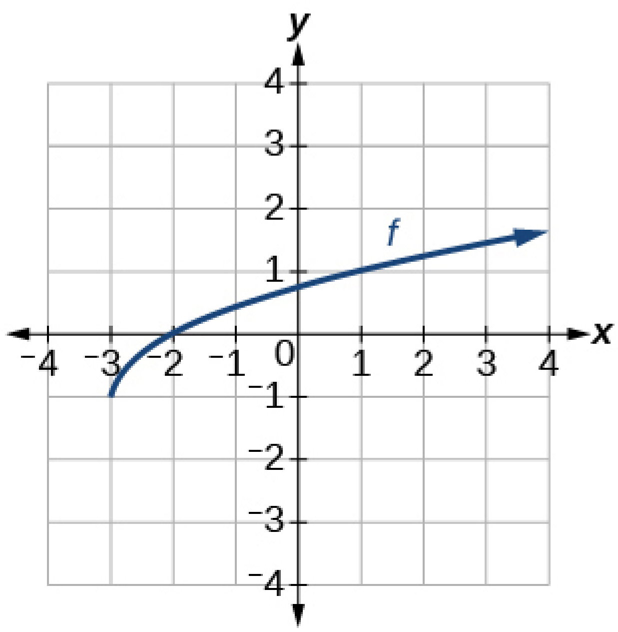
Exercise:

Problem:



Exercise:

Problem:

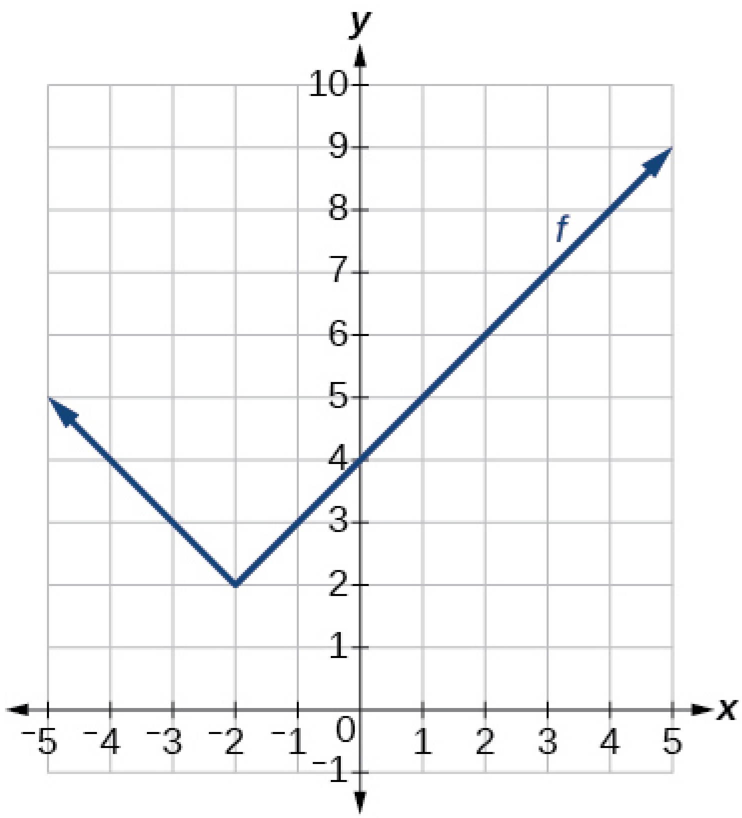


Solution:

$$f(x) = \sqrt{x+3} - 1$$

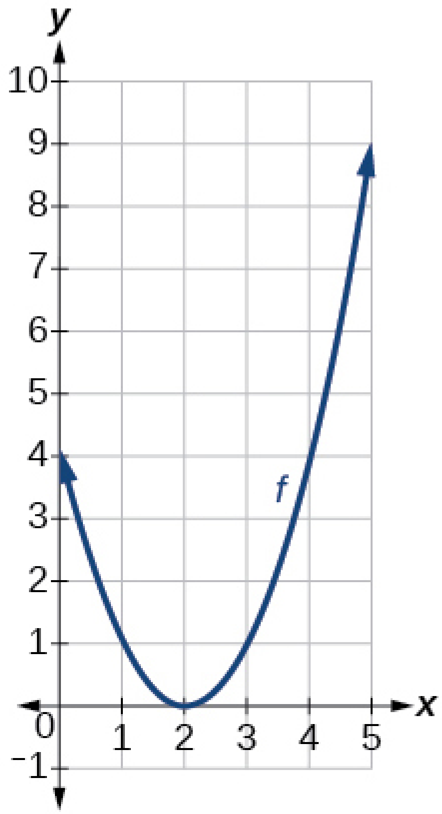
Exercise:

Problem:



Exercise:

Problem:

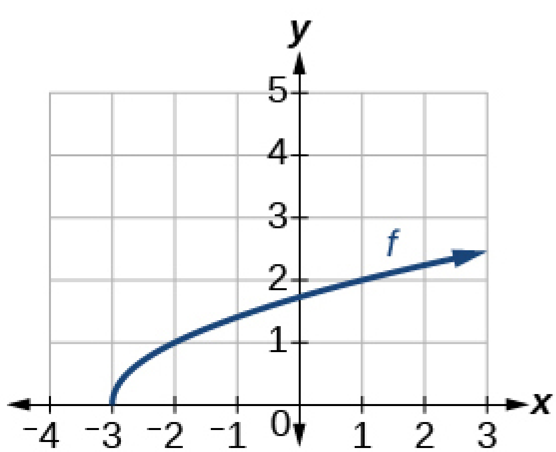


Solution:

$f(x) = (x-2)^2$

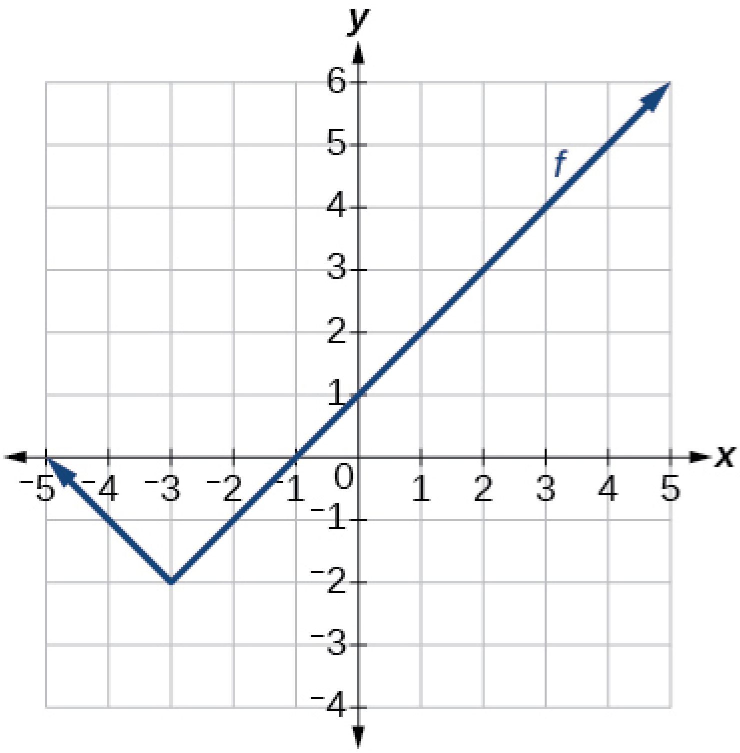
Exercise:

Problem:



Exercise:

Problem:

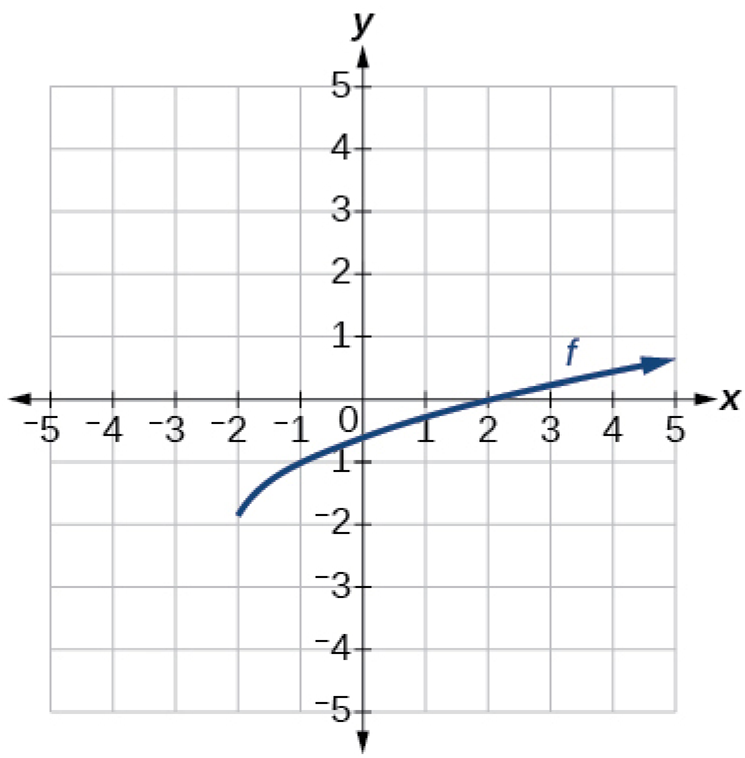


Solution:

$f(x) = |x + 3| - 2$

Exercise:

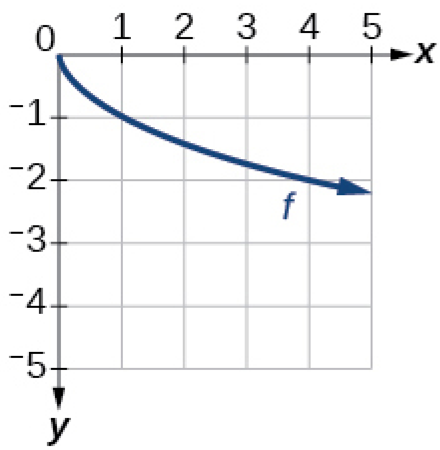
Problem:



For the following exercises, use the graphs of transformations of the square root function to find a formula for each of the functions.

Exercise:

Problem:

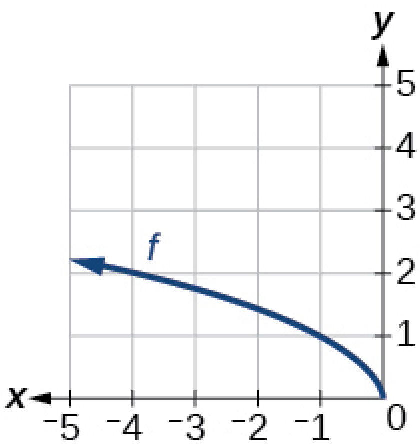


Solution:

$f(x) = -\sqrt{x}$

Exercise:

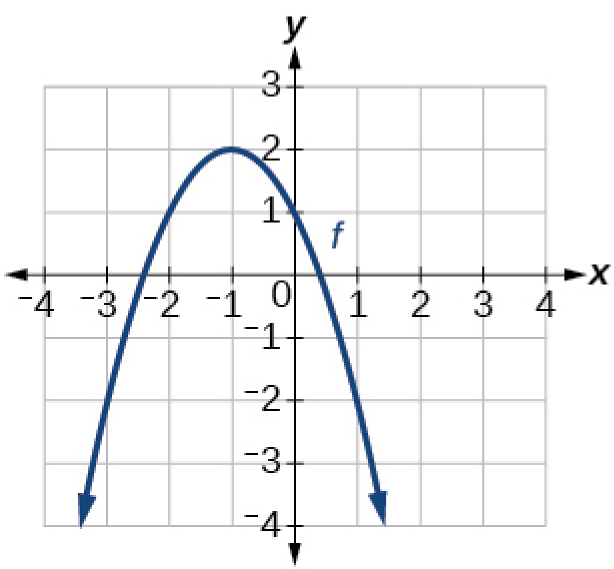
Problem:



For the following exercises, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

Exercise:

Problem:

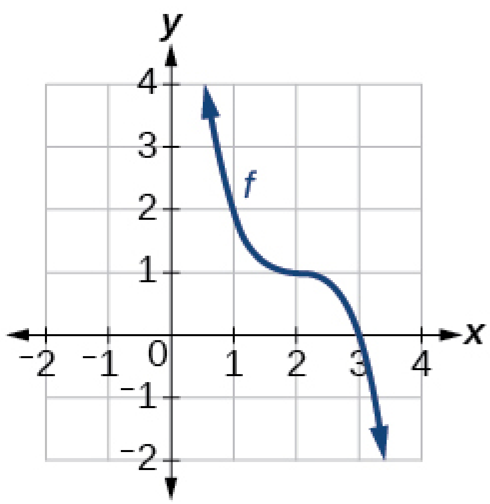


Solution:

$$f(x) = -(x+1)^2 + 2$$

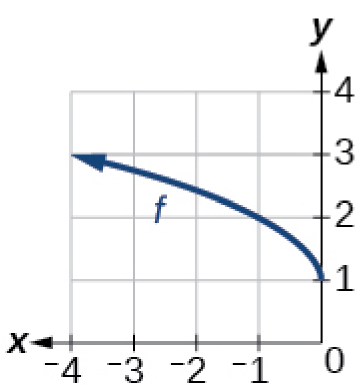
Exercise:

Problem:



Exercise:

Problem:

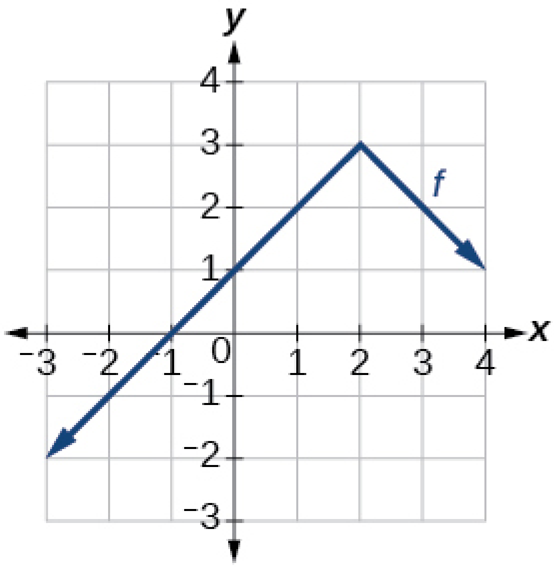


Solution:

$$f(x) = \sqrt{-x} + 1$$

Exercise:

Problem:



For the following exercises, determine whether the function is odd, even, or neither.

Exercise:

Problem: $f(x) = 3x^4$

Solution:

even

Exercise:

Problem:$g(x) = \sqrt{x}$

Exercise:

Problem: $h(x) = \frac{1}{x} + 3x$

Solution:

odd

Exercise:

Problem: $f(x) = (x-2)^2$

Exercise:

Problem: $g(x) = 2x^4$

Solution:

even

Exercise:

Problem: $h(x) = 2x - x^3$

For the following exercises, describe how the graph of each function is a transformation of the graph of the original function f.

Exercise:

Problem: $g(x) = -f(x)$

Solution:

The graph of g is a vertical reflection (across the x-axis) of the graph of f.

Exercise:

Problem: $g(x) = f(-x)$

Exercise:

Problem: $g(x) = 4f(x)$

Solution:

The graph of g is a vertical stretch by a factor of 4 of the graph of f.

Exercise:

Problem: $g(x) = 6f(x)$

Exercise:

Problem: $g(x) = f(5x)$

Solution:

The graph of g is a horizontal compression by a factor of $\frac{1}{5}$ of the graph of f.

Exercise:

Problem: $g(x) = f(2x)$

Exercise:

Problem: $g(x) = f\left(\frac{1}{3}x\right)$

Solution:

The graph of g is a horizontal stretch by a factor of 3 of the graph of f.

Exercise:

Problem: $g(x) = f\left(\frac{1}{5}x\right)$

Exercise:

Problem: $g(x) = 3f\left(-x\right)$

Solution:

The graph of g is a horizontal reflection across the y-axis and a vertical stretch by a factor of 3 of the graph of f.

Exercise:

Problem: $g(x) = -f(3x)$

For the following exercises, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

Exercise:

Problem:

The graph of $f(x) = |x|$ is reflected over the y-axis and horizontally compressed by a factor of $\frac{1}{4}$.

Solution:

$g(x) = |-4x|$

Exercise:

Problem:

The graph of $f(x) = \sqrt{x}$ is reflected over the x-axis and horizontally stretched by a factor of 2.

Exercise:

Problem:

The graph of $f(x) = \frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.

Solution:

$g(x) = \frac{1}{3(x+2)^2} - 3$

Exercise:

Problem:

The graph of $f(x) = \frac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

Exercise:

Problem:

The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.

Solution:

$g(x) = \frac{1}{2}(x-5)^2 + 1$

Exercise:

Problem:

The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

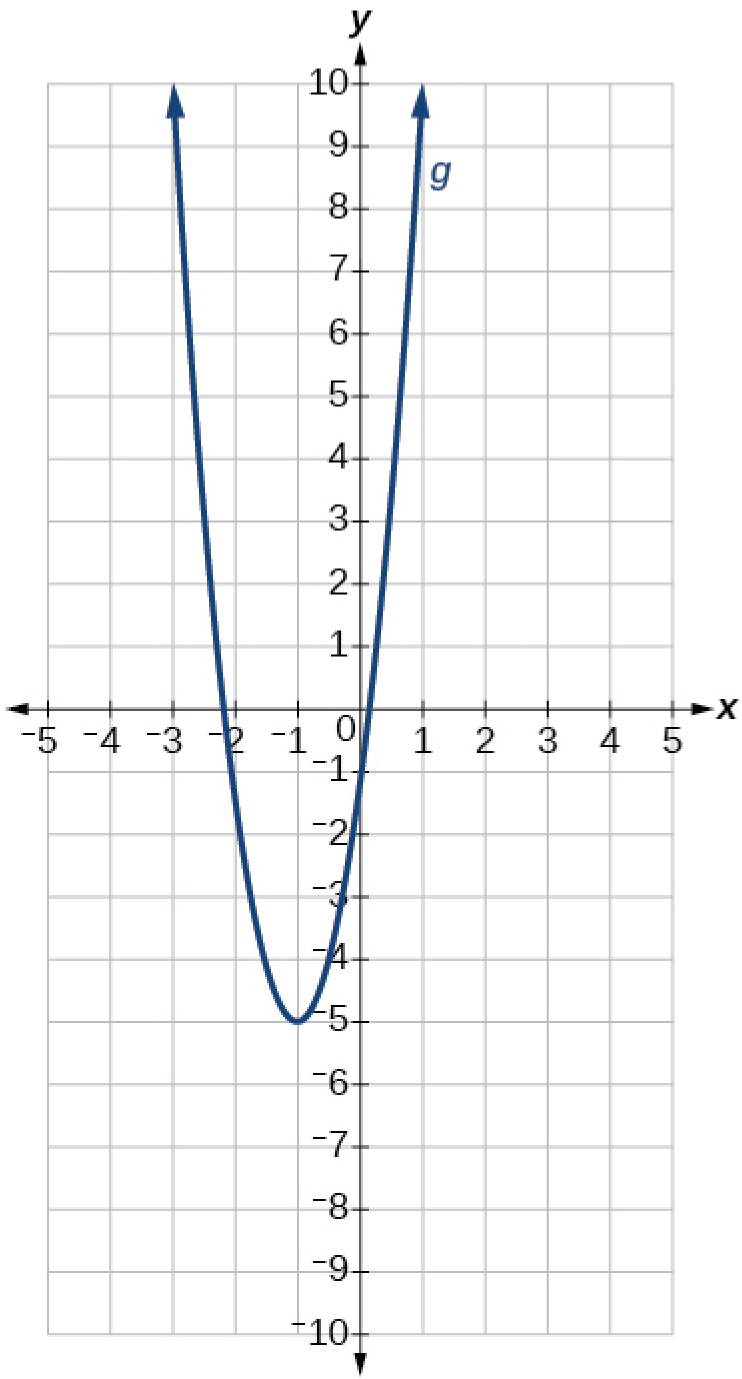
For the following exercises, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

Exercise:

Problem: $g(x) = 4(x+1)^2 - 5$

Solution:

The graph of the function $f(x) = x^2$ is shifted to the left 1 unit, stretched vertically by a factor of 4, and shifted down 5 units.



Exercise:

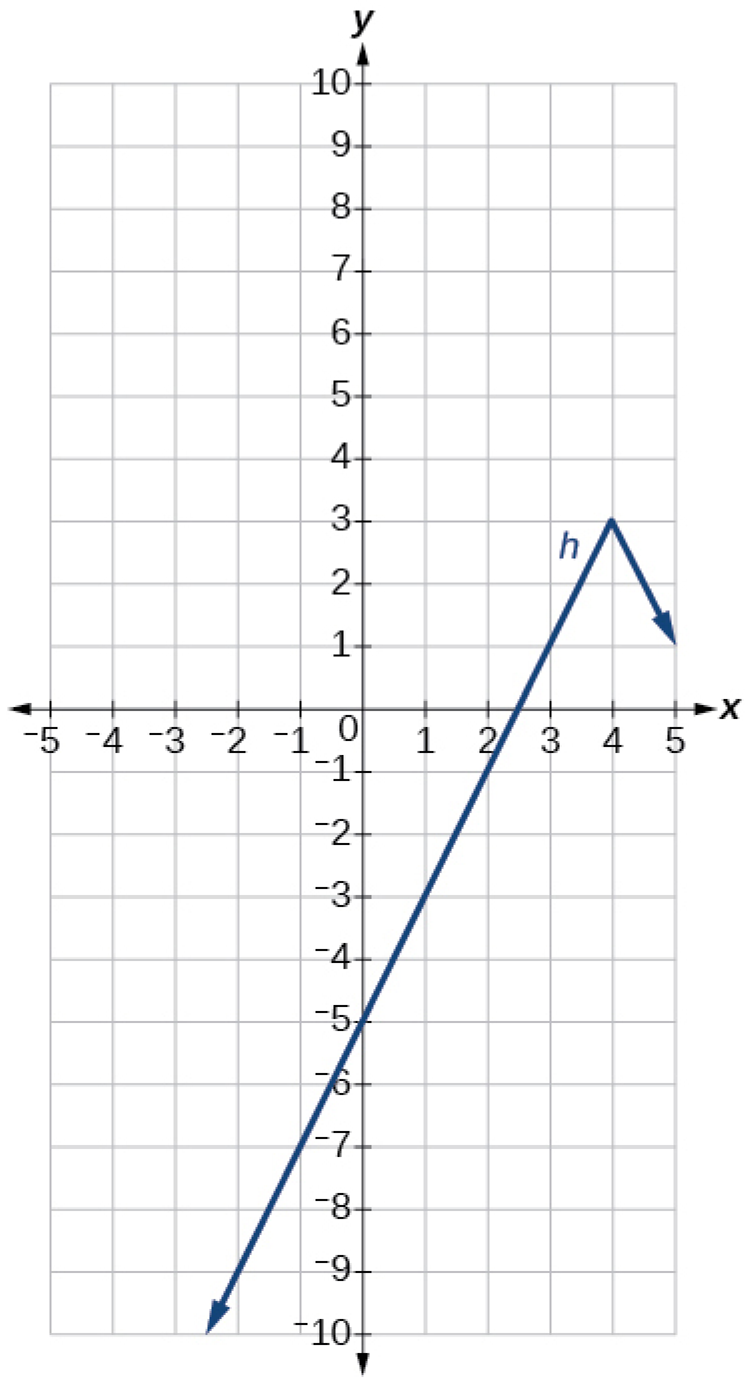
Problem: $g(x) = 5(x+3)^2 - 2$

Exercise:

Problem: $h(x) = -2|x-4|+3$

Solution:

The graph of $f(x) = |x|$ is stretched vertically by a factor of 2, shifted horizontally 4 units to the right, reflected across the horizontal axis, and then shifted vertically 3 units up.



Exercise:

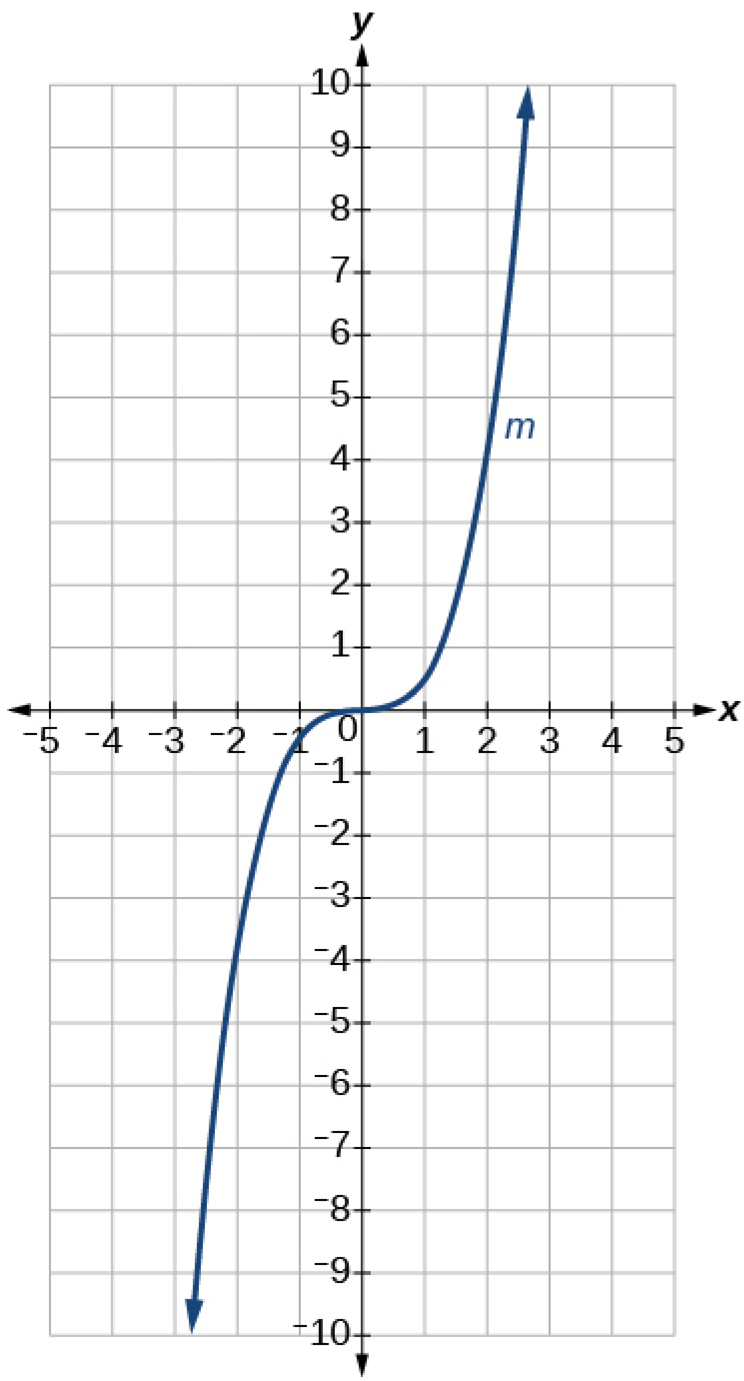
Problem: $k(x) = -3\sqrt{x} - 1$

Exercise:

Problem: $m(x) = \frac{1}{2}x^3$

Solution:

The graph of the function $f(x) = x^3$ is compressed vertically by a factor of $\frac{1}{2}$.



Exercise:

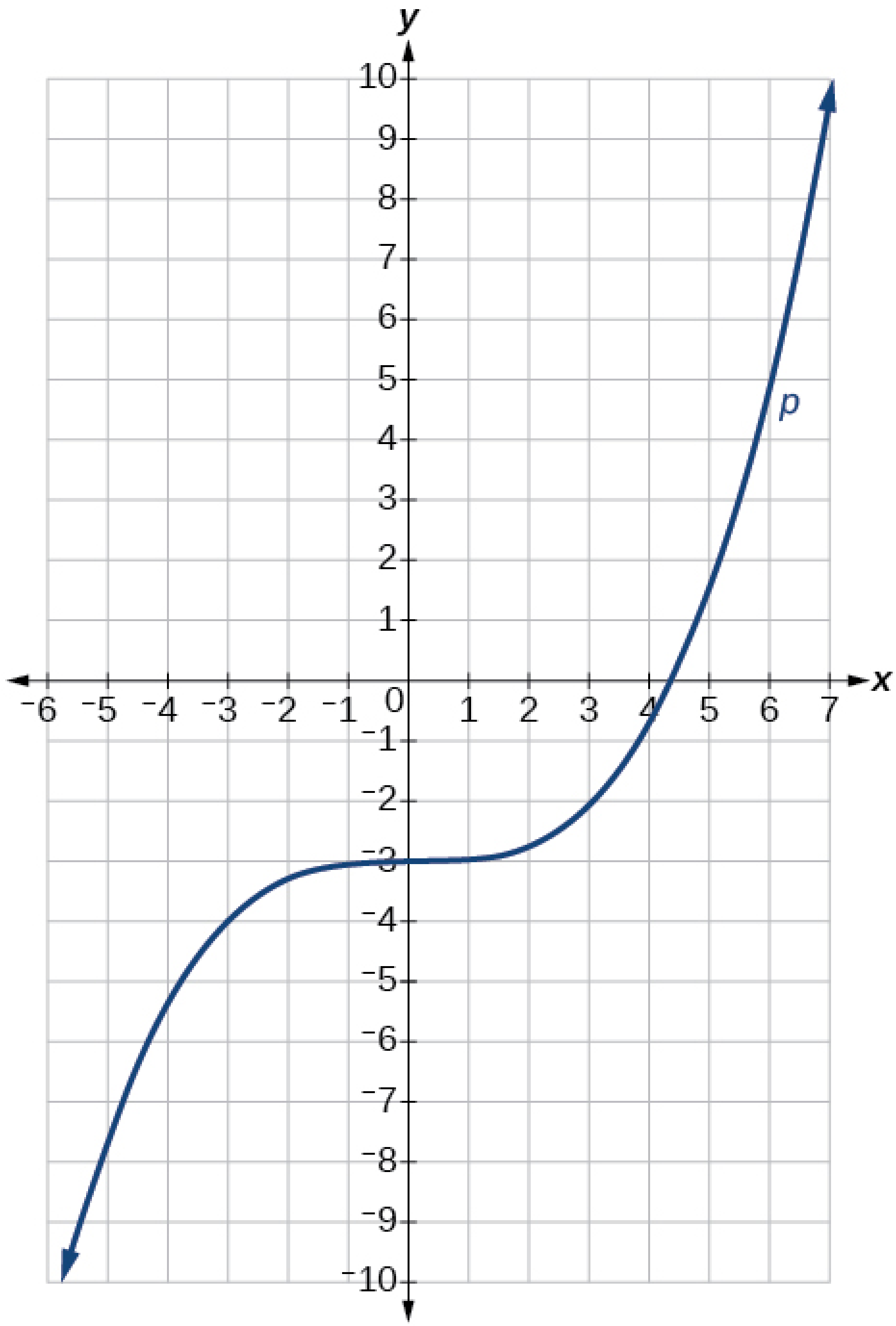
Problem: $n(x) = \frac{1}{3}\left|x - 2\right|$

Exercise:

Problem: $p\left(x\right)=\left(\frac{1}{3}x\right)^3-3$

Solution:

The graph of the function is stretched horizontally by a factor of 3 and then shifted vertically downward by 3 units.



Exercise:

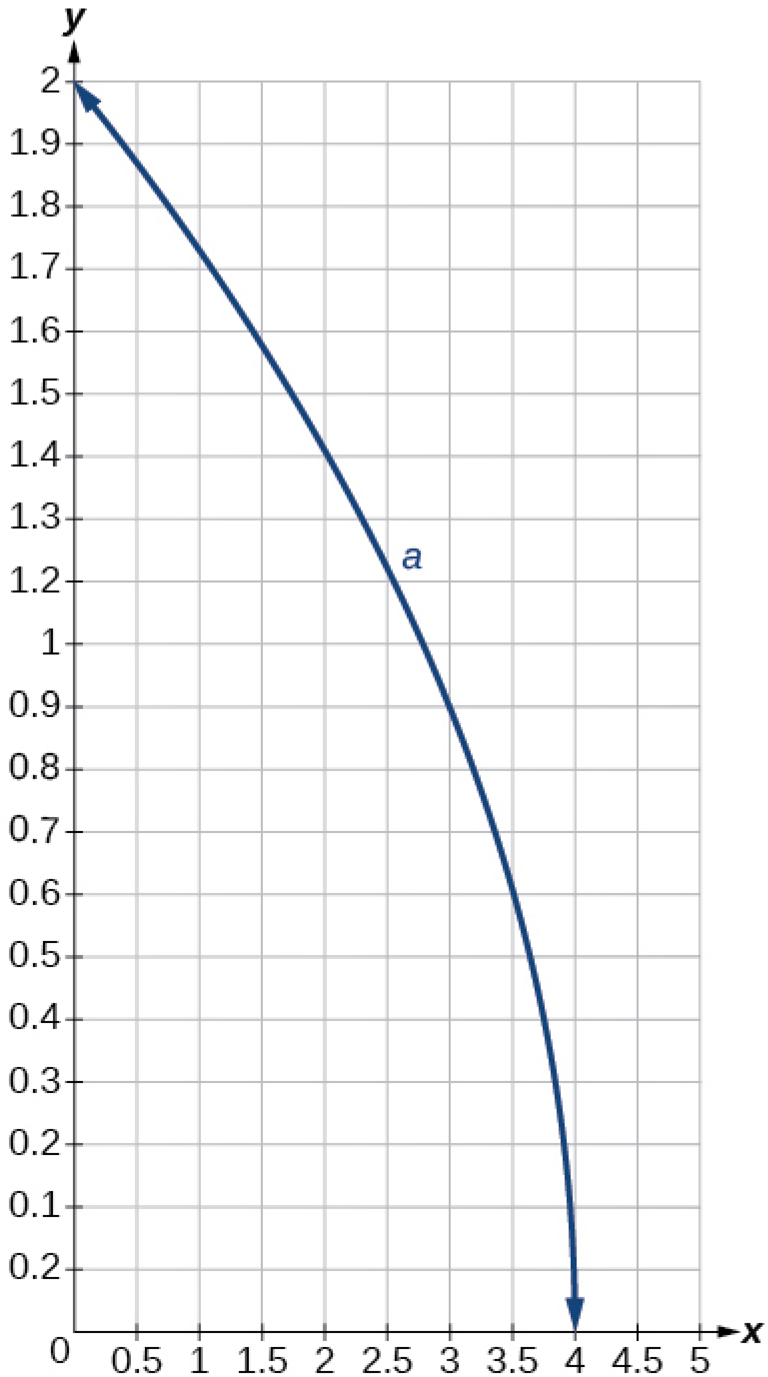
Problem: $q(x) = \left(\frac{1}{4}x\right)^3 + 1$

Exercise:

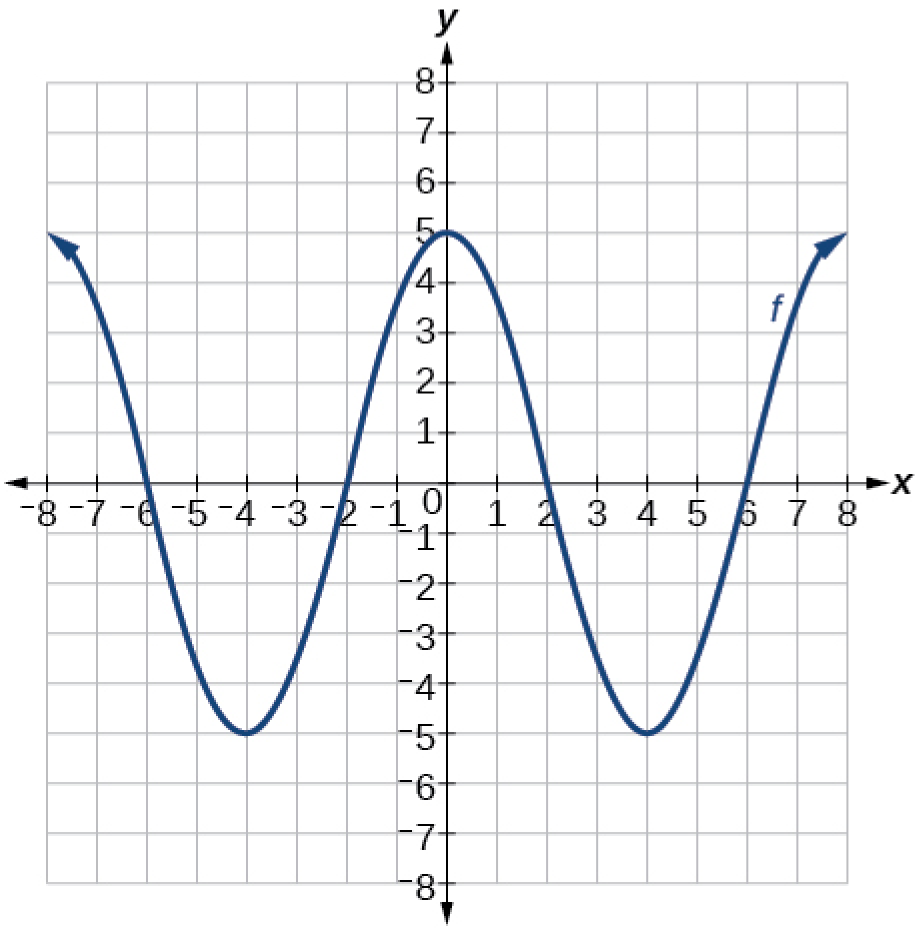
Problem: $a(x) = \sqrt{-x+4}$

Solution:

The graph of $f(x) = \sqrt{x}$ is shifted right 4 units and then reflected across the vertical line $x = 4$.



For the following exercises, use the graph in [link] to sketch the given transformations.



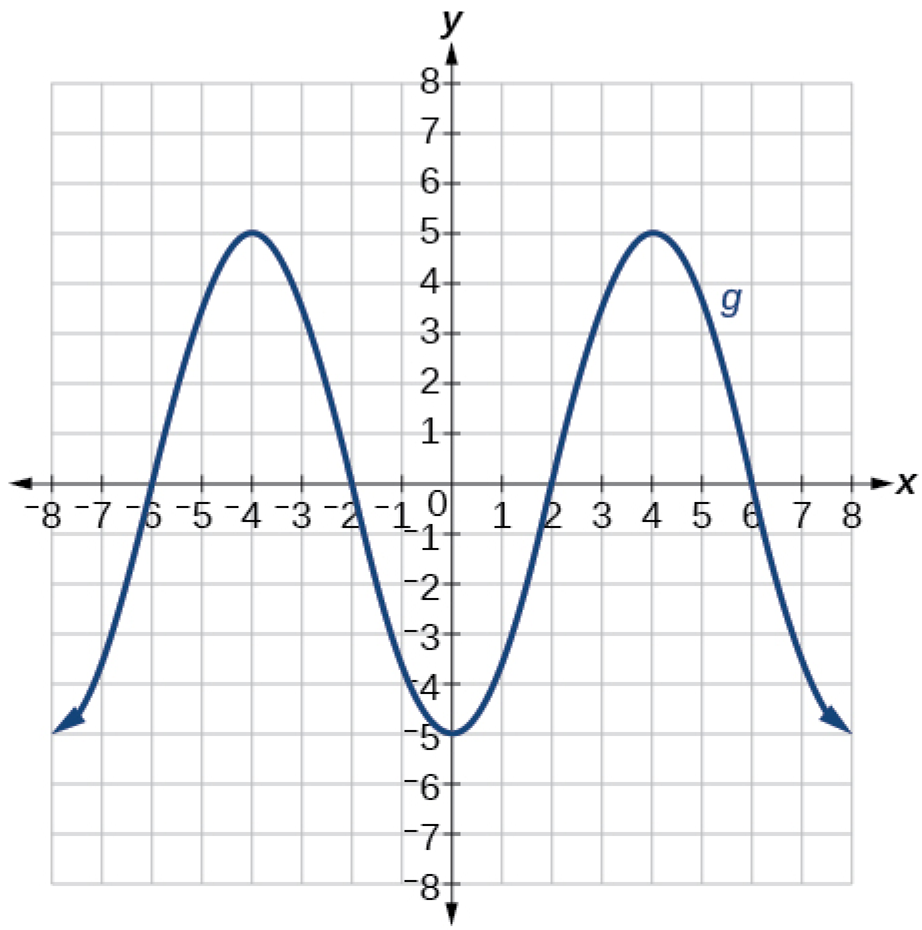
Exercise:

Problem: $g(x) = f(x) - 2$

Exercise:

Problem: $g(x) = -f(x)$

Solution:



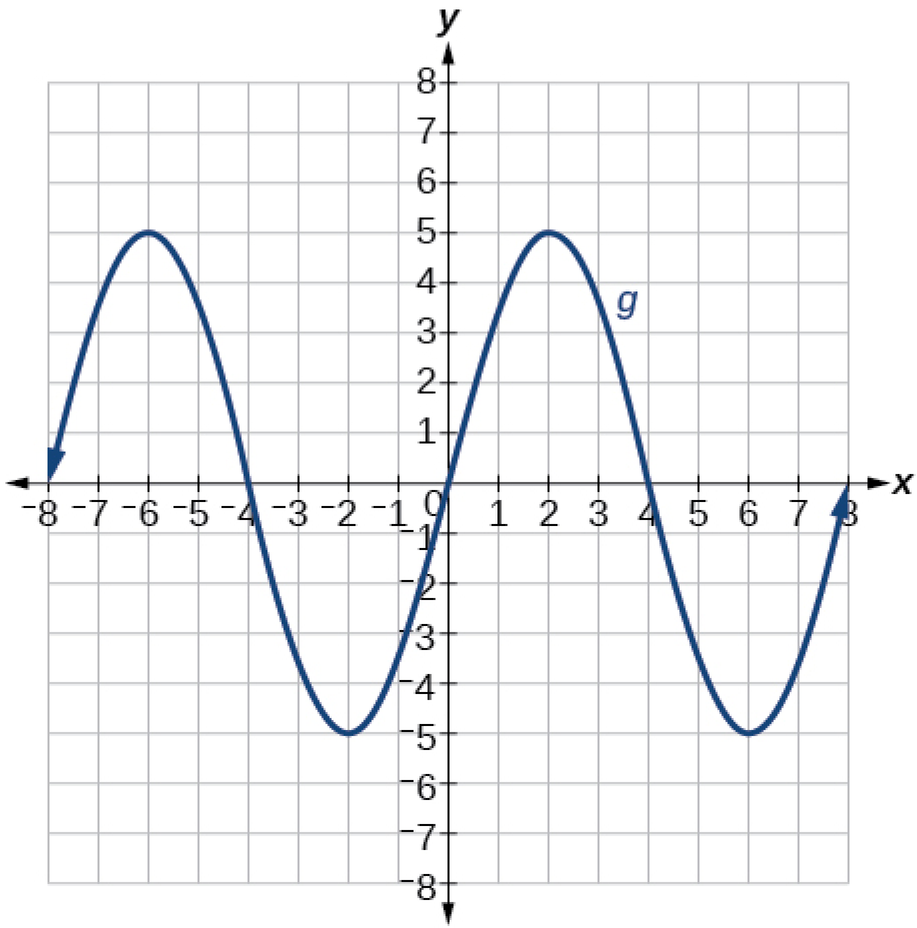
Exercise:

Problem: $g(x) = f(x+1)$

Exercise:

Problem: $g(x) = f(x-2)$

Solution:



Glossary

even function
: a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis

horizontal compression
: a transformation that compresses a function's graph horizontally, by multiplying the input by a constant $b > 1$

horizontal reflection
: a transformation that reflects a function's graph across the y-axis by multiplying the input by -1

horizontal shift

a transformation that shifts a function’s graph left or right by adding a positive or negative constant to the input

horizontal stretch
: a transformation that stretches a function’s graph horizontally by multiplying the input by a constant $0 < b < 1$

odd function
: a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$, and is symmetric about the origin

vertical compression
: a function transformation that compresses the function’s graph vertically by multiplying the output by a constant $0 < a < 1$

vertical reflection
: a transformation that reflects a function’s graph across the *x*-axis by multiplying the output by -1

vertical shift
: a transformation that shifts a function’s graph up or down by adding a positive or negative constant to the output

vertical stretch
: a transformation that stretches a function’s graph vertically by multiplying the output by a constant $a > 1$

Graphs of Exponential Functions

- Graph exponential functions.
- Graph exponential functions using transformations.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x)=b^x$ whose base is greater than one. We'll use the function $f(x)=2^x$. Observe how the output values in [link] change as the input increases by 1.

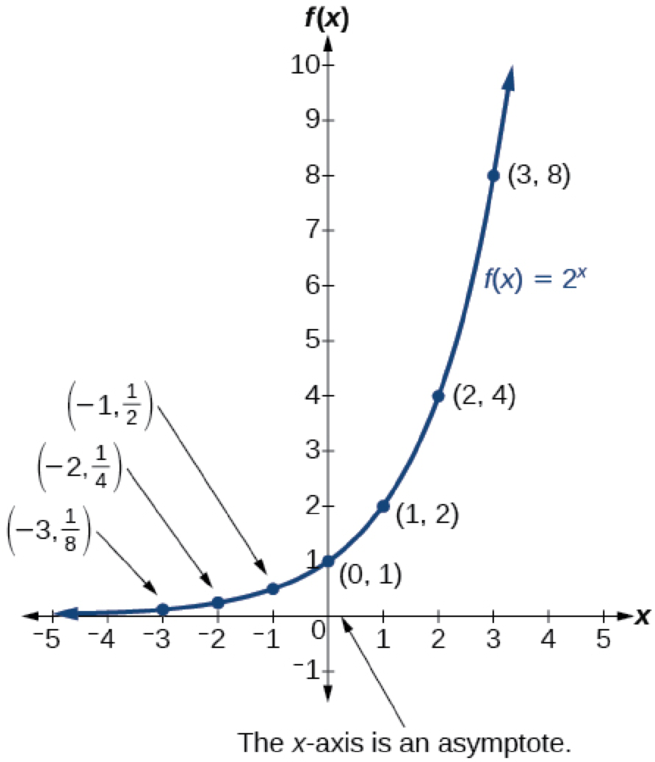
x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x)=ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

[link] shows the exponential growth function $f(x)=2^x$.



Notice that the graph gets close to the *x*-axis, but never touches it.

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of **exponential decay**, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in [link] change as the input increases by 1.

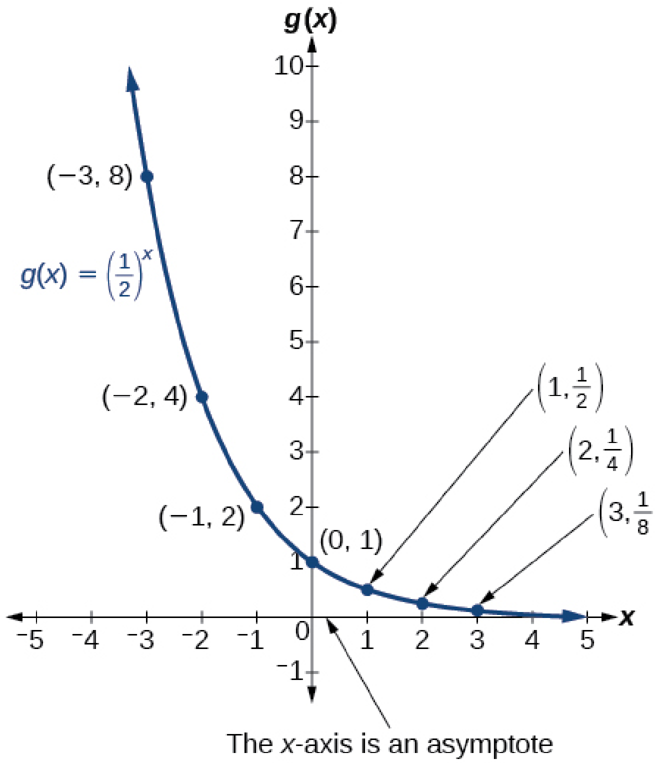
x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

[link] shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

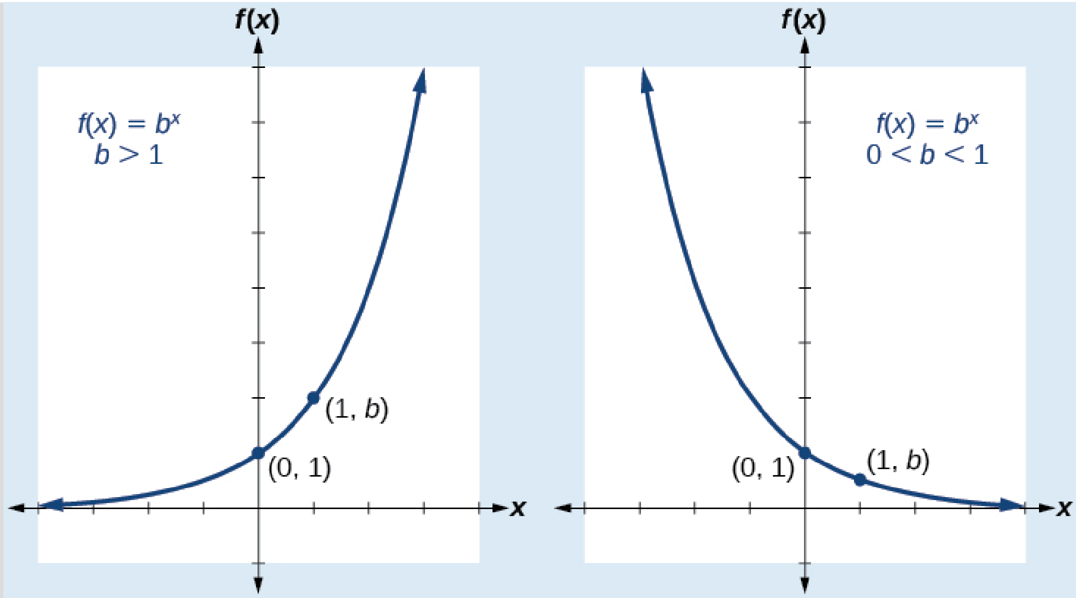


The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $(0, \infty)$,and the horizontal asymptote is $y = 0$.

Note:
Characteristics of the Graph of the Parent Function $f(x) = b^x$
An exponential function with the form $f(x) = b^x, b > 0, b \neq 1$,has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty,\ \infty)$
- range: $(0, \infty)$
- *x*-intercept: none
- *y*-intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$

[link] compares the graphs of exponential growth and decay functions.



Note:
Given an exponential function of the form $f(x) = b^x$,graph the function.

1. Create a table of points.
2. Plot at least 3 point from the table, including the *y*-intercept $(0, 1)$.
3. Draw a smooth curve through the points.
4. State the domain, $(-\infty, \infty)$,the range, $(0, \infty)$,and the horizontal asymptote, $y = 0$.

Example:
Exercise:

Problem:
Sketching the Graph of an Exponential Function of the Form *f*(*x*) = *b*^{x}

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

Solution:

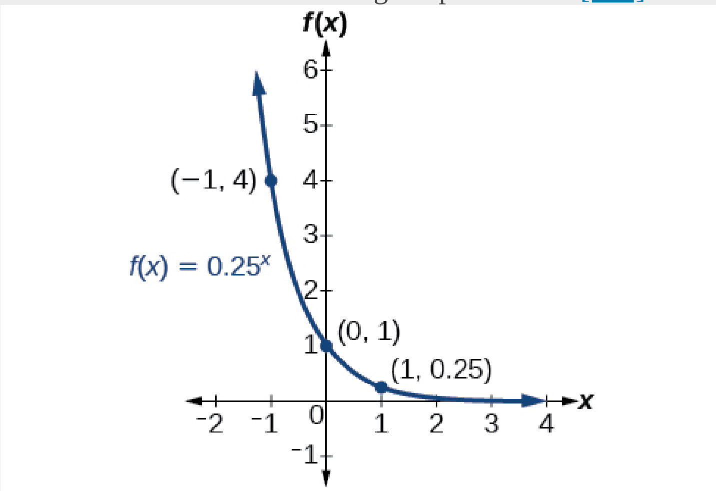
Before graphing, identify the behavior and create a table of points for the graph.

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in [link].

x	-3	-2	-1	0	1	2	3
$f(x) = 0.25^x$	64	16	4	1	0.25	0.0625	0.015625

- Plot the *y*-intercept, $(0, 1)$,along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in [link].



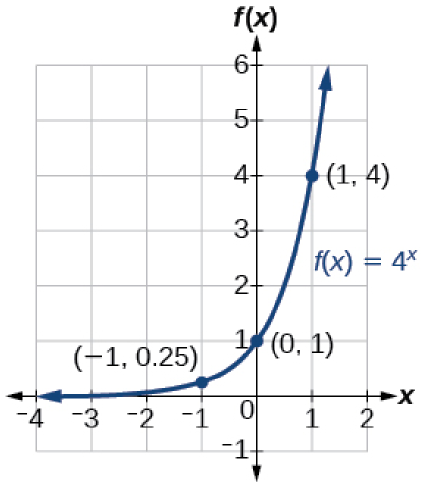
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Note:
Exercise:

Problem: Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Solution:

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

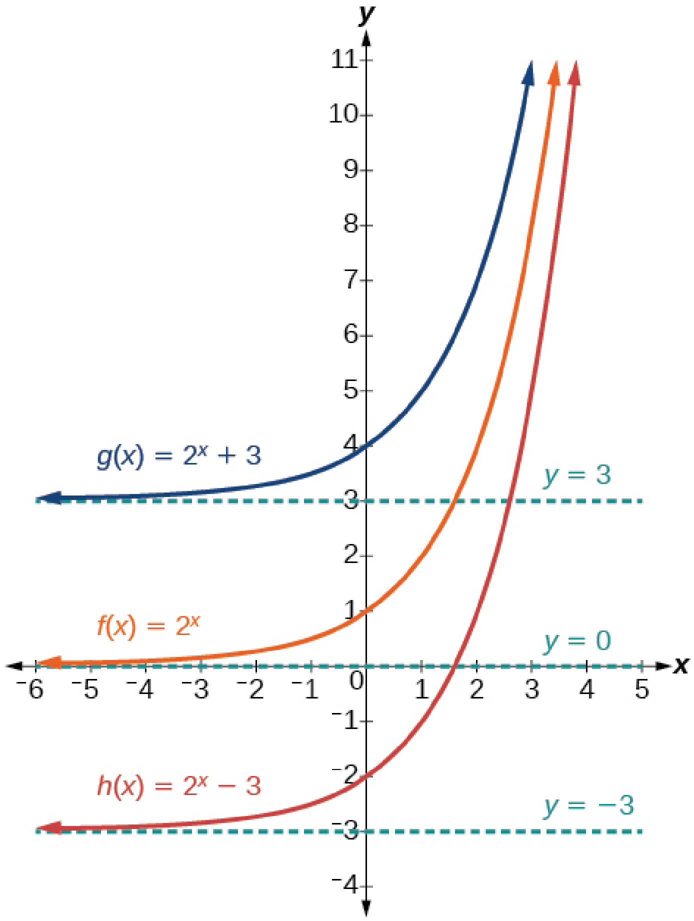


Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the parent function $f(x) = b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3:$ the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in [link].



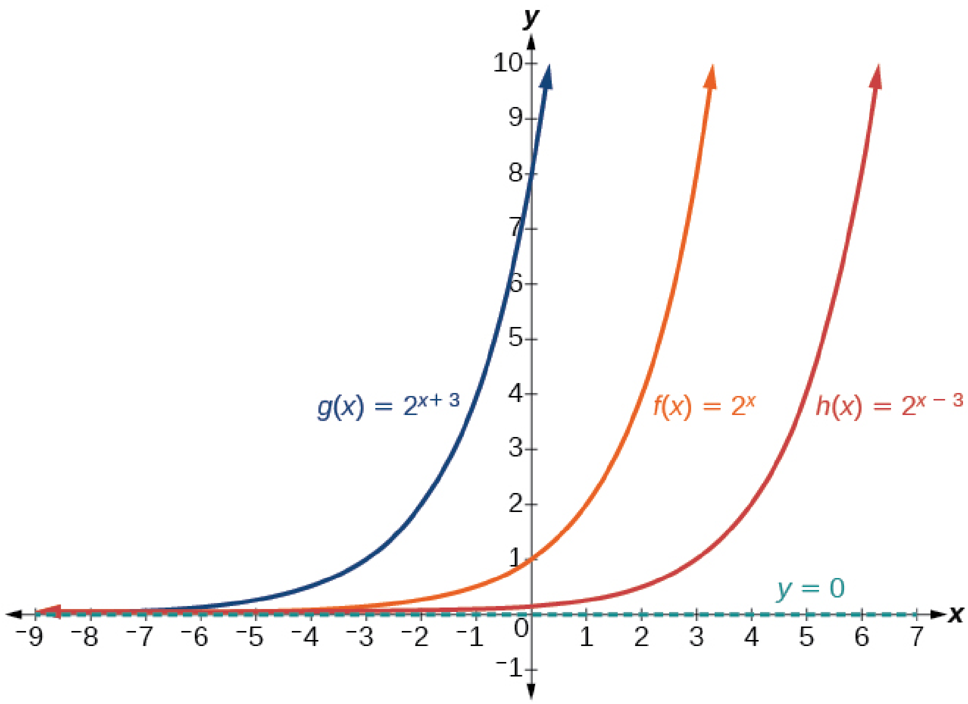
Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The *y*-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The *y*-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the parent function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using

$c = 3:$ the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. Both horizontal shifts are shown in [link].



Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$,remains unchanged.
- The asymptote, $y = 0$,remains unchanged.
- The *y*-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$,the *y*-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)2^x$,so the initial value of the function is 8.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$,the *y*-intercept becomes $\left(0, \frac{1}{8}\right)$. Again, see that $2^{x-3} = \left(\frac{1}{8}\right)2^x$,so the initial value of the function is $\frac{1}{8}$.

Note:
Shifts of the Parent Function *f*(*x*) = *b*^{x}
For any constants c and d,the function $f(x) = b^{x+c} + d$ shifts the parent function $f(x) = b^x$

- vertically d units, in the *same* direction of the sign of d.
- horizontally c units, in the *opposite* direction of the sign of c.
- The *y*-intercept becomes $(0, b^c + d)$.
- The horizontal asymptote becomes $y = d$.
- The range becomes (d, ∞).
- The domain, $(-\infty, \infty)$,remains unchanged.

Note:
Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation.

1. Draw the horizontal asymptote $y = d$.
2. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left c units if c is positive, and right c units if c is negative.
3. Shift the graph of $f(x) = b^x$ up d units if d is positive, and down d units if d is negative.
4. State the domain, $(-\infty, \infty)$, the range, (d, ∞), and the horizontal asymptote $y = d$.

Example:
Exercise:

Problem:
Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

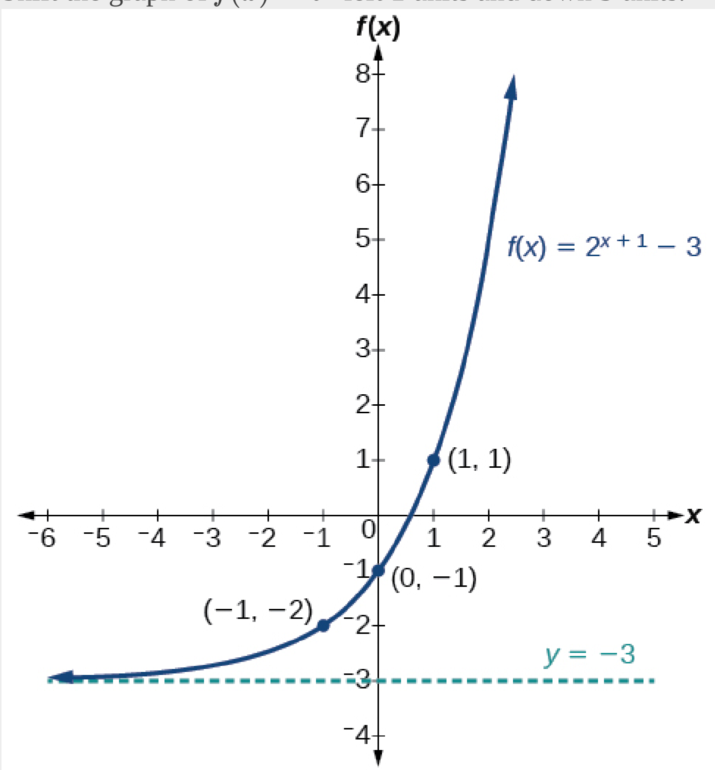
Solution:

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

Draw the horizontal asymptote $y = d$, so draw $y = -3$.

Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.

Shift the graph of $f(x) = b^x$ left 1 units and down 3 units.



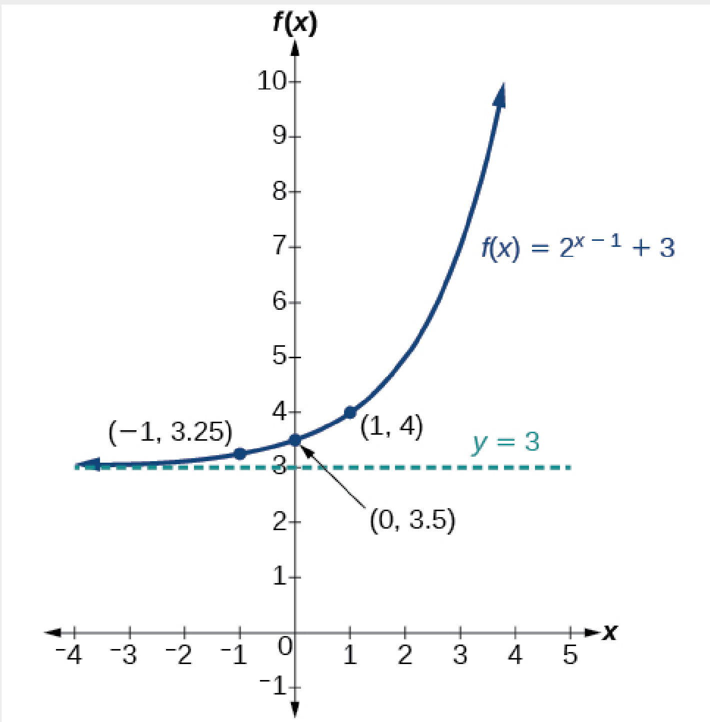
The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

Note:
Exercise:

Problem: Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

Solution:

The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.



Note:
Given an equation of the form $f(x) = b^{x+c} + d$ for x, use a graphing calculator to approximate the solution.

- Press **[Y=]**. Enter the given exponential equation in the line headed "**Y_1=**".
- Enter the given value for $f(x)$ in the line headed "**Y_2=**".
- Press **[WINDOW]**. Adjust the *y*-axis so that it includes the value entered for "**Y_2=**".
- Press **[GRAPH]** to observe the graph of the exponential function along with the line for the specified value of $f(x)$.
- To find the value of x,we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of *x* for the indicated value of the function.

Example:
Exercise:

Problem:
Approximating the Solution of an Exponential Equation

Solve $42=1.2(5)^x+2.8$ graphically. Round to the nearest thousandth.

Solution:

Press **[Y=]** and enter $1.2(5)^x+2.8$ next to **Y$_1$**=. Then enter 42 next to **Y2=**. For a window, use the values –3 to 3 for x and –5 to 55 for y. Press **[GRAPH]**. The graphs should intersect somewhere near $x=2$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 2.1661943. (Your answer may be different if you use a different window or use a different value for **Guess?**) To the nearest thousandth, $x\approx 2.166$.

Note:
Exercise:

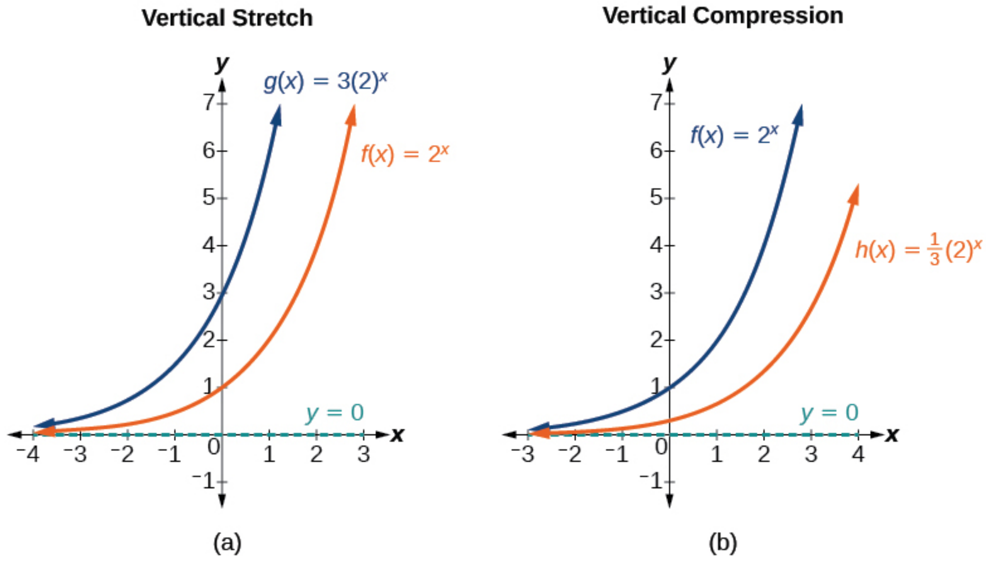
Problem: Solve $4=7.85(1.15)^x-2.27$ graphically. Round to the nearest thousandth.

Solution:

$x\approx -1.608$

Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a stretch or compression occurs when we multiply the parent function $f(x)=b^x$ by a constant $|a|>0$. For example, if we begin by graphing the parent function $f(x)=2^x$,we can then graph the stretch, using $a=3$,to get $g(x)=3(2)^x$ as shown on the left in [link], and the compression, using $a=\frac{1}{3}$,to get $h(x)=\frac{1}{3}(2)^x$ as shown on the right in [link].



(a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

Note:
Stretches and Compressions of the Parent Function $f(x) = b^x$
For any factor $a > 0$,the function $f(x) = a(b)^x$

- is stretched vertically by a factor of a if $|a| > 1$.
- is compressed vertically by a factor of a if $|a| < 1$.
- has a *y*-intercept of $(0, a)$.
- has a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$,which are unchanged from the parent function.

Example:
Exercise:

Problem: Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Solution:

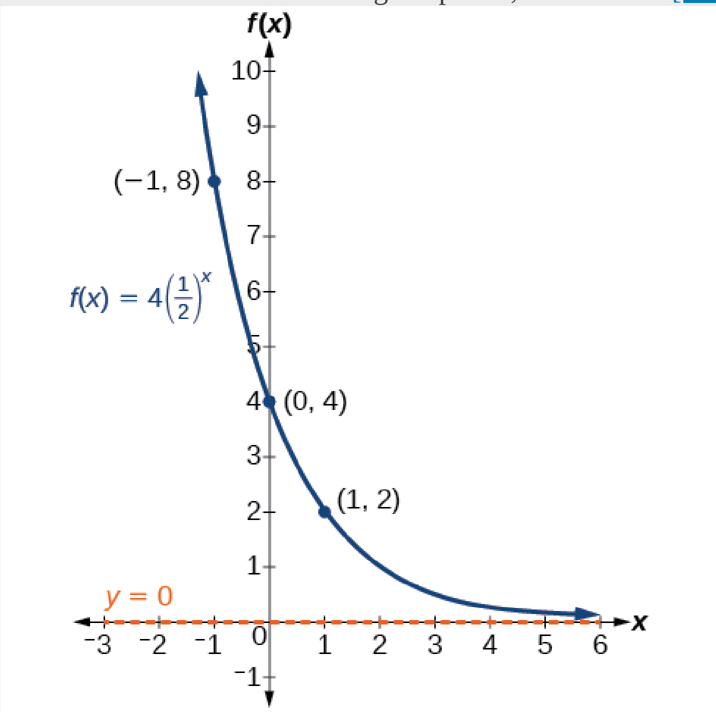
Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the x-axis as x increases.
- Since $a = 4$,the graph of $f(x) = \left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.
- Create a table of points as shown in [link].

x	-3	-2	-1	0	1	2	3
$f(x) = 4\left(\dfrac{1}{2}\right)^x$	32	16	8	4	2	1	0.5

- Plot the y-intercept, $(0, 4)$,along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points, as shown in [link].



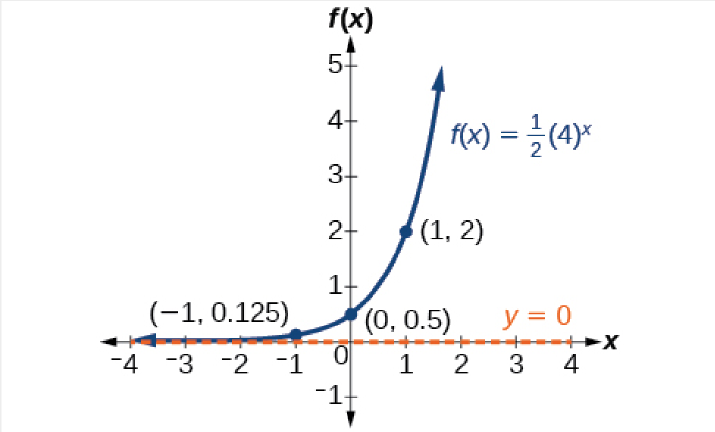
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Note:
Exercise:

Problem: Sketch the graph of $f(x)=\frac{1}{2}(4)^x$. State the domain, range, and asymptote.

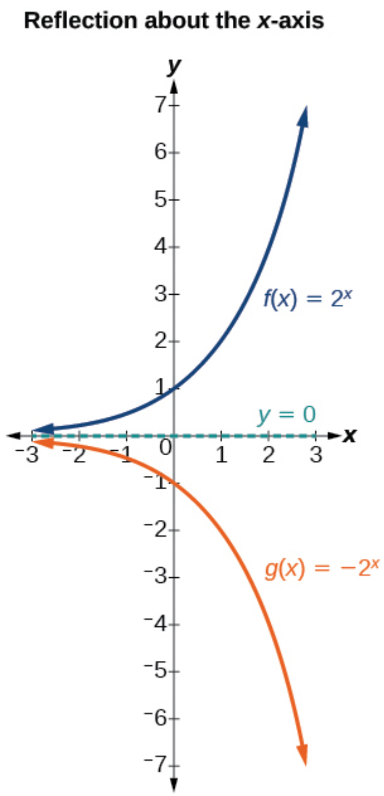
Solution:

The domain is $(-\infty,\infty)$; the range is $(0,\infty)$; the horizontal asymptote is $y=0$.



Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the *x*-axis or the *y*-axis. When we multiply the parent function $f(x)=b^x$ by -1,we get a reflection about the *x*-axis. When we multiply the input by -1,we get a reflection about the *y*-axis. For example, if we begin by graphing the parent function $f(x)=2^x$, we can then graph the two reflections alongside it. The reflection about the *x*-axis, $g(x)=-2^x$,is shown on the left side of [link], and the reflection about the *y*-axis $h(x)=2^{-x}$, is shown on the right side of [link].



(a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the *y*-axis.

Note:
Reflections of the Parent Function $f(x) = b^x$
The function $f(x) = -b^x$

- reflects the parent function $f(x) = b^x$ about the *x*-axis.
- has a *y*-intercept of $(0, -1)$.
- has a range of $(-\infty, 0)$.
- has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$,which are unchanged from the parent function.

The function $f(x) = b^{-x}$

- reflects the parent function $f(x) = b^x$ about the *y*-axis.
- has a *y*-intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

Example:
Exercise:

Problem:
Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$,that reflects $f(x) = \left(\frac{1}{4}\right)^x$ about the *x*-axis. State its domain, range, and asymptote.

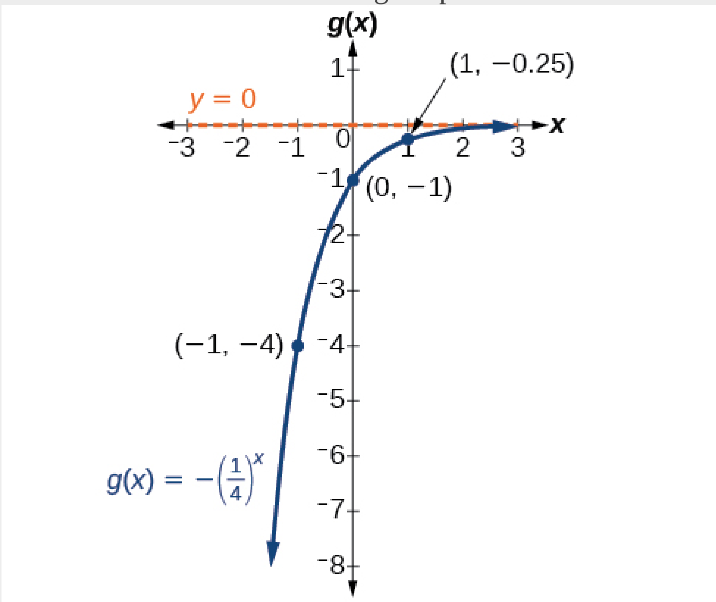
Solution:

Since we want to reflect the parent function $f(x) = \left(\frac{1}{4}\right)^x$ about the *x*-axis, we multiply $f(x)$ by -1 to get, $g(x) = -\left(\frac{1}{4}\right)^x$. Next we create a table of points as in [link].

x	-3	-2	-1	0	1	2	3
$g(x) = -\left(\frac{1}{4}\right)^x$	-64	-16	-4	-1	-0.25	-0.0625	-0.0156

Plot the *y*-intercept, $(0, -1)$,along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:



The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

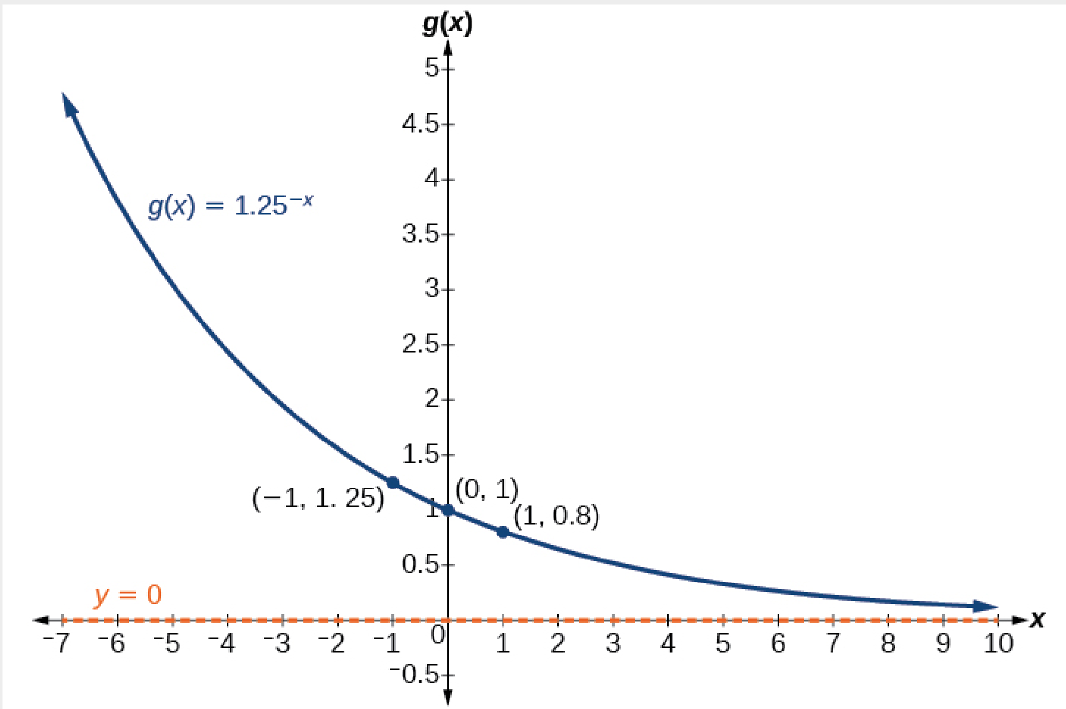
Note:
Exercise:

Problem:

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the y-axis. State its domain, range, and asymptote.

Solution:

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in [link] to arrive at the general equation for translating exponential functions.

Translations of the Parent Function $f(x) = b^x$	
Translation	**Form**

Translations of the Parent Function $f(x)=b^x$	
Translation	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$f(x)=b^{x+c}+d$
Stretch and Compress • Stretch if $\lvert a\rvert>1$ • Compression if $0<\lvert a\rvert<1$	$f(x)=ab^x$
Reflect about the *x*-axis	$f(x)=-b^x$
Reflect about the *y*-axis	$f(x)=b^{-x}=\left(\frac{1}{b}\right)^x$
General equation for all translations	$f(x)=ab^{x+c}+d$

Note:
Translations of Exponential Functions
A translation of an exponential function has the form
Equation:

$$f(x)=ab^{x+c}+d$$

Where the parent function, $y=b^x$, $b>1$,is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $|a|$ if $|a|>0$.
- compressed vertically by a factor of $|a|$ if $0<|a|<1$.
- shifted vertically d units.

- reflected about the *x*-axis when $a < 0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

Example:
Exercise:

Problem: Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the *y*-axis, and then shifted up 4 units.

Solution:

We want to find an equation of the general form $f(x) = ab^{x+c} + d$. We use the description provided to find a, b, c, and d.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of 2, so $a = 2$.
- The function is reflected about the *y*-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,
Equation:

$$\begin{aligned} f(x) &= ab^{x+c} + d \\ &= 2e^{-x+0} + 4 \\ &= 2e^{-x} + 4 \end{aligned}$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

Note:
Exercise:

Problem:

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the *x*-axis and then shifted down 2 units.

Solution:

$f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, 2)$; the horizontal asymptote is $y = 2$.

Note:
Access this online resource for additional instruction and practice with graphing exponential functions.

- Graph Exponential Functions

Key Equations

General Form for the Translation of the Parent Function $f(x) = b^x$	$f(x) = ab^{x+c} + d$

Key Concepts

- The graph of the function $f(x) = b^x$ has a *y*-intercept at $(0,\ 1)$, domain $(-\infty,\ \infty)$, range $(0,\ \infty)$, and horizontal asymptote $y = 0$. See [link].
- If $b > 1$, the function is increasing. The left tail of the graph will approach the asymptote $y = 0$, and the right tail will increase without bound.
- If $0 < b < 1$, the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- The equation $f(x) = b^x + d$ represents a vertical shift of the parent function $f(x) = b^x$.
- The equation $f(x) = b^{x+c}$ represents a horizontal shift of the parent function $f(x) = b^x$. See [link].
- Approximate solutions of the equation $f(x) = b^{x+c} + d$ can be found using a graphing calculator. See [link].
- The equation $f(x) = ab^x$, where $a > 0$, represents a vertical stretch if $|a| > 1$ or compression if $0 < |a| < 1$ of the parent function $f(x) = b^x$. See [link].
- When the parent function $f(x) = b^x$ is multiplied by -1, the result, $f(x) = -b^x$, is a reflection about the *x*-axis. When the input is multiplied by -1, the result, $f(x) = b^{-x}$, is a reflection about the *y*-axis. See [link].
- All translations of the exponential function can be summarized by the general equation $f(x) = ab^{x+c} + d$. See [link].
- Using the general equation $f(x) = ab^{x+c} + d$, we can write the equation of a function given its description. See [link].

Section Exercises

Verbal

Exercise:

Problem:

What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?

Solution:

An asymptote is a line that the graph of a function approaches, as x either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small.

Exercise:

Problem:

What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

Algebraic

Exercise:

Problem:

The graph of $f(x) = 3^x$ is reflected about the *y*-axis and stretched vertically by a factor of 4. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

Solution:

$g(x) = 4(3)^{-x}$; *y*-intercept: $(0, 4)$; Domain: all real numbers; Range: all real numbers greater than 0.

Exercise:

Problem:

The graph of $f(x) = \left(\frac{1}{2}\right)^{-x}$ is reflected about the *y*-axis and compressed vertically by a factor of $\frac{1}{5}$. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

Exercise:

Problem:

The graph of $f(x) = 10^x$ is reflected about the *x*-axis and shifted upward 7 units. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

Solution:

$g(x) = -10^x + 7$; *y*-intercept: $(0, 6)$; Domain: all real numbers; Range: all real numbers less than 7.

Exercise:

Problem:

The graph of $f(x) = (1.68)^x$ is shifted right 3 units, stretched vertically by a factor of 2, reflected about the *x*-axis, and then shifted downward 3 units. What is the equation of the new function, $g(x)$? State its *y*-intercept (to the nearest thousandth), domain, and range.

Exercise:

Problem:

The graph of $f(x) = 2\left(\frac{1}{4}\right)^{x-20}$ is shifted downward 4 units, and then shifted left 2 units, stretched vertically by a factor of 4, and reflected about the *x*-axis. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

Solution:

$g(x) = 2\left(\frac{1}{4}\right)^x;$ *y*-intercept: $(0,\ 2);$ Domain: all real numbers; Range: all real numbers greater than 0.

Graphical

For the following exercises, graph the function and its reflection about the *y*-axis on the same axes, and give the *y*-intercept.

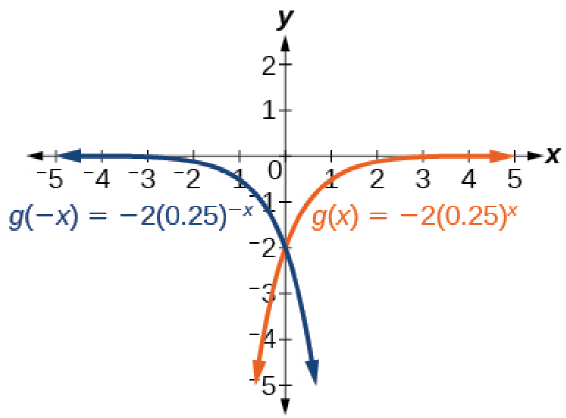
Exercise:

Problem: $f(x) = 3\left(\frac{1}{2}\right)^x$

Exercise:

Problem: $g(x) = -2(0.25)^x$

Solution:



y-intercept: $(0, -2)$

Exercise:

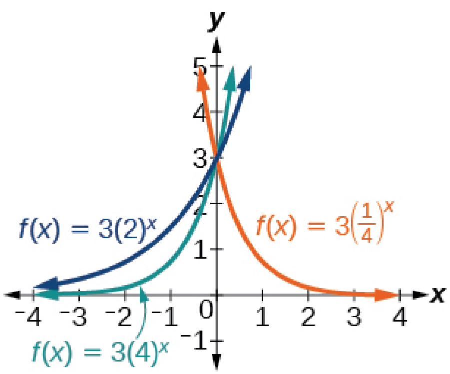
Problem: $h(x) = 6(1.75)^{-x}$

For the following exercises, graph each set of functions on the same axes.

Exercise:

Problem: $f(x) = 3\left(\frac{1}{4}\right)^{x}$,$g(x) = 3(2)^{x}$,and $h(x) = 3(4)^{x}$

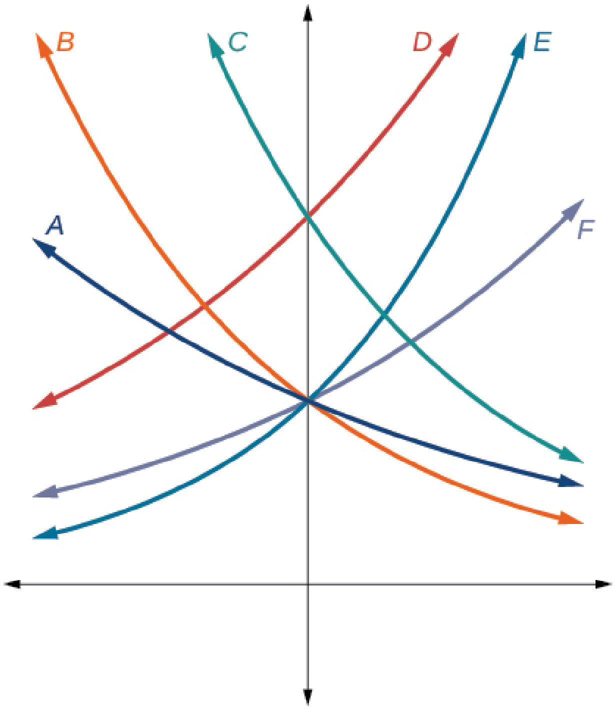
Solution:



Exercise:

Problem: $f(x) = \frac{1}{4}(3)^{x}$,$g(x) = 2(3)^{x}$,and $h(x) = 4(3)^{x}$

For the following exercises, match each function with one of the graphs in [link].



Exercise:

Problem: $f\left(x\right)=2(0.69)^{x}$

Solution:

B

Exercise:

Problem: $f\left(x\right)=2(1.28)^{x}$

Exercise:

Problem: $f\left(x\right)=2(0.81)^{x}$

Solution:

A

Exercise:

Problem: $f\left(x\right)=4(1.28)^{x}$

Exercise:

Problem: $f\left(x\right)=2(1.59)^{x}$

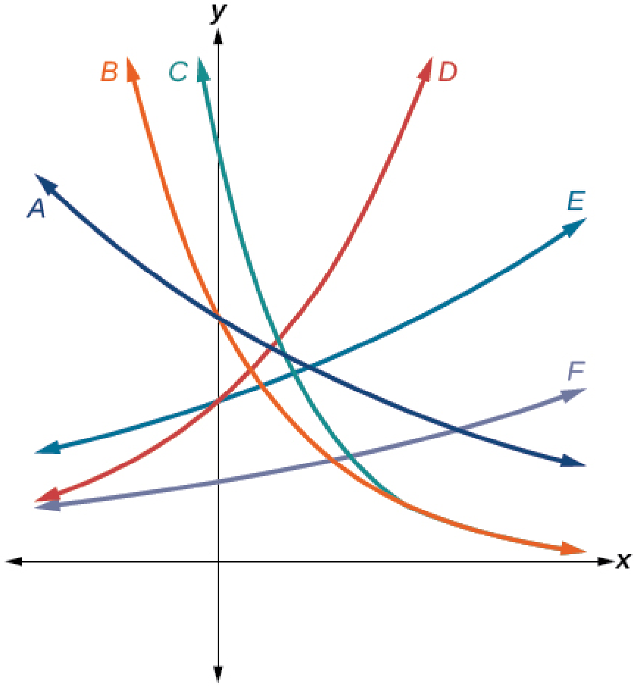
Solution:

E

Exercise:

Problem: $f\left(x\right)=4{\left(0.69\right)}^{x}$

For the following exercises, use the graphs shown in [link]. All have the form $f\left(x\right)=a{b}^{x}$.



Exercise:

Problem: Which graph has the largest value for b?

Solution:

D

Exercise:

Problem: Which graph has the smallest value for b?

Exercise:

Problem: Which graph has the largest value for a?

Solution:

C

Exercise:

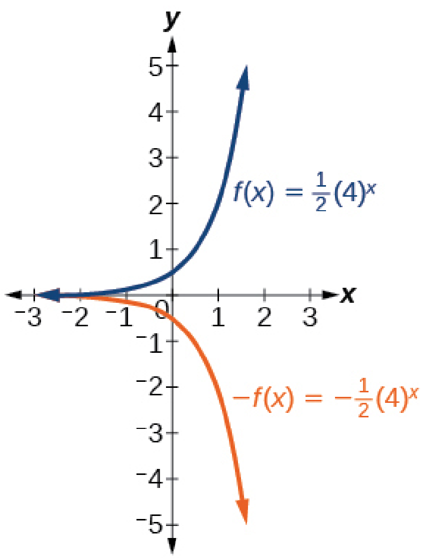
Problem: Which graph has the smallest value for a?

For the following exercises, graph the function and its reflection about the *x*-axis on the same axes.

Exercise:

Problem: $f(x) = \frac{1}{2}(4)^x$

Solution:



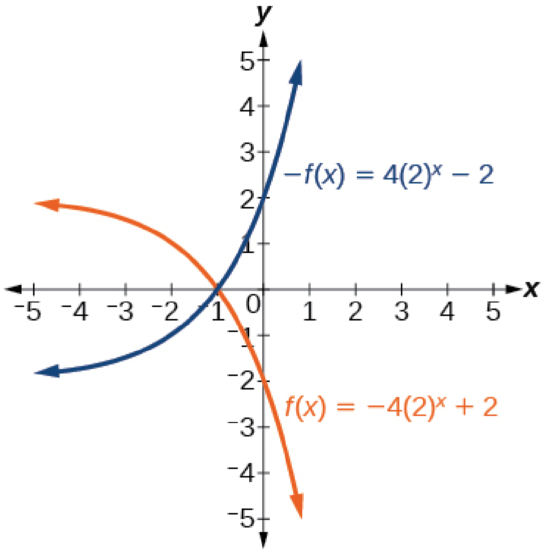
Exercise:

Problem: $f(x) = 3(0.75)^x - 1$

Exercise:

Problem: $f(x) = -4(2)^x + 2$

Solution:



For the following exercises, graph the transformation of $f(x) = 2^x$. Give the horizontal asymptote, the domain, and the range.

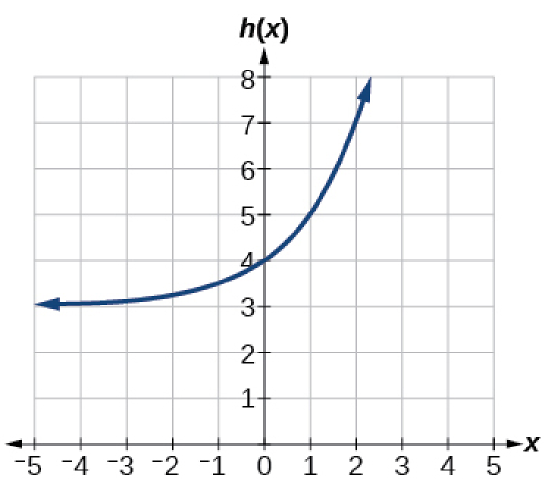
Exercise:

Problem: $f\left(x\right) = 2^{-x}$

Exercise:

Problem: $h\left(x\right) = 2^x + 3$

Solution:



Horizontal asymptote: $h(x) = 3;$ Domain: all real numbers; Range: all real numbers strictly greater than 3.

Exercise:

Problem: $f(x) = 2^{x-2}$

For the following exercises, describe the end behavior of the graphs of the functions.
Exercise:

Problem: $f(x) = -5(4)^{x} - 1$

Solution:

As $x \to \infty$, $f(x) \to -\infty$;
As $x \to -\infty$, $f(x) \to -1$

Exercise:

Problem: $f(x) = 3\left(\frac{1}{2}\right)^{x} - 2$

Exercise:

Problem: $f(x) = 3(4)^{-x} + 2$

Solution:

As $x \to \infty$, $f(x) \to 2$;
As $x \to -\infty$, $f(x) \to \infty$

For the following exercises, start with the graph of $f(x) = 4^{x}.$ Then write a function that results from the given transformation.
Exercise:

Problem: Shift $f(x)$ 4 units upward

Exercise:

Problem: Shift $f(x)$ 3 units downward

Solution:

$f(x) = 4^{x} - 3$

Exercise:

Problem: Shift $f(x)$ 2 units left

Exercise:

Problem: Shift $f(x)$ 5 units right

Solution:

$f(x) = 4^{x-5}$

Exercise:

Problem: Reflect $f(x)$ about the *x*-axis

Exercise:

Problem: Reflect $f(x)$ about the *y*-axis

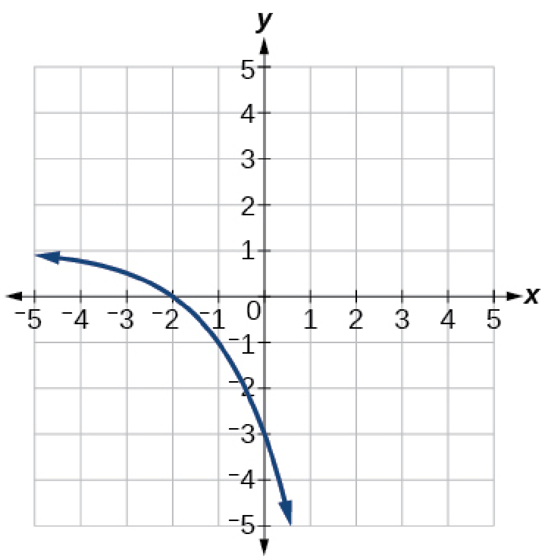
Solution:

$f\left(x\right) = 4^{-x}$

For the following exercises, each graph is a transformation of $y = 2^{x}.$ Write an equation describing the transformation.

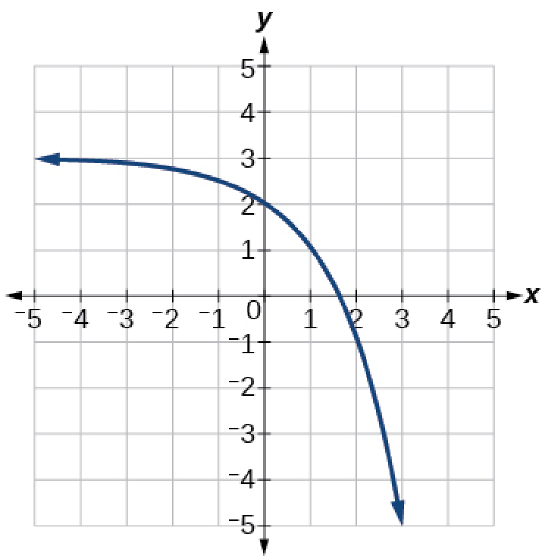
Exercise:

Problem:



Exercise:

Problem:

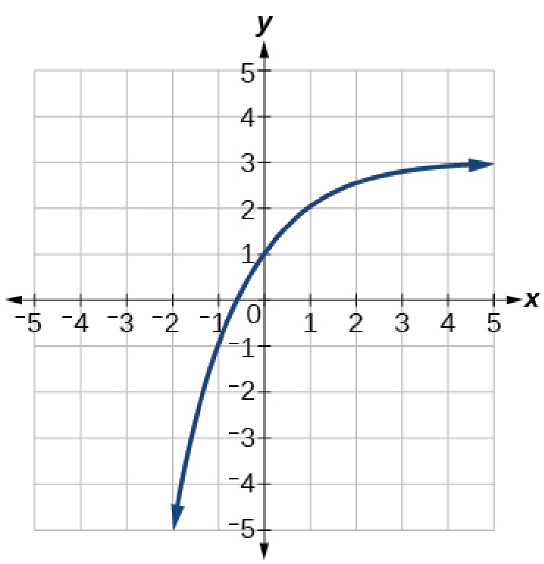


Solution:

$y = -2^x + 3$

Exercise:

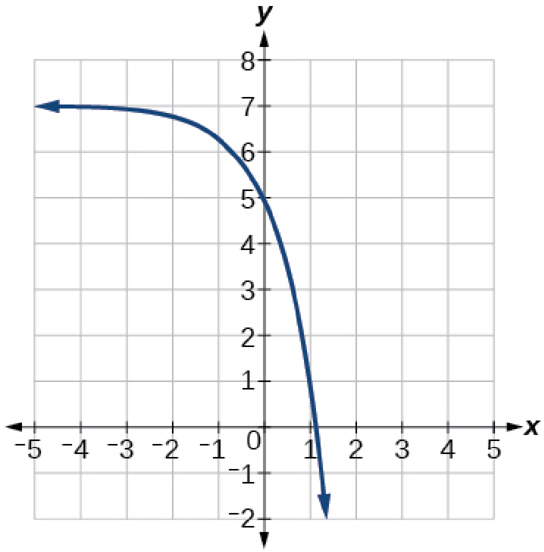
Problem:



For the following exercises, find an exponential equation for the graph.

Exercise:

Problem:

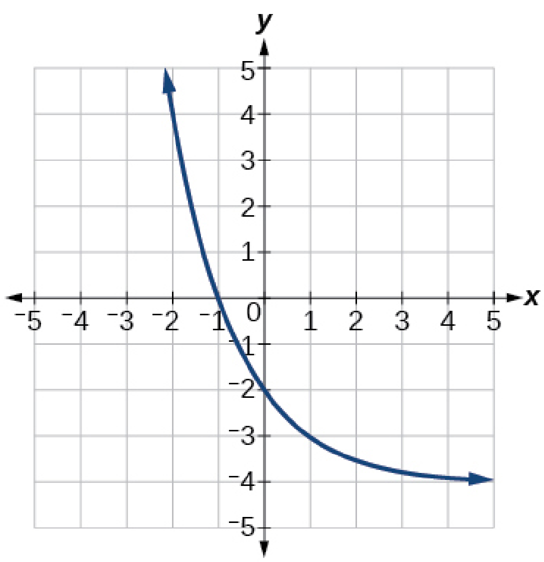


Solution:

$y = -2(3)^x + 7$

Exercise:

Problem:



Numeric

For the following exercises, evaluate the exponential functions for the indicated value of x.

Exercise:

Problem: $g(x) = \frac{1}{3}(7)^{x-2}$ for $g(6)$.

Solution:

$g(6) = 800 + \frac{1}{3} \approx 800.3333$

Exercise:

Problem: $f(x) = 4(2)^{x-1} - 2$ for $f(5)$.

Exercise:

Problem: $h(x) = -\frac{1}{2}\left(\frac{1}{2}\right)^{x} + 6$ for $h(-7)$.

Solution:

$h(-7) = -58$

Technology

For the following exercises, use a graphing calculator to approximate the solutions of the equation. Round to the nearest thousandth.

Exercise:

Problem: $-50 = -\left(\frac{1}{2}\right)^{-x}$

Exercise:

Problem: $116 = \frac{1}{4}\left(\frac{1}{8}\right)^{x}$

Solution:

$x \approx -2.953$

Exercise:

Problem: $12 = 2(3)^{x} + 1$

Exercise:

Problem: $5 = 3\left(\frac{1}{2}\right)^{x-1} - 2$

Solution:

$x \approx -0.222$

Exercise:

Problem: $-30 = -4(2)^{x+2} + 2$

Extensions

Exercise:

Problem:

Explore and discuss the graphs of $F(x) = (b)^x$ and $G(x) = \left(\frac{1}{b}\right)^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b}\right)^x$ for any real number $b > 0$.

Solution:

The graph of $G(x) = \left(\frac{1}{b}\right)^x$ is the refelction about the *y*-axis of the graph of $F(x) = b^x$; For any real number $b > 0$ and function $f(x) = b^x$,the graph of $\left(\frac{1}{b}\right)^x$ is the the reflection about the *y*-axis, $F(-x)$.

Exercise:

Problem: Prove the conjecture made in the previous exercise.

Exercise:

Problem:

Explore and discuss the graphs of $f(x) = 4^x$, $g(x) = 4^{x-2}$,and $h(x) = \left(\frac{1}{16}\right)4^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b^n}\right)b^x$ for any real number *n* and real number $b > 0$.

Solution:

The graphs of $g(x)$ and $h(x)$ are the same and are a horizontal shift to the right of the graph of $f(x)$; For any real number *n*, real number $b > 0$, and function $f(x) = b^x$, the graph of $\left(\frac{1}{b^n}\right)b^x$ is the horizontal shift $f(x - n)$.

Exercise:

Problem: Prove the conjecture made in the previous exercise.

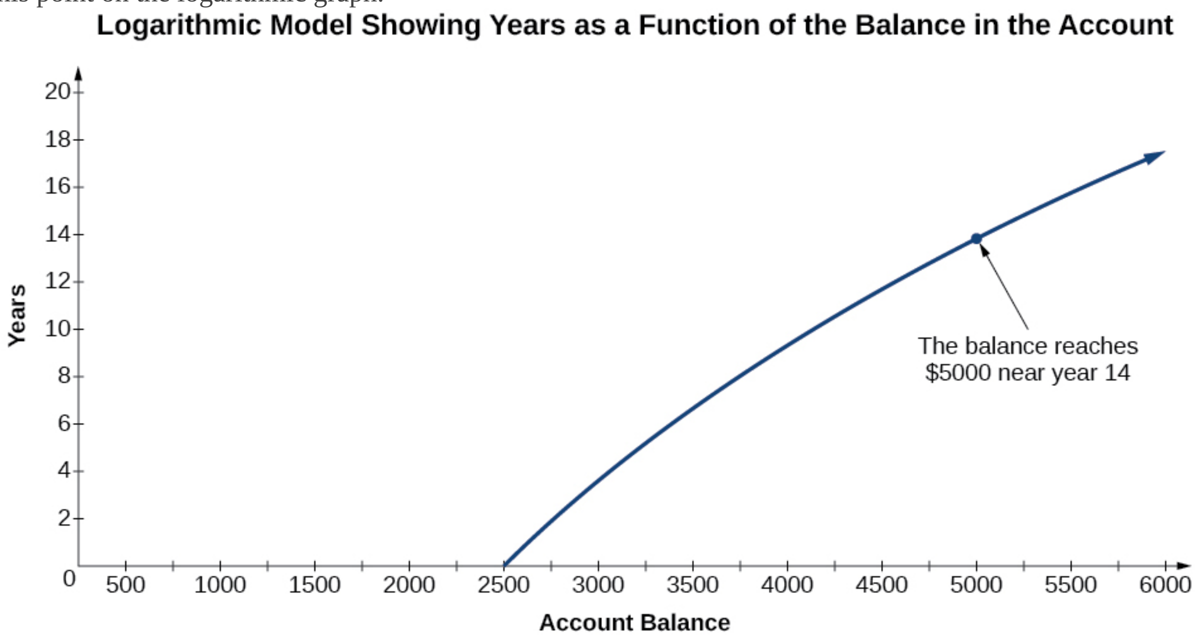
Graphs of Logarithmic Functions
In this section, you will:

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

In Graphs of Exponential Functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5%,compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A=2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? [link] shows this point on the logarithmic graph.



In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y=b^x$ for any real number x and constant $b>0$, $b\neq 1$, where

- The domain of y is $(-\infty,\infty)$.

- The range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$, is greater than zero. To find the domain, we set up an inequality and solve for x:

Equation:

$$\begin{array}{rl} 2x - 3 > 0 & \text{Show the argument greater than zero.} \\ 2x > 3 & \text{Add 3.} \\ x > 1.5 & \text{Divide by 2.} \end{array}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

Note:
Given a logarithmic function, identify the domain.

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

Example:
Exercise:

Problem:
Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x + 3)$?

Solution:

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

Equation:

$$\begin{aligned} x+3&>0 && \text{The input must be positive.} \\ x&>-3 && \text{Subtract 3.} \end{aligned}$$

The domain of $f(x)=\log_2(x+3)$ is $(-3,\infty)$.

Note:
Exercise:

Problem: What is the domain of $f(x)=\log_5(x-2)+1$?

Solution:

$(2,\infty)$

Example:
Exercise:

Problem:
Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x)=\log(5-2x)$?

Solution:

The logarithmic function is defined only when the input is positive, so this function is defined when $5–2x>0.$ Solving this inequality,
Equation:

$$\begin{aligned} 5-2x&>0 && \text{The input must be positive.} \\ -2x&>-5 && \text{Subtract 5.} \\ x&<\frac{5}{2} && \text{Divide by } -2 \text{ and switch the inequality.} \end{aligned}$$

The domain of $f(x)=\log(5-2x)$ is $\left(-\infty,\frac{5}{2}\right)$.

Note:
Exercise:

Problem: What is the domain of $f(x)=\log(x-5)+2$?

Solution:

$(5,\infty)$

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y=\log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

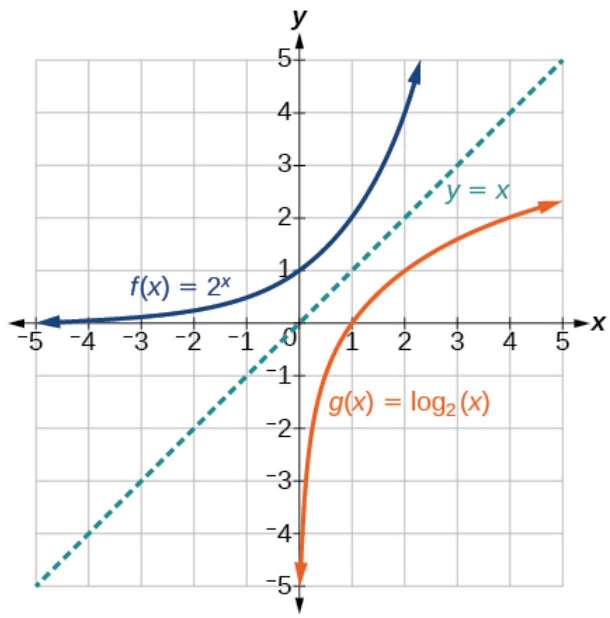
We begin with the parent function $y=\log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y=b^x$, their graphs will be reflections of each other across the line $y=x$. To illustrate this, we can observe the relationship between the input and output values of $y=2^x$ and its equivalent $x=\log_2(y)$ in [link].

x	-3	-2	-1	0	1	2	3
$2^x=y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y)=x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from [link], we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x)=2^x$ and $g(x)=\log_2(x)$. See [link].

$f(x)=2^x$	$\left(-3,\frac{1}{8}\right)$	$\left(-2,\frac{1}{4}\right)$	$\left(-1,\frac{1}{2}\right)$	$(0,1)$	$(1,2)$	$(2,4)$	$(3,8)$
$g(x)=\log_2(x)$	$\left(\frac{1}{8},-3\right)$	$\left(\frac{1}{4},-2\right)$	$\left(\frac{1}{2},-1\right)$	$(1,0)$	$(2,1)$	$(4,2)$	$(8,3)$

As we’d expect, the *x*- and *y*-coordinates are reversed for the inverse functions. [link] shows the graph of f and g.



Notice that the graphs of $f\left(x\right)={2}^{x}$ and $g\left(x\right)={\mathrm{log}}_{2}\left(x\right)$ are reflections about the line $y=x$.

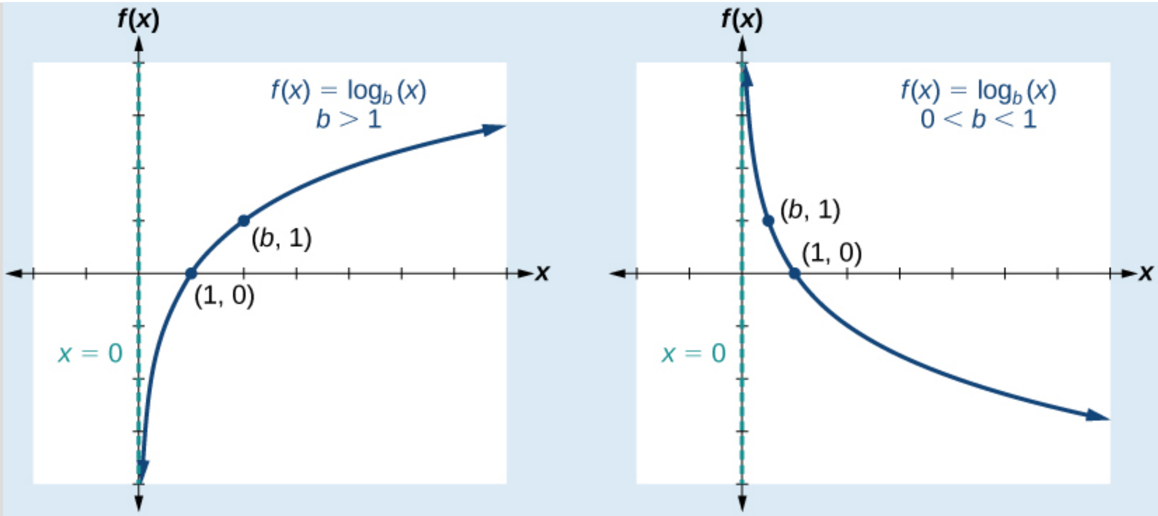
Observe the following from the graph:

- $f(x)={2}^{x}$ has a y-intercept at $(0,1)$ and $g(x)={\mathrm{log}}_{2}\left(x\right)$ has an x- intercept at $(1,0)$.
- The domain of $f(x)={2}^{x}$, $(-\infty ,\infty)$, is the same as the range of $g(x)={\mathrm{log}}_{2}\left(x\right)$.
- The range of $f(x)={2}^{x}$, $(0,\infty)$, is the same as the domain of $g(x)={\mathrm{log}}_{2}\left(x\right)$.

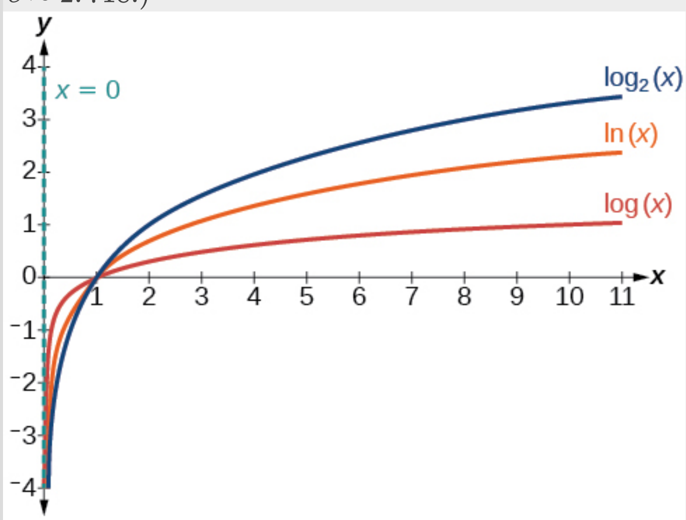
Note:
Characteristics of the Graph of the Parent Function, $f(x)={\mathrm{log}}_{b}\left(x\right):$
For any real number x and constant $b>0$,$b\ne 1$, we can see the following characteristics in the graph of $f(x)={\mathrm{log}}_{b}\left(x\right):$

- one-to-one function
- vertical asymptote: $x=0$
- domain: $(0,\infty)$
- range: $(-\infty ,\infty)$
- x-intercept: $(1,0)$ and key point $(b,1)$
- y-intercept: none
- increasing if $b>1$
- decreasing if $0<b<1$

See [link].



[link] shows how changing the base b in $f(x)=\log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (*Note:* recall that the function $\ln(x)$ has base $e\approx 2.718$.)



The graphs of three logarithmic functions with different bases, all greater than 1.

Note:
Given a logarithmic function with the form $f(x)=\log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x=0$.
2. Plot the *x*-intercept, $(1,0)$.
3. Plot the key point $(b,1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0,\infty)$,the range, $(-\infty,\infty)$,and the vertical asymptote, $x=0$.

Example:
Exercise:

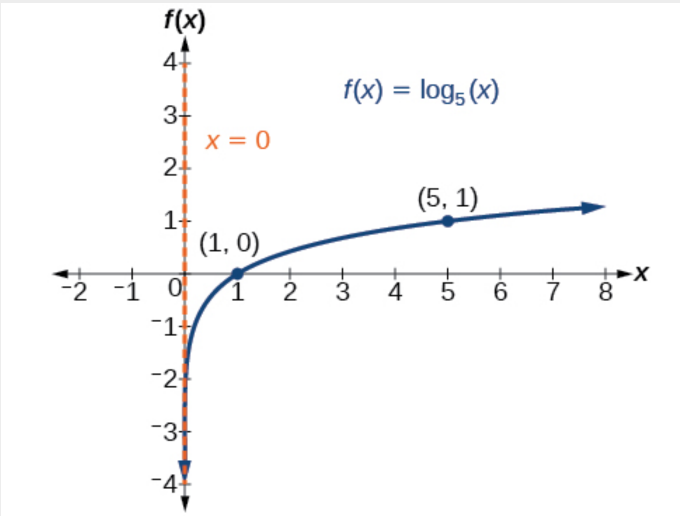
Problem:
Graphing a Logarithmic Function with the Form *f*(*x*) = log*b*(*x*).

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution:

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The *x*-intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see [link]).

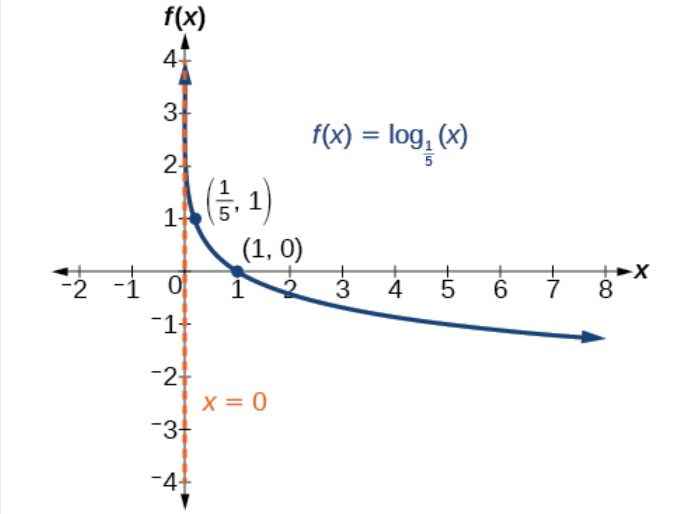


The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Note:
Exercise:

Problem: Graph $f(x) = \log_{\frac{1}{5}}(x)$. State the domain, range, and asymptote.

Solution:



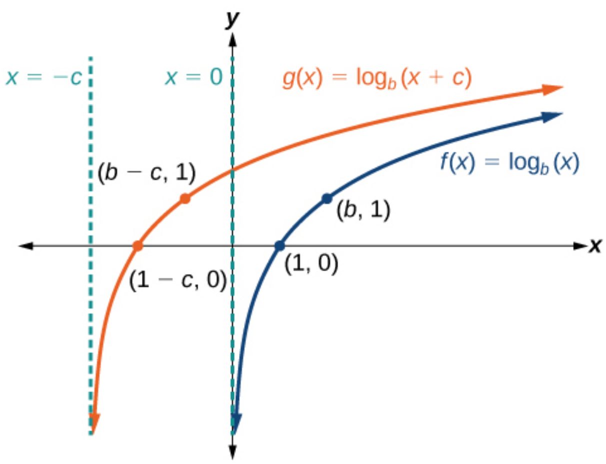
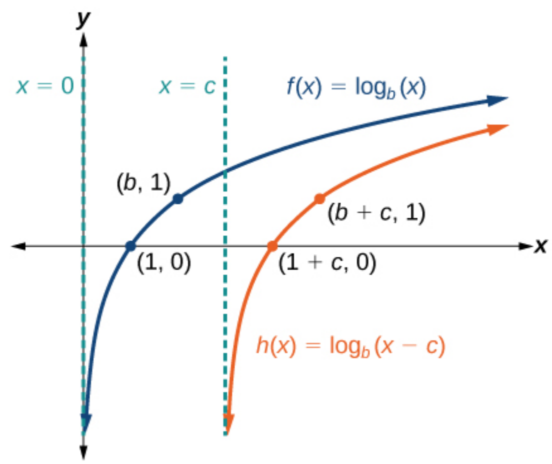
The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of *f*(*x*) = log*b*(*x*)

When a constant c is added to the input of the parent function $f(x) = log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See [link].

Shift left $g(x) = \log_b(x + c)$	**Shift right** $h(x) = \log_b(x - c)$
 •The asymptote changes to $x = -c$. •The domain changes to $(-c, \infty)$. •The range remains $(-\infty, \infty)$.	 •The asymptote changes to $x = c$. •The domain changes to (c, ∞). •The range remains $(-\infty, \infty)$.

Note:
Horizontal Shifts of the Parent Function $f(x) = \log_b(x)$
For any constant c,the function $f(x) = \log_b(x + c)$

- shifts the parent function $y = \log_b(x)$ left c units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right c units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

Note:
Given a logarithmic function with the form $f(x) = \log_b(x + c)$, graph the translation.

1. Identify the horizontal shift:

    a. If $c > 0$,shift the graph of $f(x) = \log_b(x)$ left c units.
    b. If $c < 0$,shift the graph of $f(x) = \log_b(x)$ right c units.

2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting c from the x coordinate.
4. Label the three points.
5. The Domain is $(-c, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

Example:
Exercise:

Problem:
Graphing a Horizontal Shift of the Parent Function *y* = log*ᵦ*(*x*)

Sketch the horizontal shift $f(x)=\log_3(x-2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution:

Since the function is $f(x)=\log_3(x-2)$, we notice $x+(-2)=x–2$.

Thus $c=-2$,so $c<0$. This means we will shift the function $f(x)=\log_3(x)$ right 2 units.

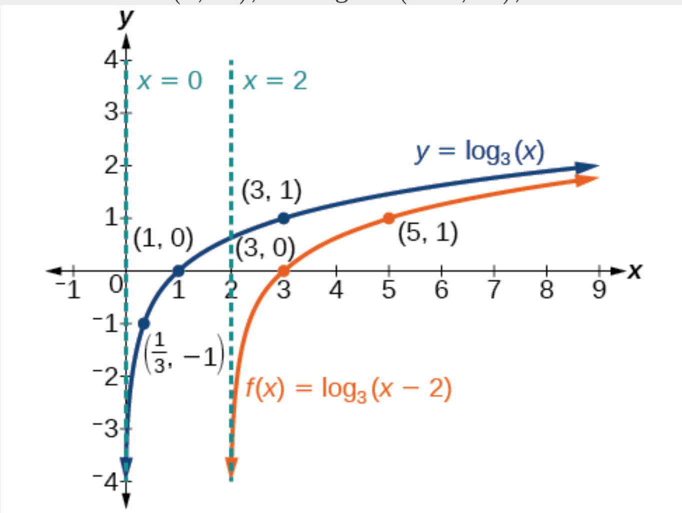
The vertical asymptote is $x=-(-2)$ or $x=2$.

Consider the three key points from the parent function, $\left(\frac{1}{3},-1\right)$,$(1,0)$,and $(3,1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $\left(\frac{7}{3},-1\right)$,$(3,0)$,and $(5,1)$.

The domain is $(2,\infty)$,the range is $(-\infty,\infty)$,and the vertical asymptote is $x=2$.

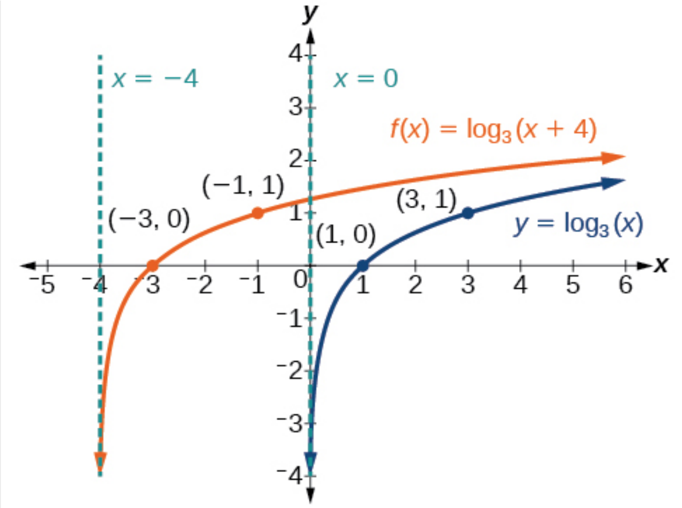


Note:
Exercise:

Problem:

Sketch a graph of $f(x)=\log_3(x+4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

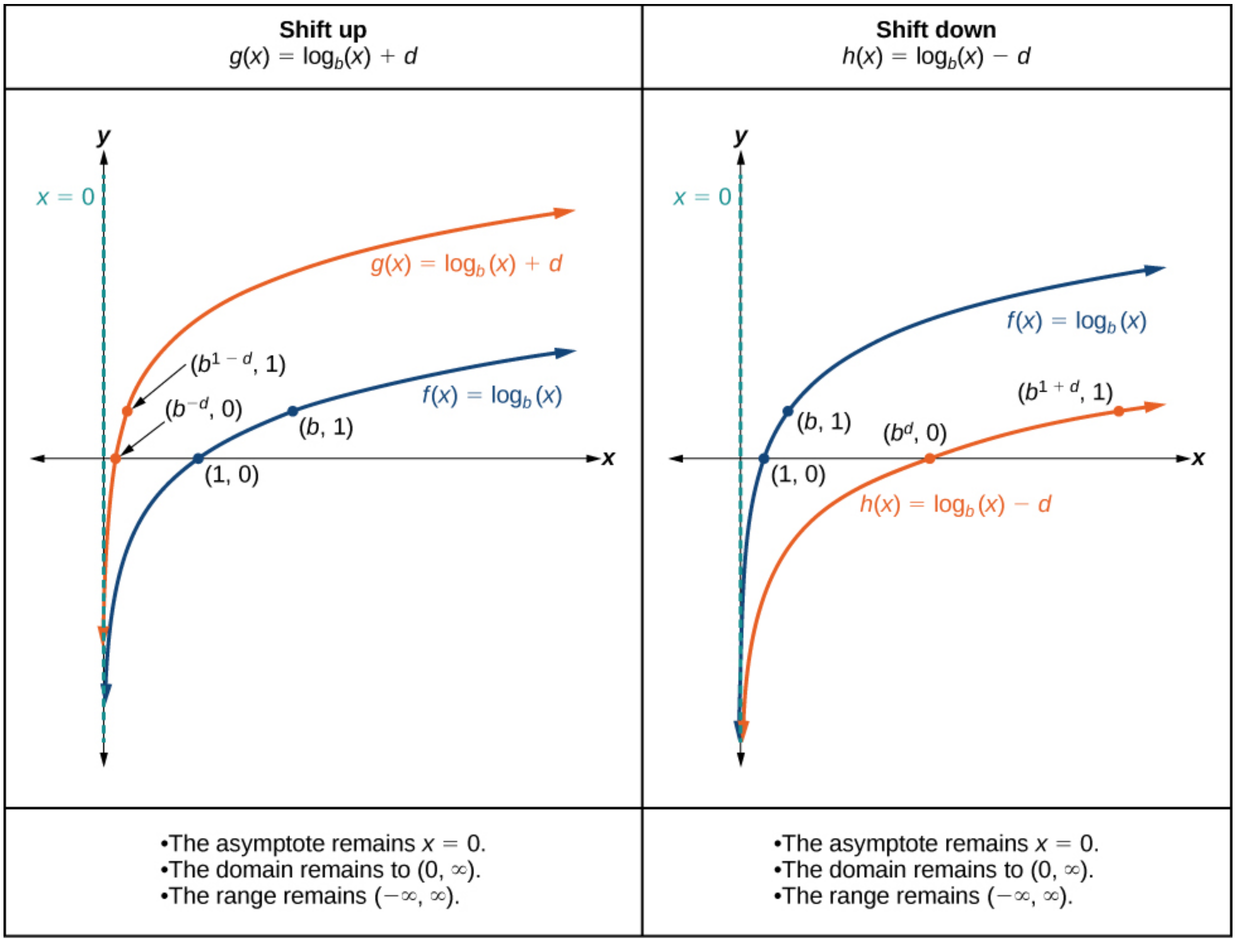
Solution:



The domain is $(-4, \infty)$,the range $(-\infty, \infty)$,and the asymptote $x = -4$.

Graphing a Vertical Shift of *y* = log*b*(*x*)

When a constant d is added to the parent function $f(x) = \log_b(x)$,the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$.See [link].



Note:

Vertical Shifts of the Parent Function $y = \log_b(x)$

For any constant d,the function $f(x) = \log_b(x) + d$

- shifts the parent function $y = \log_b(x)$ up d units if $d > 0$.
- shifts the parent function $y = \log_b(x)$ down d units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

Note:

Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up d units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$down d units.

2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding d to the y coordinate.
4. Label the three points.
5. The domain is $(0,\infty)$,the range is $(-\infty,\infty)$,and the vertical asymptote is $x = 0$.

Example:
Exercise:

Problem:
Graphing a Vertical Shift of the Parent Function *y* = log*b*(*x*)

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:

Since the function is $f(x) = \log_3(x) - 2$,we will notice $d =-2$. Thus $d < 0$.

This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

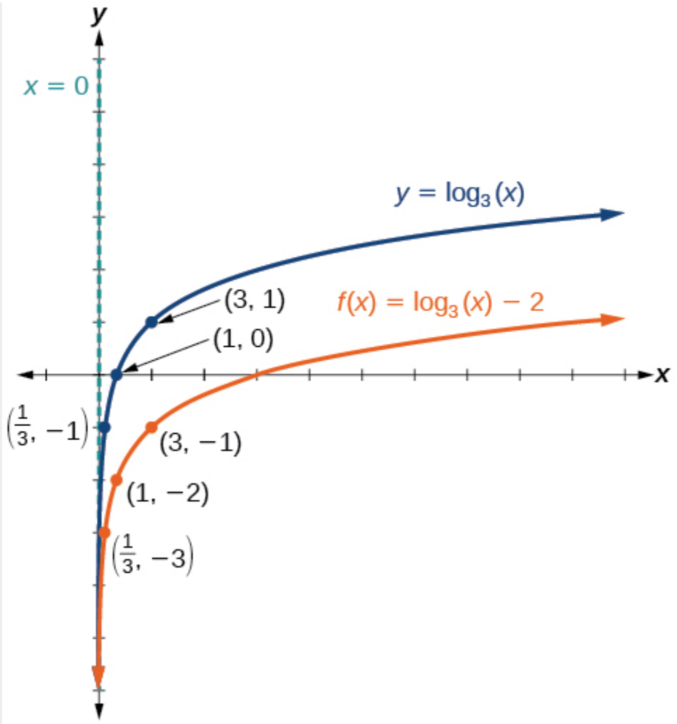
The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$,$(1, 0)$,and $(3, 1)$.

The new coordinates are found by subtracting 2 from the *y* coordinates.

Label the points $\left(\frac{1}{3}, -3\right)$,$(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.



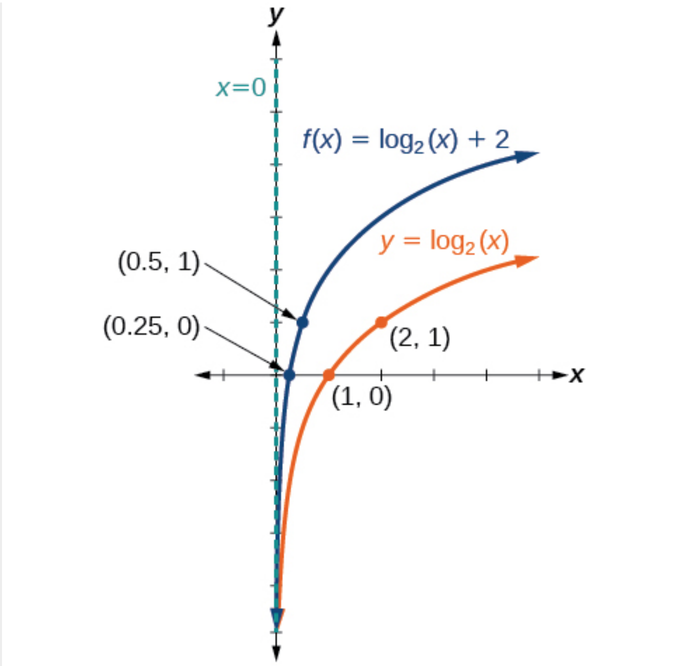
The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Note:
Exercise:

Problem:

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

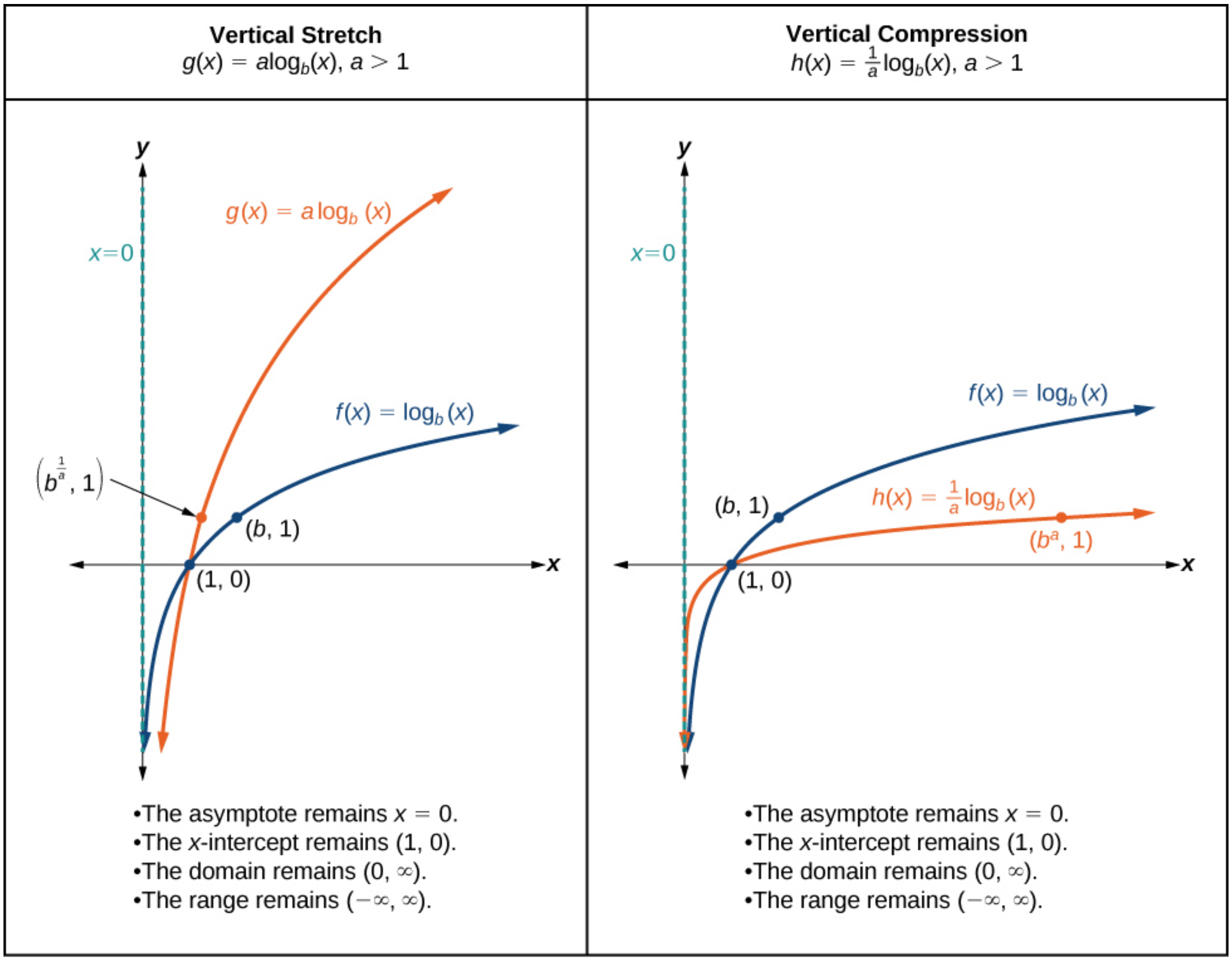
Solution:



The domain is $(0,\infty)$,the range is $(-\infty,\infty)$,and the vertical asymptote is $x=0$.

Graphing Stretches and Compressions of *y* = log*b*(*x*)

When the parent function $f(x)=\log_b(x)$ is multiplied by a constant $a>0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a>1$ and observe the general graph of the parent function $f(x)=\log_b(x)$ alongside the vertical stretch, $g(x)=a\log_b(x)$ and the vertical compression, $h(x)=\frac{1}{a}\log_b(x)$.See [link].



Note:
Vertical Stretches and Compressions of the Parent Function $y = \log_b(x)$
For any constant $a > 1$,the function $f(x) = a\log_b(x)$

- stretches the parent function $y = \log_b(x)$ vertically by a factor of a if $a > 1$.
- compresses the parent function $y = \log_b(x)$ vertically by a factor of a if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the x-intercept $(1, 0)$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

Note:
Given a logarithmic function with the form $f(x) = a\log_b(x)$,$a > 0$,graph the translation.

1. Identify the vertical stretch or compressions:
   - If $|a| > 1$,the graph of $f(x) = \log_b(x)$ is stretched by a factor of a units.
   - If $|a| < 1$,the graph of $f(x) = \log_b(x)$ is compressed by a factor of a units.

2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying the y coordinates by a.
4. Label the three points.
5. The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Example:
Exercise:

Problem:
Graphing a Stretch or Compression of the Parent Function *y* = log$_b$(*x*)

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:

Since the function is $f(x) = 2\log_4(x)$,we will notice $a = 2$.

This means we will stretch the function $f(x) = \log_4(x)$ by a factor of 2.

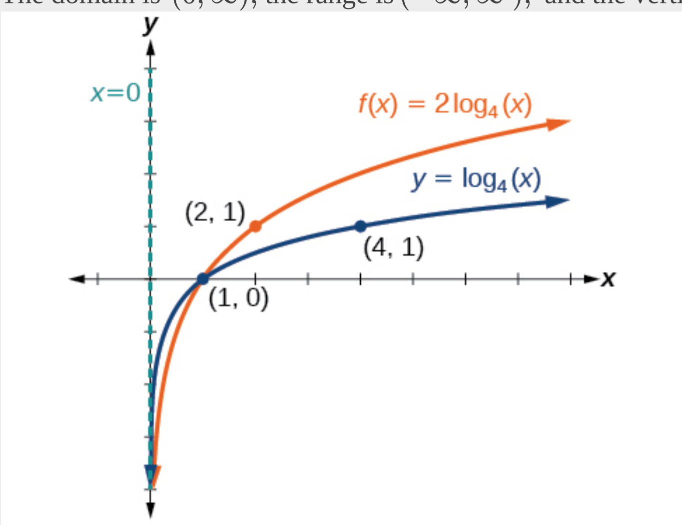
The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{4}, -1\right)$,$(1, 0\ )$, and $(4, 1)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $\left(\frac{1}{4}, -2\right)$,$(1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty\ )$, and the vertical asymptote is $x = 0$. See [link].



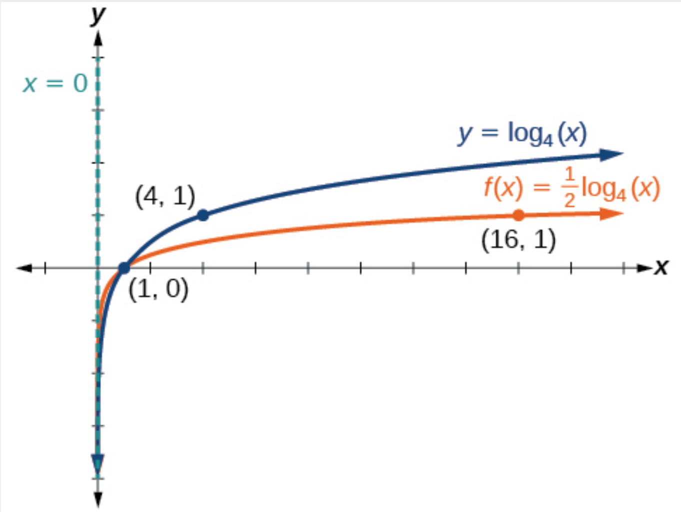
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Note:
Exercise:

Problem:

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:



The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

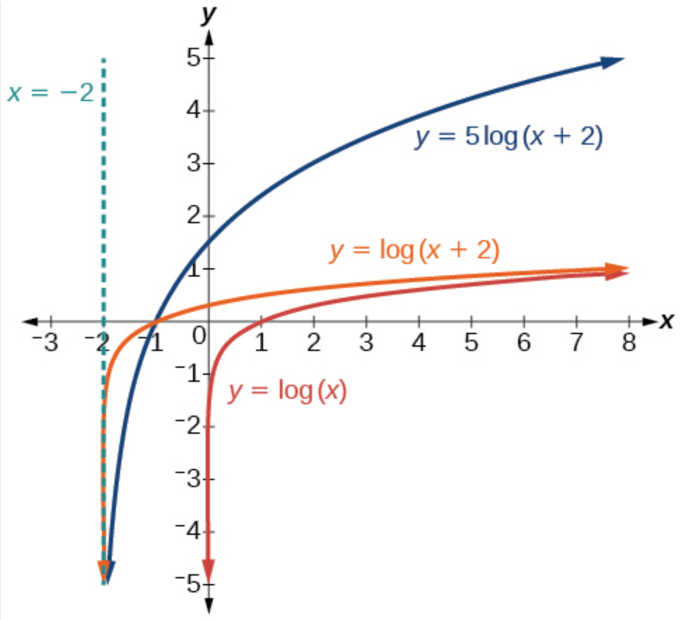
Example:
Exercise:

Problem:
Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x + 2)$. State the domain, range, and asymptote.

Solution:

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in [link]. The vertical asymptote will be shifted to $x = -2$. The *x*-intercept will be $(-1,0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5\ )$. We chose $x = 8$ as the *x*-coordinate of one point to graph because when $x = 8,\ x + 2 = 10,$ the base of the common logarithm.



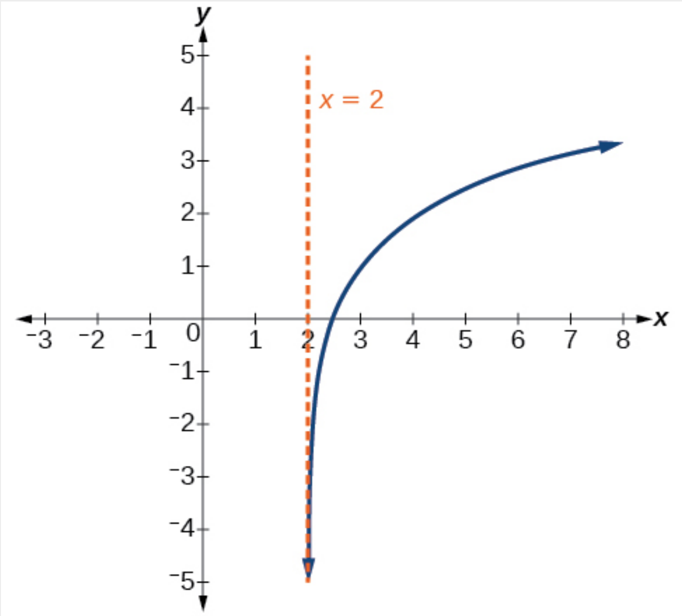
The domain is $(-2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = -2$.

Note:
Exercise:

Problem:

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

Solution:

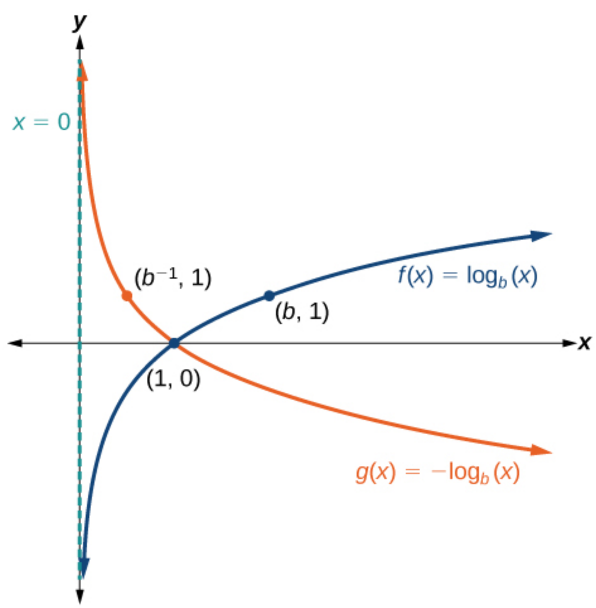


The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

Graphing Reflections of *f*(*x*) = log*b*(*x*)

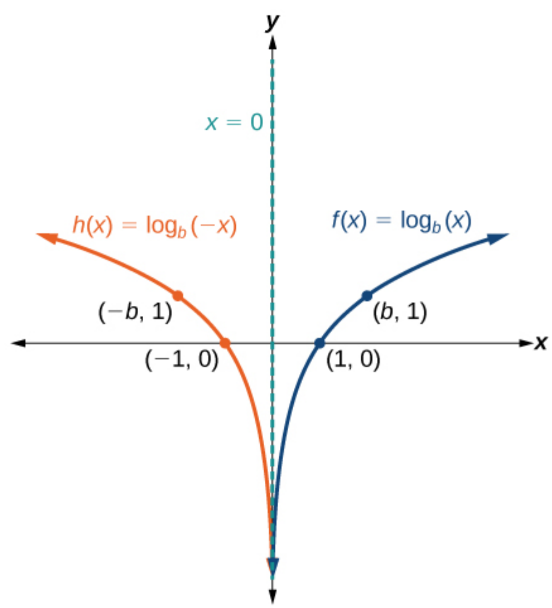
When the parent function $f(x) = \log_b(x)$ is multiplied by -1,the result is a reflection about the *x*-axis. When the *input* is multiplied by -1,the result is a reflection about the *y*-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the *x*-axis, $g(x) = -\log_b(x)$ and the reflection about the *y*-axis, $h(x) = \log_b(-x)$.

Reflection about the *x*-axis
$g(x) = \log_b(x)$, $b > 1$



- The reflected function is decreasing as *x* moves from zero to infinity.
- The asymptote remains $x = 0$.
- The *x*-intercept remains (1, 0).
- The key point changes to $(b^{-1}, 1)$
- The domain remains $(0, \infty)$.
- The range remains $(-\infty, \infty)$.

Reflection about the *y*-axis
$h(x) = \log_b(-x)$, $b > 1$



- The reflected function is decreasing as *x* moves from negative infinity to zero.
- The asymptote remains $x = 0$.
- The *x*-intercept changes to $(-1, 0)$.
- The key point changes to $(-b, 1)$
- The domain changes to $(-\infty, 0)$.
- The range remains $(-\infty, \infty)$.

Note:
Reflections of the Parent Function $y = \log_b(x)$
The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the *x*-axis.

- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

- reflects the parent function $y = \log_b(x)$ about the *y*-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

Note:
Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

If $f(x) = -\log_b(x)$	If $f(x) = \log_b(-x)$
1. Draw the vertical asymptote, $x = 0$.	1. Draw the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.	2. Plot the *x*-intercept, $(1, 0)$.
3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *x*-axis.	3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *y*-axis.
4. Draw a smooth curve through the points.	4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.	5. State the domain, $(-\infty, 0)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.

Example:
Exercise:

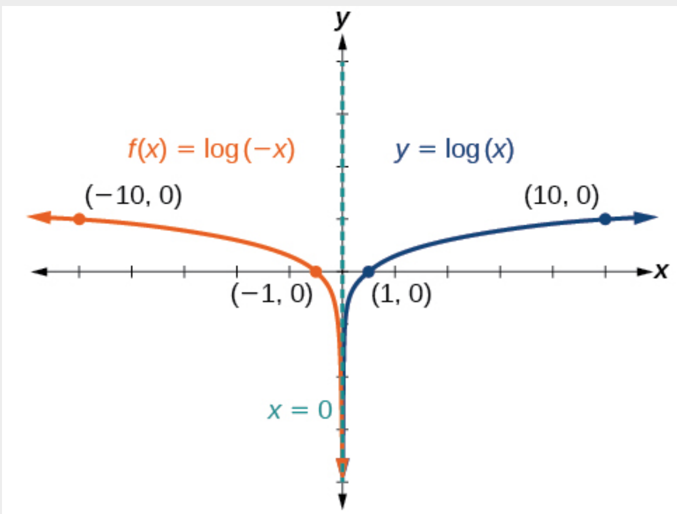
Problem:
Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution:

Before graphing $f(x) = \log(-x)$,identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by -1,f is a reflection of the parent graph about the *y*-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The *x*-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

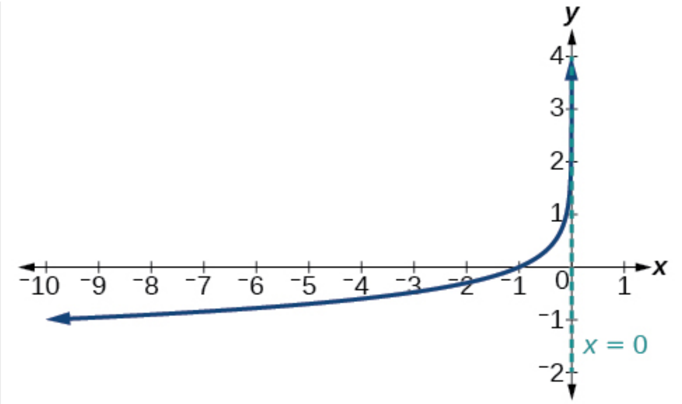


The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Note:
Exercise:

Problem: Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Solution:



The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Note:
Given a logarithmic equation, use a graphing calculator to approximate solutions.

1. Press **[Y=]**. Enter the given logarithm equation or equations as **Y_1=** and, if needed, **Y_2=**.
2. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.
3. To find the value of x, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of x,for the point(s) of intersection.

Example:
Exercise:

Problem:
Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x) + 1 = -2\ln(x-1)$ graphically. Round to the nearest thousandth.

Solution:

Press **[Y=]** and enter $4\ln(x) + 1$ next to **Y_1**=. Then enter $-2\ln(x-1)$ next to **Y_2=**. For a window, use the values 0 to 5 for x and –10 to 10 for y. Press **[GRAPH]**. The graphs should intersect somewhere a little to right of $x = 1$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 1.3385297. (Your answer may be different if you use a different window or use a different value for **Guess?**) So, to the nearest thousandth, $x \approx 1.339$.

Note:
Exercise:

Problem: Solve $5\log(x+2) = 4 - \log(x)$ graphically. Round to the nearest thousandth.

Solution:

$x \approx 3.049$

Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in [link] to arrive at the general equation for translating exponential functions.

Translations of the Parent Function $y = \log_b(x)$	
Translation	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$y = \log_b(x+c) + d$
Stretch and Compress • Stretch if $\lvert a \rvert > 1$ • Compression if $\lvert a \rvert < 1$	$y = a\log_b(x)$
Reflect about the *x*-axis	$y = -\log_b(x)$
Reflect about the *y*-axis	$y = \log_b(-x)$
General equation for all translations	$y = a\log_b(x+c) + d$

Note:
Translations of Logarithmic Functions
All translations of the parent logarithmic function, $y = \log_b(x)$, have the form
Equation:

$$f(x) = a\log_b(x+c) + d$$

where the parent function, $y = \log_b(x), b > 1$,is

- shifted vertically up d units.

- shifted horizontally to the left c units.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- reflected about the *x*-axis when $a < 0$.

For $f\left(x\right) = \log\left(-x\right)$, the graph of the parent function is reflected about the *y*-axis.

Example:
Exercise:

Problem:
Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x+4) + 5$?

Solution:

The vertical asymptote is at $x = -4$.

Analysis

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to $x = -4$.

Note:
Exercise:

Problem: What is the vertical asymptote of $f(x) = 3 + \ln(x-1)$?

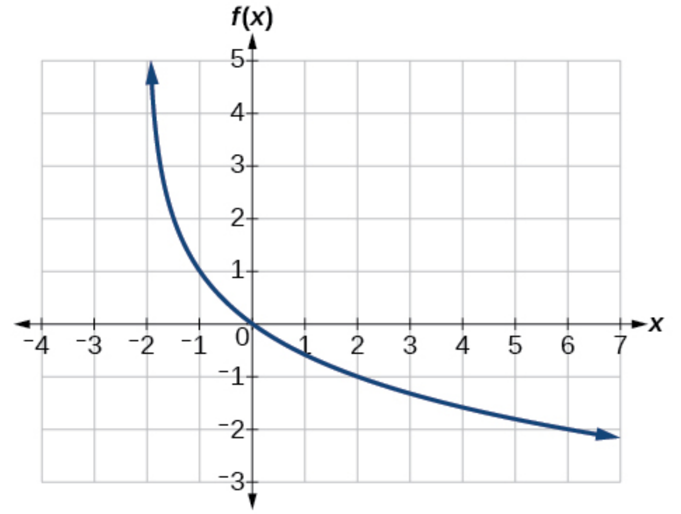
Solution:

$x = 1$

Example:
Exercise:

Problem:
Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in [link].



Solution:

This graph has a vertical asymptote at $x = –2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:
Equation:

$$f(x) = -a\log(x+2) + k$$

It appears the graph passes through the points $(–1, 1)$ and $(2, –1)$. Substituting $(–1, 1)$,
Equation:

$$\begin{array}{ll} 1 = -a\log(-1+2) + k & \text{Substitute } (-1, 1). \\ 1 = -a\log(1) + k & \text{Arithmetic.} \\ 1 = k & \log(1) = 0. \end{array}$$

Next, substituting in $(2, –1)$,
Equation:

$$\begin{array}{ll} -1 = -a\log(2+2) + 1 & \text{Plug in } (2, -1). \\ -2 = -a\log(4) & \text{Arithmetic.} \\ a = \frac{2}{\log(4)} & \text{Solve for } a. \end{array}$$

This gives us the equation $f(x) = -\frac{2}{\log(4)}\log(x+2) + 1$.

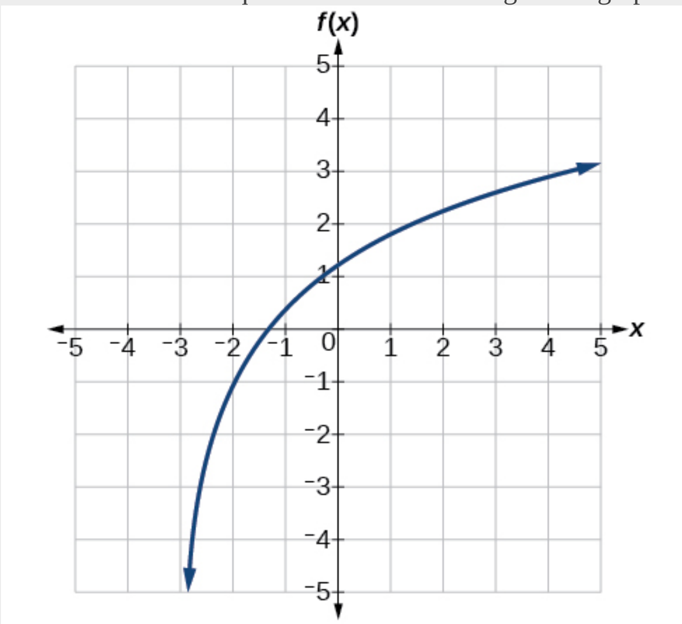
Analysis

We can verify this answer by comparing the function values in [link] with the points on the graph in [link].

x	−1	0	1	2	3
$f(x)$	1	0	−0.58496	−1	−1.3219
x	4	5	6	7	8
$f(x)$	−1.5850	−1.8074	−2	−2.1699	−2.3219

Note:
Exercise:

Problem: Give the equation of the natural logarithm graphed in [link].



Solution:

$f(x) = 2\ln(x+3) - 1$

Note:
Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?
Yes, if we know the function is a general logarithmic function. For example, look at the graph in [link]. The graph approaches $x = -3$ *(or thereabouts) more and more closely, so* $x = -3$ *is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right,* $\{x \mid x > -3\}$. *The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as* $x \to -3^+, f(x) \to -\infty$ *and as* $x \to \infty, f(x) \to \infty$.

Note:
Access these online resources for additional instruction and practice with graphing logarithms.

- [Graph an Exponential Function and Logarithmic Function]
- [Match Graphs with Exponential and Logarithmic Functions]
- [Find the Domain of Logarithmic Functions]

Key Equations

General Form for the Translation of the Parent Logarithmic Function $f(x)=\log_b(x)$	$f(x)=a\log_b(x+c)+d$

Key Concepts

- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for x. See [link] and [link]
- The graph of the parent function $f(x)=\log_b(x)$ has an *x*-intercept at $(1,0)$,domain $(0,\infty)$,range $(-\infty,\infty)$,vertical asymptote $x=0$,and
  - if $b>1$,the function is increasing.
  - if $0<b<1$, the function is decreasing.

  See [link].
- The equation $f(x)=\log_b(x+c)$ shifts the parent function $y=\log_b(x)$ horizontally
  - left c units if $c>0$.
  - right c units if $c<0$.

  See [link].
- The equation $f(x)=\log_b(x)+d$ shifts the parent function $y=\log_b(x)$ vertically
  - up d units if $d>0$.
  - down d units if $d<0$.

  See [link].
- For any constant $a>0$, the equation $f(x)=a\log_b(x)$
  - stretches the parent function $y=\log_b(x)$ vertically by a factor of a if $|a|>1$.
  - compresses the parent function $y=\log_b(x)$ vertically by a factor of a if $|a|<1$.

  See [link] and [link].
- When the parent function $y=\log_b(x)$ is multiplied by -1, the result is a reflection about the *x*-axis. When the input is multiplied by -1, the result is a reflection about the *y*-axis.
  - The equation $f(x)=-\log_b(x)$ represents a reflection of the parent function about the *x*-axis.

- The equation $f(x) = \log_b(-x)$ represents a reflection of the parent function about the y-axis.

See [link].

- A graphing calculator may be used to approximate solutions to some logarithmic equations See [link].

- All translations of the logarithmic function can be summarized by the general equation $f(x) = a\log_b(x + c) + d$. See [link].
- Given an equation with the general form $f(x) = a\log_b(x + c) + d$,we can identify the vertical asymptote $x = -c$ for the transformation. See [link].
- Using the general equation $f(x) = a\log_b(x + c) + d$,we can write the equation of a logarithmic function given its graph. See [link].

Section Exercises

Verbal

Exercise:

Problem:

The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

Solution:

Since the functions are inverses, their graphs are mirror images about the line $y = x$. So for every point (a, b) on the graph of a logarithmic function, there is a corresponding point (b, a) on the graph of its inverse exponential function.

Exercise:

Problem: What type(s) of translation(s), if any, affect the range of a logarithmic function?

Exercise:

Problem: What type(s) of translation(s), if any, affect the domain of a logarithmic function?

Solution:

Shifting the function right or left and reflecting the function about the y-axis will affect its domain.

Exercise:

Problem: Consider the general logarithmic function $f(x) = \log_b(x)$. Why can’t x be zero?

Exercise:

Problem: Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

Solution:

No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

Algebraic

For the following exercises, state the domain and range of the function.
Exercise:

Problem: $f(x) = \log_3(x+4)$

Exercise:

Problem: $h(x) = \ln\left(\frac{1}{2} - x\right)$

Solution:

Domain: $\left(-\infty, \frac{1}{2}\right)$; Range: $(-\infty, \infty)$

Exercise:

Problem: $g(x) = \log_5(2x+9) - 2$

Exercise:

Problem: $h(x) = \ln(4x+17) - 5$

Solution:

Domain: $\left(-\frac{17}{4}, \infty\right)$; Range: $(-\infty, \infty)$

Exercise:

Problem: $f(x) = \log_2(12 - 3x) - 3$

For the following exercises, state the domain and the vertical asymptote of the function.
Exercise:

Problem: $f(x) = \log_b(x-5)$

Solution:

Domain: $(5, \infty)$; Vertical asymptote: $x = 5$

Exercise:

Problem: $g(x) = \ln(3 - x)$

Exercise:

Problem: $f(x) = \log(3x+1)$

Solution:

Domain: $\left(-\frac{1}{3}, \infty\right);$ Vertical asymptote: $x = -\frac{1}{3}$

Exercise:

Problem: $f(x) = 3\log(-x) + 2$

Exercise:

Problem: $g(x) = -\ln(3x+9) - 7$

Solution:

Domain: $(-3, \infty);$ Vertical asymptote: $x = -3$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

Exercise:

Problem: $f(x) = \ln(2-x)$

Exercise:

Problem: $f(x) = \log\left(x - \frac{3}{7}\right)$

Solution:

Domain: $\left(\frac{3}{7}, \infty\right);$
Vertical asymptote: $x = \frac{3}{7}$; End behavior: as $x \to \left(\frac{3}{7}\right)^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$

Exercise:

Problem: $h(x) = -\log(3x-4) + 3$

Exercise:

Problem: $g(x) = \ln(2x+6) - 5$

Solution:

Domain: $(-3, \infty)$; Vertical asymptote: $x = -3$;
End behavior: as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$

Exercise:

Problem: $f(x) = \log_3(15-5x) + 6$

For the following exercises, state the domain, range, and *x*- and *y*-intercepts, if they exist. If they do not exist, write DNE.

Exercise:

Problem: $h(x) = \log_4(x-1) + 1$

Solution:

Domain: $(1, \infty)$; Range: $(-\infty, \infty)$; Vertical asymptote: $x = 1$; x-intercept: $\left(\frac{5}{4}, 0\right)$; y-intercept: DNE

Exercise:

Problem: $f(x) = \log(5x + 10) + 3$

Exercise:

Problem: $g(x) = \ln(-x) - 2$

Solution:

Domain: $(-\infty, 0)$; Range: $(-\infty, \infty)$; Vertical asymptote: $x = 0$; x-intercept: $\left(-e^2, 0\right)$; y-intercept: DNE

Exercise:

Problem: $f(x) = \log_2(x + 2) - 5$

Exercise:

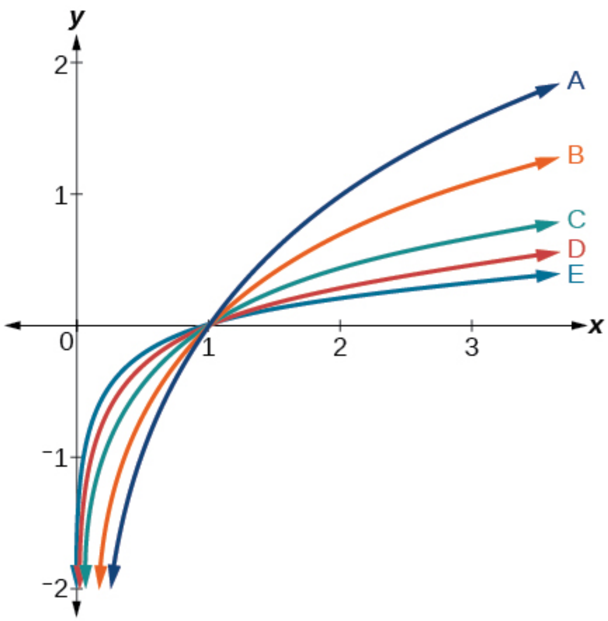
Problem: $h(x) = 3\ln(x) - 9$

Solution:

Domain: $(0, \infty)$; Range: $(-\infty, \infty)$; Vertical asymptote: $x = 0$; x-intercept: $\left(e^3, 0\right)$; y-intercept: DNE

Graphical

For the following exercises, match each function in [link] with the letter corresponding to its graph.



Exercise:

Problem: $d(x) = \log(x)$

Exercise:

Problem: $f(x) = \ln(x)$

Solution:

B

Exercise:

Problem: $g(x) = \log_2(x)$

Exercise:

Problem: $h(x) = \log_5(x)$

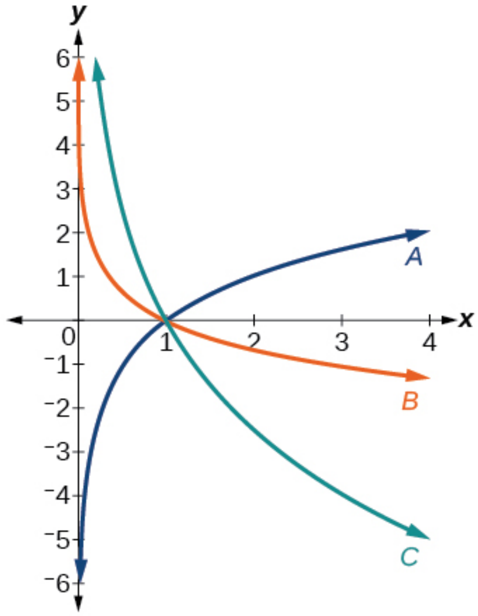
Solution:

C

Exercise:

Problem: $j(x) = \log_{25}(x)$

For the following exercises, match each function in [link] with the letter corresponding to its graph.



Exercise:

Problem: $f(x) = \log_{\frac{1}{3}}(x)$

Solution:

B

Exercise:

Problem: $g(x) = \log_2(x)$

Exercise:

Problem: $h(x) = \log_{\frac{3}{4}}(x)$

Solution:

C

For the following exercises, sketch the graphs of each pair of functions on the same axis.

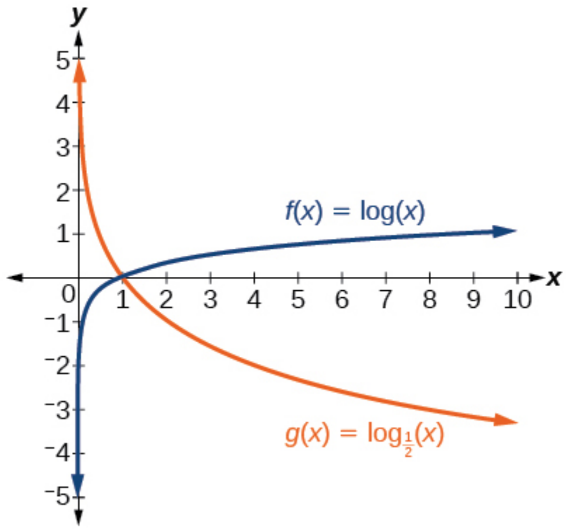
Exercise:

Problem: $f(x) = \log(x)$ and $g(x) = 10^x$

Exercise:

Problem: $f(x) = \log(x)$ and $g(x) = \log_{\frac{1}{2}}(x)$

Solution:



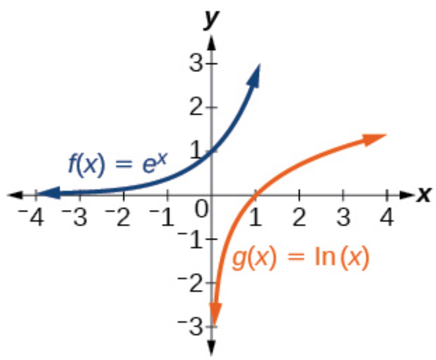
Exercise:

Problem: $f(x) = \log_4(x)$ and $g(x) = \ln(x)$

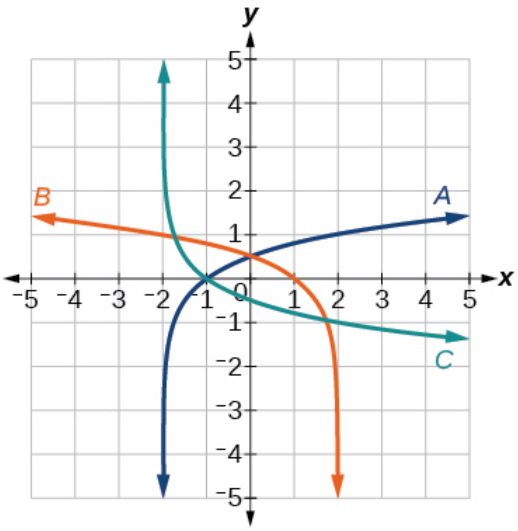
Exercise:

Problem: $f(x) = e^x$ and $g(x) = \ln(x)$

Solution:



For the following exercises, match each function in [link] with the letter corresponding to its graph.



Exercise:

Problem: $f(x) = \log_4(-x+2)$

Exercise:

Problem: $g(x) = -\log_4(x+2)$

Solution:

C

Exercise:

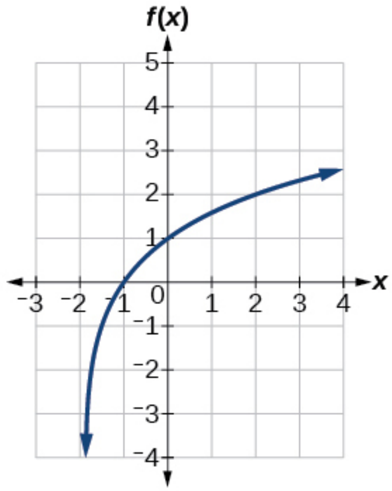
Problem: $h(x) = \log_4(x+2)$

For the following exercises, sketch the graph of the indicated function.

Exercise:

Problem: $f(x) = \log_2(x+2)$

Solution:



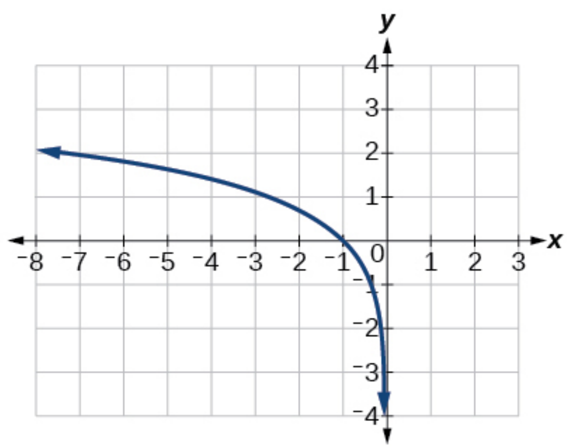
Exercise:

Problem: $f(x)=2\log(x)$

Exercise:

Problem: $f(x)=\ln(-x)$

Solution:



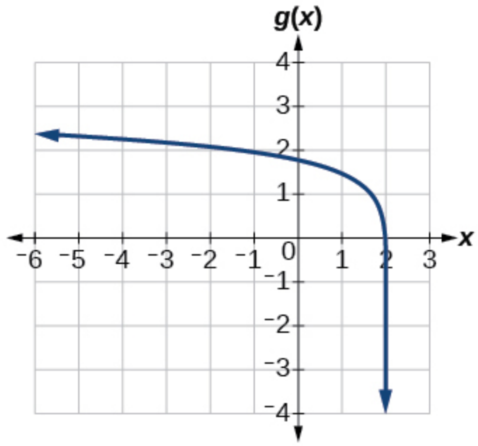
Exercise:

Problem: $g(x)=\log\left(4x+16\right)+4$

Exercise:

Problem: $g(x)=\log\left(6-3x\right)+1$

Solution:



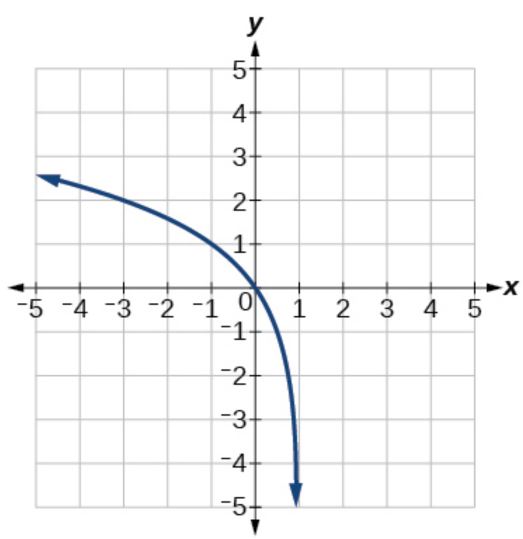
Exercise:

Problem: $h(x) = -\frac{1}{2}\ln(x+1) - 3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

Exercise:

Problem: Use $y = \log_2(x)$ as the parent function.

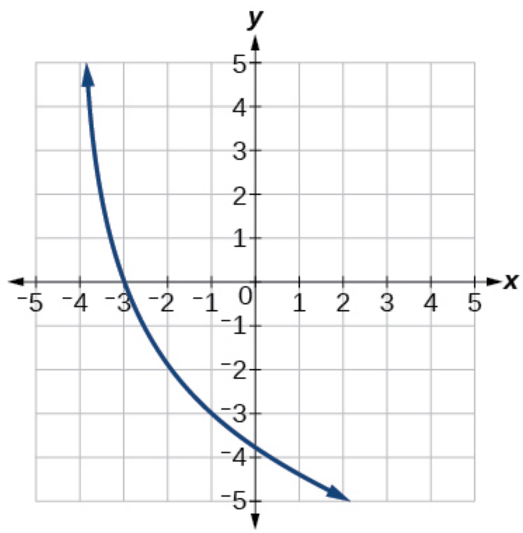


Solution:

$f(x) = \log_2(-(x-1))$

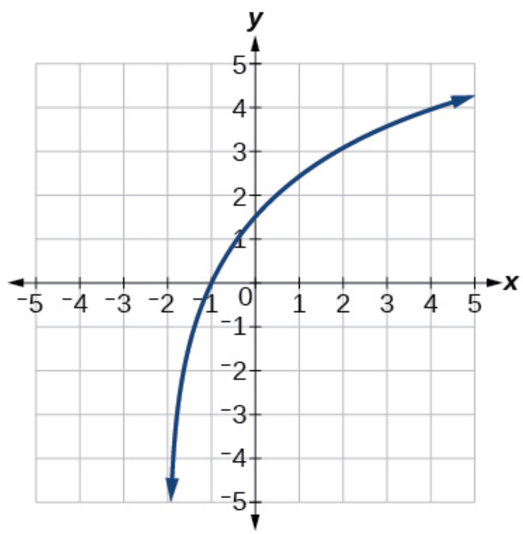
Exercise:

Problem: Use $f(x) = \log_3(x)$ as the parent function.



Exercise:

Problem: Use $f(x) = \log_4(x)$ as the parent function.

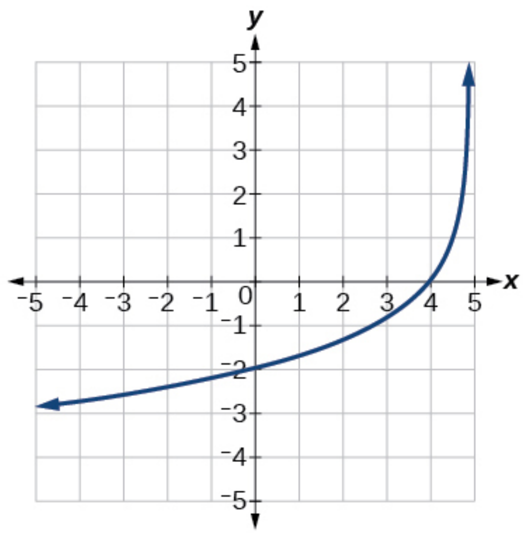


Solution:

$f(x) = 3\log_4(x+2)$

Exercise:

Problem: Use $f(x) = \log_5(x)$ as the parent function.



Technology

For the following exercises, use a graphing calculator to find approximate solutions to each equation.

Exercise:

Problem: $\log(x-1)+2=\ln(x-1)+2$

Solution:

$x=2$

Exercise:

Problem: $\log(2x-3)+2=-\log(2x-3)+5$

Exercise:

Problem: $\ln(x-2)=-\ln(x+1)$

Solution:

$x\approx 2.303$

Exercise:

Problem: $2\ln(5x+1)=\frac{1}{2}\ln(-5x)+1$

Exercise:

Problem: $\frac{1}{3}\log(1-x)=\log(x+1)+\frac{1}{3}$

Solution:

$x \approx -0.472$

Extensions

Exercise:

Problem:

Let b be any positive real number such that $b \neq 1$. What must $\log_b 1$ be equal to? Verify the result.

Exercise:

Problem:

Explore and discuss the graphs of $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = -\log_2(x)$. Make a conjecture based on the result.

Solution:

The graphs of $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = -\log_2(x)$ appear to be the same; Conjecture: for any positive base $b \neq 1$, $\log_b(x) = -\log_{\frac{1}{b}}(x)$.

Exercise:

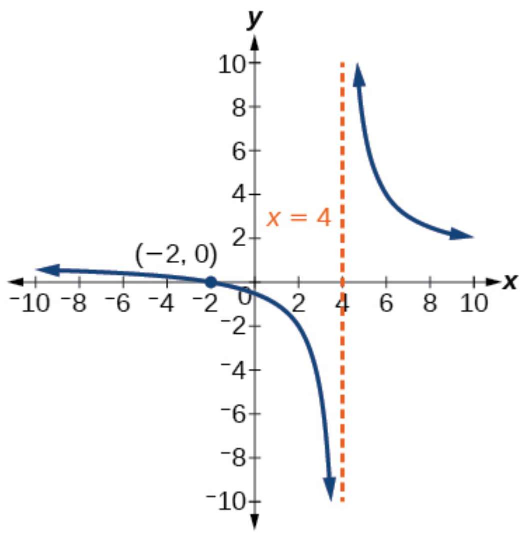
Problem: Prove the conjecture made in the previous exercise.

Exercise:

Problem: What is the domain of the function $f(x) = \ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.

Solution:

Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4} > 0$. From the graph of the function $f(x) = \frac{x+2}{x-4}$, note that the graph lies above the *x*-axis on the interval $(-\infty, -2)$ and again to the right of the vertical asymptote, that is $(4, \infty)$. Therefore, the domain is $(-\infty, -2) \cup (4, \infty)$.



Exercise:

Problem:

Use properties of exponents to find the *x*-intercepts of the function $f(x) = \log\left(x^2 + 4x + 4\right)$ algebraically. Show the steps for solving, and then verify the result by graphing the function.

Systems of Linear Equations: Two Variables
In this section, you will:

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

(credit: Thomas Sørenes)

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. A **system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

Equation:

$$\begin{aligned} 2x + y = &\ 15 \\ 3x - y = &\ 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair (4, 7) is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

Equation:

$$\begin{aligned} 2(4) + (7) = 15 \quad & \text{True} \\ 3(4) - (7) = 5 \quad & \text{True} \end{aligned}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A **consistent system** of equations has at least one solution. A consistent

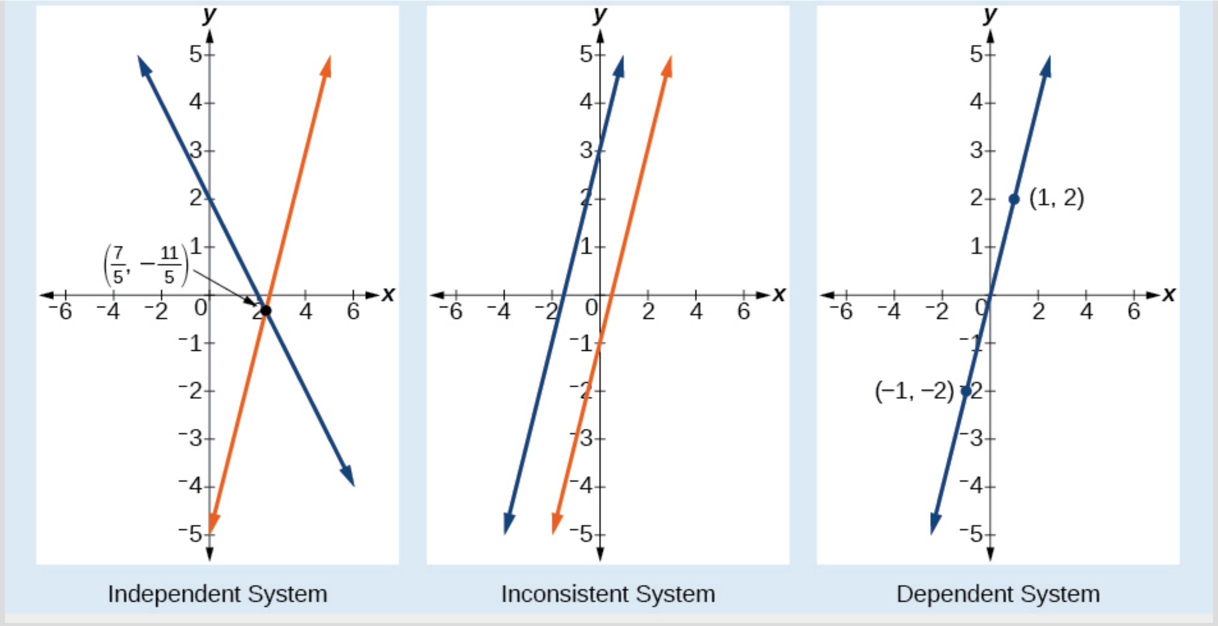
system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same *y*-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different *y*-intercepts. There are no points common to both lines; hence, there is no solution to the system.

Note:
Types of Linear Systems
There are three types of systems of linear equations in two variables, and three types of solutions.

- An **independent system** has exactly one solution pair (x, y). The point where the two lines intersect is the only solution.
- An **inconsistent system** has no solution. Notice that the two lines are parallel and will never intersect.
- A **dependent system** has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

[link] compares graphical representations of each type of system.



Note:
Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution.

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Example:
Exercise:

Problem:
Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.
Equation:

$$x + 3y = 8$$
$$2x - 9 = y$$

Solution:

Substitute the ordered pair $(5, 1)$ into both equations.

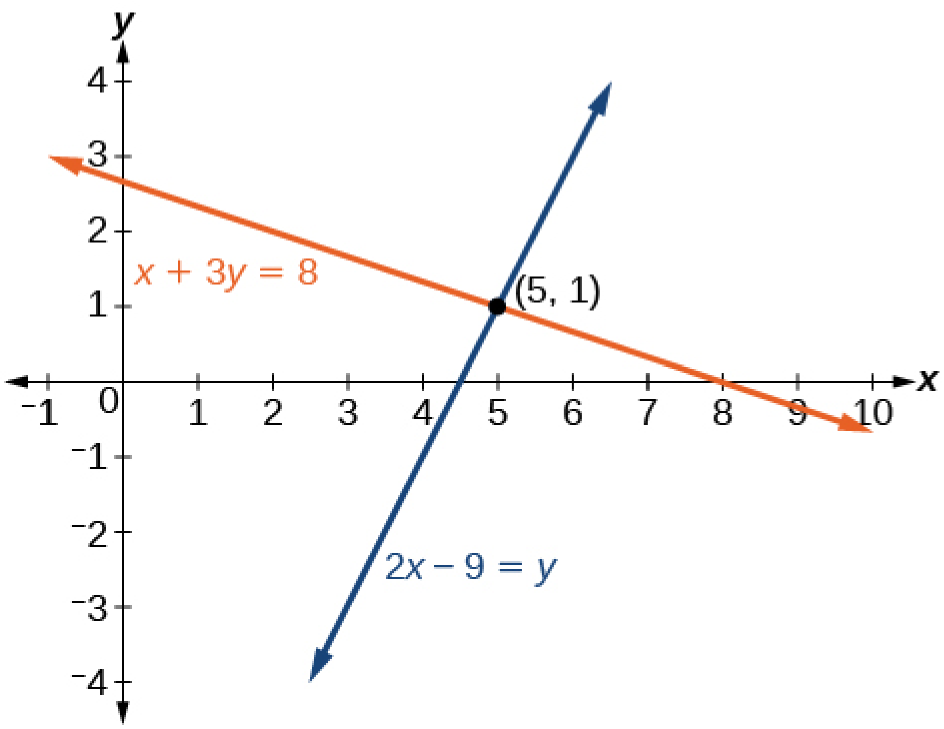
Equation:

$$\begin{aligned} (5) + 3(1) &= 8 \\ 8 &= 8 \qquad \text{True} \\ 2(5) - 9 &= (1) \\ 1 &= 1 \qquad \text{True} \end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See [link].



Note:

Exercise:

Problem:

Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

Equation:

$$\begin{array}{c}5x-4y=20\\ 2x+1=3y\end{array}$$

Solution:

Not a solution.

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

Example:
Exercise:

Problem:
Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

Equation:

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

Solution:

Solve the first equation for y.

Equation:

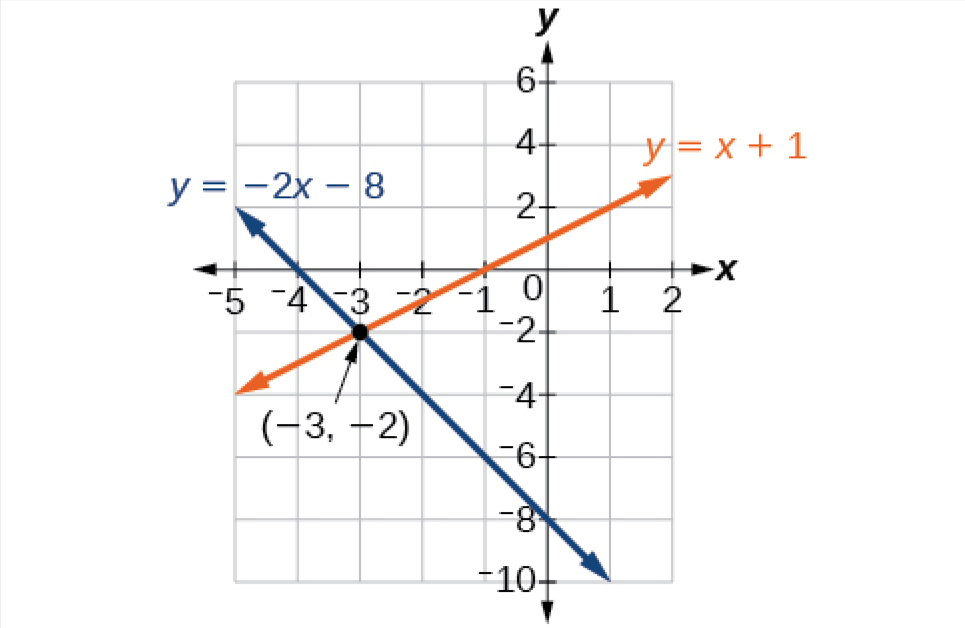
$$\begin{aligned} 2x + y &= -8 \\ y &= -2x - 8 \end{aligned}$$

Solve the second equation for y.

Equation:

$$\begin{aligned} x - y &= -1 \\ y &= x + 1 \end{aligned}$$

Graph both equations on the same set of axes as in [link].



The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

Equation:

$$\begin{array}{rl} 2(-3)+(-2)=-8 & \\ -8=-8 & \text{True} \\ (-3)-(-2)=-1 & \\ -1=-1 & \text{True} \end{array}$$

The solution to the system is the ordered pair $(-3,-2)$,so the system is independent.

Note:
Exercise:

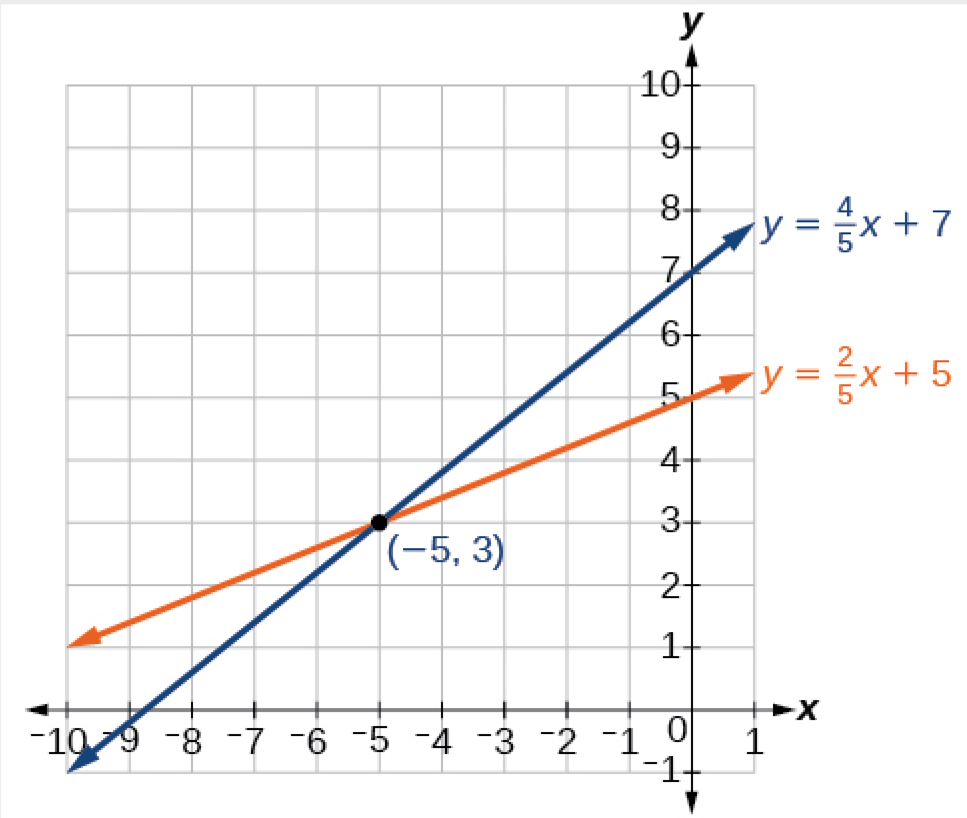
Problem: Solve the following system of equations by graphing.

Equation:

$$\begin{aligned} 2x - 5y &= -25 \\ -4x + 5y &= 35 \end{aligned}$$

Solution:

The solution to the system is the ordered pair $(-5, 3)$.



Note:

Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no

solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a system of linear equations that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

Note:
Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Example:
Exercise:

Problem:

Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

Equation:

$$\begin{aligned} -x + y &= -5 \\ 2x - 5y &= 1 \end{aligned}$$

Solution:

First, we will solve the first equation for y.

Equation:

$$\begin{aligned} -x + y &= -5 \\ y &= x - 5 \end{aligned}$$

Now we can substitute the expression $x - 5$ for y in the second equation.

Equation:

$$\begin{aligned} 2x - 5y &= 1 \\ 2x - 5(x - 5) &= 1 \\ 2x - 5x + 25 &= 1 \\ -3x &= -24 \\ x &= 8 \end{aligned}$$

Now, we substitute $x = 8$ into the first equation and solve for y.

Equation:

$$\begin{aligned} -(8) + y &= -5 \\ y &= 3 \end{aligned}$$

Our solution is $(8, 3)$.

Check the solution by substituting $(8, 3)$ into both equations.
Equation:

$$\begin{aligned} -x+y&=-5 \\ -(8)+(3)&=-5 \qquad \text{True} \\ 2x-5y&=1 \\ 2(8)-5(3)&=1 \qquad \text{True} \end{aligned}$$

Note:
Exercise:

Problem: Solve the following system of equations by substitution.
Equation:

$$\begin{aligned} x&=y+3 \\ 4&=3x-2y \end{aligned}$$

Solution:

$(-2, -5)$

Note:
Can the substitution method be used to solve any linear system in two variables?
Yes, but the method works best if one of the equations contains a coefficient of 1 or –1 so that we do not have to deal with fractions.

Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the **addition method**. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

Note:
Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Example:
Exercise:

Problem:
Solving a System by the Addition Method

Solve the given system of equations by addition.
Equation:

$$\begin{aligned} x + 2y &= -1 \\ -x + y &= 3 \end{aligned}$$

Solution:

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, –1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.
Equation:

$$\begin{array}{r} x + 2y = -1 \\ -x + y = 3 \\ \hline 3y = 2 \end{array}$$

Now that we have eliminated x, we can solve the resulting equation for y.
Equation:

$$\begin{aligned} 3y &= 2 \\ y &= \tfrac{2}{3} \end{aligned}$$

Then, we substitute this value for y into one of the original equations and solve for x.
Equation:

$$
\begin{aligned}
-x+y&=3\\
-x+\frac{2}{3}&=3\\
-x&=3-\frac{2}{3}\\
-x&=\frac{7}{3}\\
x&=-\frac{7}{3}
\end{aligned}
$$

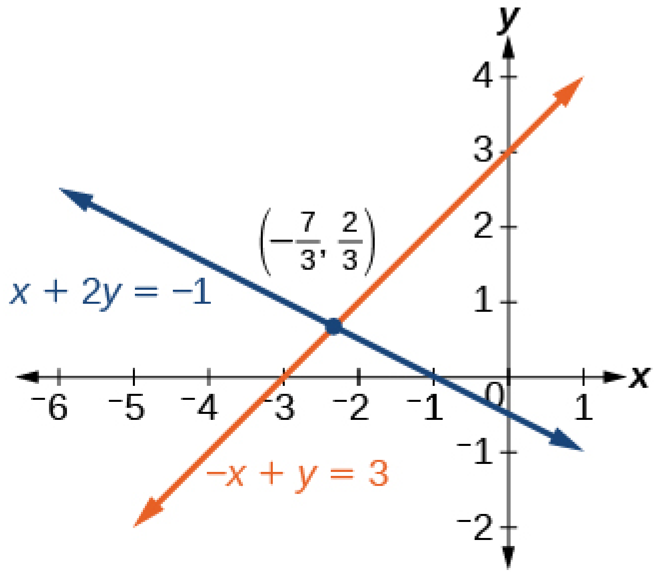
The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

Check the solution in the first equation.
Equation:

$$
\begin{aligned}
x+2y&=-1\\
\left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right)&=\\
-\frac{7}{3}+\frac{4}{3}&=\\
-\frac{3}{3}&=\\
-1&=-1 \qquad \text{True}
\end{aligned}
$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See [link] to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.



Example:
Exercise:

Problem:
Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.
Equation:

$$3x + 5y = -11$$

$$x - 2y = 11$$

Solution:

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by -3, the *x*-terms will add to zero.
Equation:

$$\begin{aligned} x - 2y &= 11 \\ -3(x - 2y) &= -3(11) && \text{Multiply both sides by } -3. \\ -3x + 6y &= -33 && \text{Use the distributive property.} \end{aligned}$$

Now, let’s add them.
Equation:

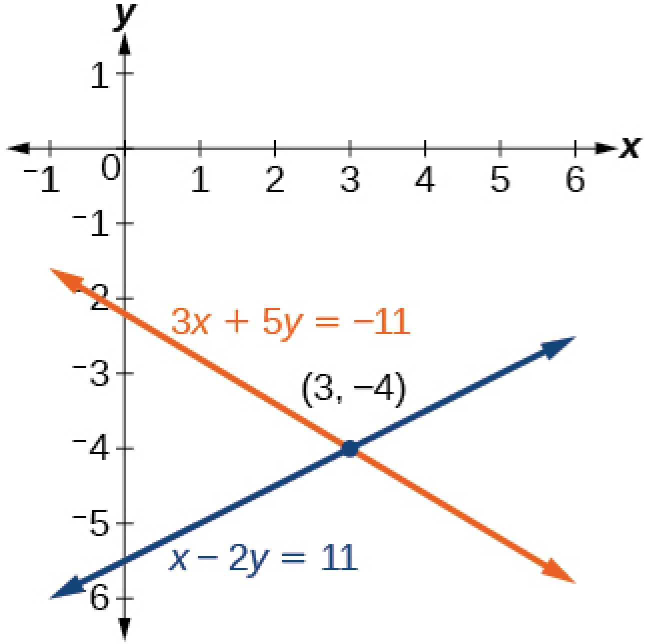
$$\begin{array}{r} 3x + 5y = -11 \\ -3x + 6y = -33 \\ \hline 11y = -44 \\ y = -4 \end{array}$$

For the last step, we substitute $y = -4$ into one of the original equations and solve for x.
Equation:

$$\begin{aligned} 3x + 5y &= -11 \\ 3x + 5(-4) &= -11 \\ 3x - 20 &= -11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Our solution is the ordered pair $(3, -4)$. See [link]. Check the solution in the original second equation.
Equation:

$$\begin{aligned} x - 2y &= 11 \\ (3) - 2(-4) &= 3 + 8 \\ 11 &= 11 && \text{True} \end{aligned}$$



Note:
Exercise:

Problem: Solve the system of equations by addition.
Equation:

$$\begin{array}{l} 2x - 7y = 2 \\ 3x + y = -20 \end{array}$$

Solution:

$(-6, -2)$

Example:
Exercise:

Problem:
Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.
Equation:

$$2x + 3y = -16$$
$$5x - 10y = 30$$

Solution:

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.
Equation:

$$-5(2x + 3y) = -5(-16)$$
$$-10x - 15y = 80$$
$$2(5x - 10y) = 2(30)$$
$$10x - 20y = 60$$

Then, we add the two equations together.
Equation:

$$\begin{array}{r} -10x - 15y = 80 \\ 10x - 20y = 60 \\ \hline -35y = 140 \\ y = -4 \end{array}$$

Substitute $y = -4$ into the original first equation.

Equation:

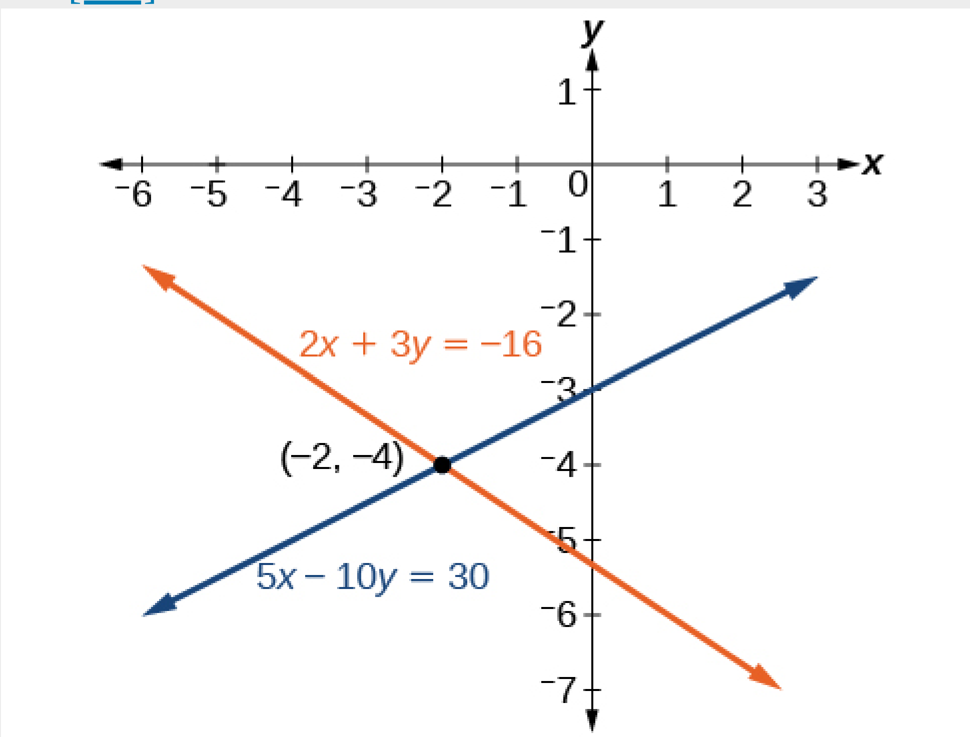
$$\begin{aligned} 2x+3(-4)&=-16 \\ 2x-12&=-16 \\ 2x&=-4 \\ x&=-2 \end{aligned}$$

The solution is $(-2,-4)$. Check it in the other equation.

Equation:

$$\begin{aligned} 5x-10y&=30 \\ 5(-2)-10(-4)&=30 \\ -10+40&=30 \\ 30&=30 \end{aligned}$$

See [link].



Example:
Exercise:

Problem:
Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.
Equation:

$$\begin{aligned}\frac{x}{3}+\frac{y}{6}&=3\\ \frac{x}{2}-\frac{y}{4}&=\ 1\end{aligned}$$

Solution:

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.
Equation:

$$\begin{aligned}6\left(\frac{x}{3}+\frac{y}{6}\right)&=6(3)\\ 2x+y&=18\\ 4\left(\frac{x}{2}-\frac{y}{4}\right)&=4(1)\\ 2x-y&=4\end{aligned}$$

Now multiply the second equation by -1 so that we can eliminate the *x*-variable.
Equation:

$$\begin{aligned}-1(2x-y)&=-1(4)\\ -2x+y&=-4\end{aligned}$$

Add the two equations to eliminate the *x*-variable and solve the resulting equation.
Equation:

$$
\begin{aligned}
2x + y &= 18 \\
\underline{-2x + y} &\underline{= -4} \\
2y &= 14 \\
y &= 7
\end{aligned}
$$

Substitute $y = 7$ into the first equation.
Equation:

$$
\begin{aligned}
2x + (7) &= 18 \\
2x &= 11 \\
x &= \frac{11}{2} \\
&= 5.5
\end{aligned}
$$

The solution is $\left(\frac{11}{2}, 7\right)$. Check it in the other equation.
Equation:

$$
\begin{aligned}
\frac{x}{2} - \frac{y}{4} &= 1 \\
\frac{\frac{11}{2}}{2} - \frac{7}{4} &= 1 \\
\frac{11}{4} - \frac{7}{4} &= 1 \\
\frac{4}{4} &= 1
\end{aligned}
$$

Note:
Exercise:

Problem: Solve the system of equations by addition.
Equation:

$$
\begin{aligned}
2x + 3y &= 8 \\
3x + 5y &= 10
\end{aligned}
$$

Solution:

$(10, -4)$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12 = 0$.

Example:
Exercise:

Problem:
Solving an Inconsistent System of Equations

Solve the following system of equations.
Equation:

$$\begin{aligned} x &= 9 - 2y \\ x + 2y &= 13 \end{aligned}$$

Solution:

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.
Equation:

$$\begin{aligned} x + 2y &= 13 \\ (9 - 2y) + 2y &= 13 \\ 9 + 0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.
Equation:

$$\begin{aligned} x &= 9 - 2y \\ 2y &= -x + 9 \\ y &= -\frac{1}{2}x + \frac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.
Equation:

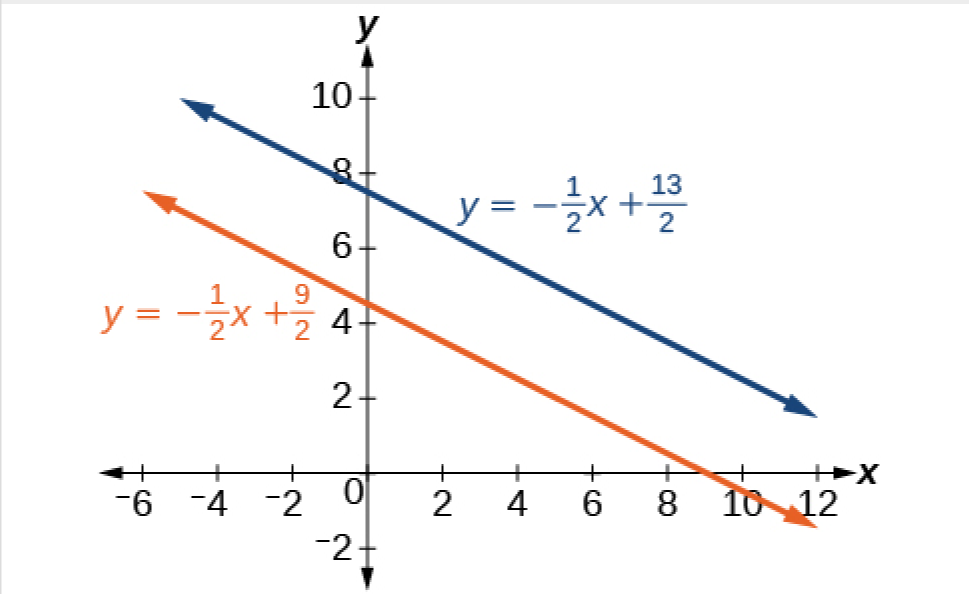
$$\begin{aligned} x + 2y &= 13 \\ 2y &= -x + 13 \\ y &= -\frac{1}{2}x + \frac{13}{2} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different *y*-intercepts. Therefore, the lines are parallel and do not intersect.
Equation:

$$y = -\frac{1}{2}x + \frac{9}{2}$$
$$y = -\frac{1}{2}x + \frac{13}{2}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in [link].



Note:
Exercise:

Problem: Solve the following system of equations in two variables.
Equation:

$$2y{-}2x = 2$$
$$2y{-}2x = 6$$

Solution:

No solution. It is an inconsistent system.

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0 = 0$.

Example:
Exercise:

Problem:
Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the addition method.
Equation:

$$\begin{aligned} x + 3y &= 2 \\ 3x + 9y &= 6 \end{aligned}$$

Solution:

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.
Equation:

$$
\begin{aligned}
x + 3y &= 2 \\
(-3)(x + 3y) &= (-3)(2) \\
-3x - 9y &= -6
\end{aligned}
$$

Now add the equations.

Equation:

$$
\begin{array}{rrl}
 & -\,3x - 9y & = -6 \\
+ & 3x + 9y & = 6 \\
\hline
 & 0 & = 0
\end{array}
$$

We can see that there will be an infinite number of solutions that satisfy both equations.

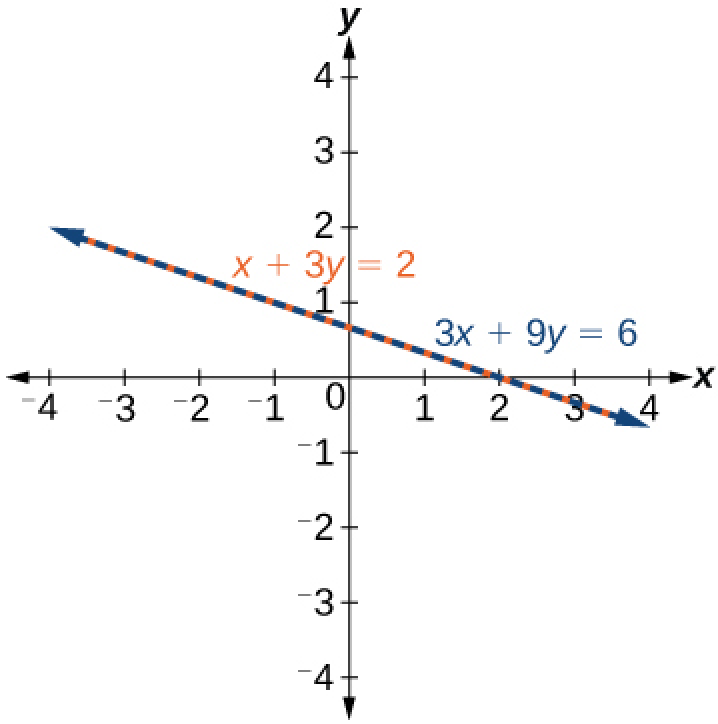
Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let’s look at what happens when we convert the system to slope-intercept form.

Equation:

$$
\begin{aligned}
x + 3y &= 2 \\
3y &= -x + 2 \\
y &= -\frac{1}{3}x + \frac{2}{3} \\
3x + 9y &= 6 \\
9y &= -3x + 6 \\
y &= -\frac{3}{9}x + \frac{6}{9} \\
y &= -\frac{1}{3}x + \frac{2}{3}
\end{aligned}
$$

See [link]. Notice the results are the same. The general solution to the system is $\left(x\ ,-\ \frac{1}{3}x + \frac{2}{3}\right)$.



Note:
Exercise:

Problem: Solve the following system of equations in two variables.
Equation:

$$y–2x = 5$$
$$-3y + 6x = -15$$

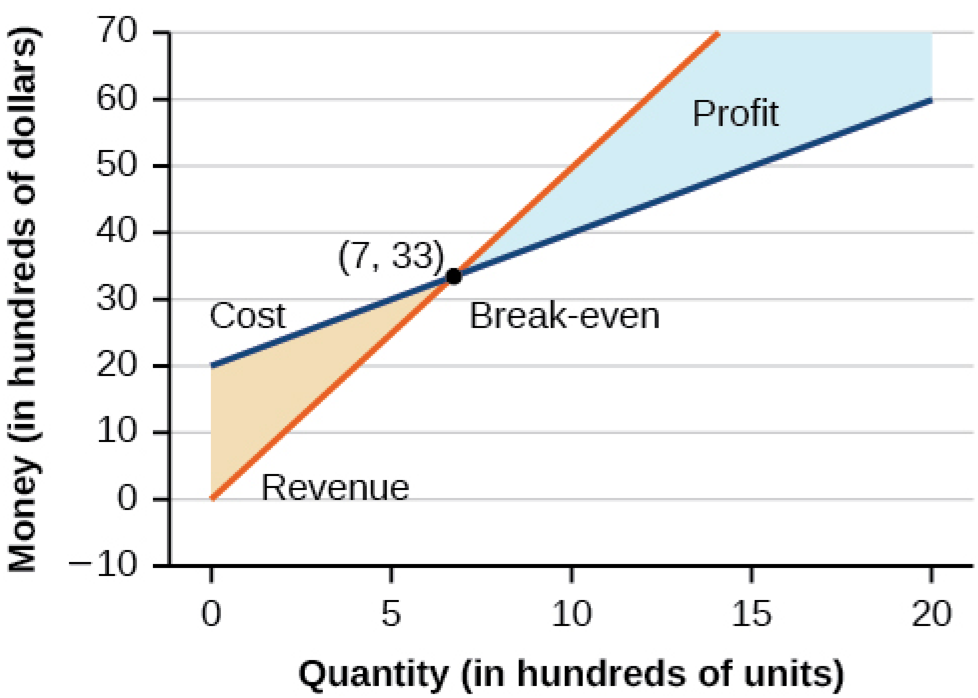
Solution:

The system is dependent so there are infinite solutions of the form $(x, 2x + 5)$.

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's **revenue function** is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where $x =$quantity and $p =$price. The revenue function is shown in orange in [link].

The **cost function** is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in [link]. The x-axis represents quantity in hundreds of units. The *y*-axis represents either cost or revenue in hundreds of dollars.



The point at which the two lines intersect is called the **break-even point**. We can see from the graph that if 700 units are produced, the cost is $3,300 and the revenue is also $3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The **profit function** is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

Example:
Exercise:

Problem:
Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35{,}000$ and the revenue function $R(x) = 1.55x$,find the break-even point and the profit function.

Solution:

Write the system of equations using y to replace function notation.
Equation:

$$\begin{aligned} y &= 0.85x + 35{,}000 \\ y &= 1.55x \end{aligned}$$

Substitute the expression $0.85x + 35{,}000$ from the first equation into the second equation and solve for x.
Equation:

$$\begin{aligned} 0.85x + 35{,}000 &= 1.55x \\ 35{,}000 &= 0.7x \\ 50{,}000 &= x \end{aligned}$$

Then, we substitute $x = 50{,}000$ into either the cost function or the revenue function.

Equation:

$$1.55\,(50{,}000) = 77{,}500$$

The break-even point is $(50{,}000, 77{,}500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

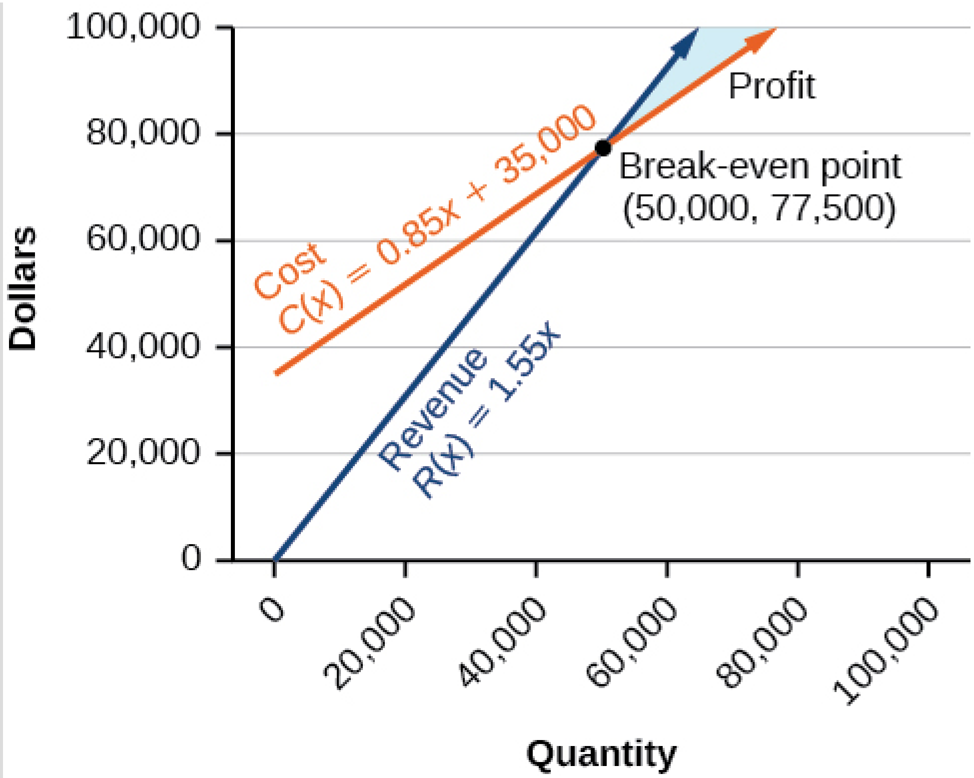
Equation:

$$\begin{aligned} P(x) &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

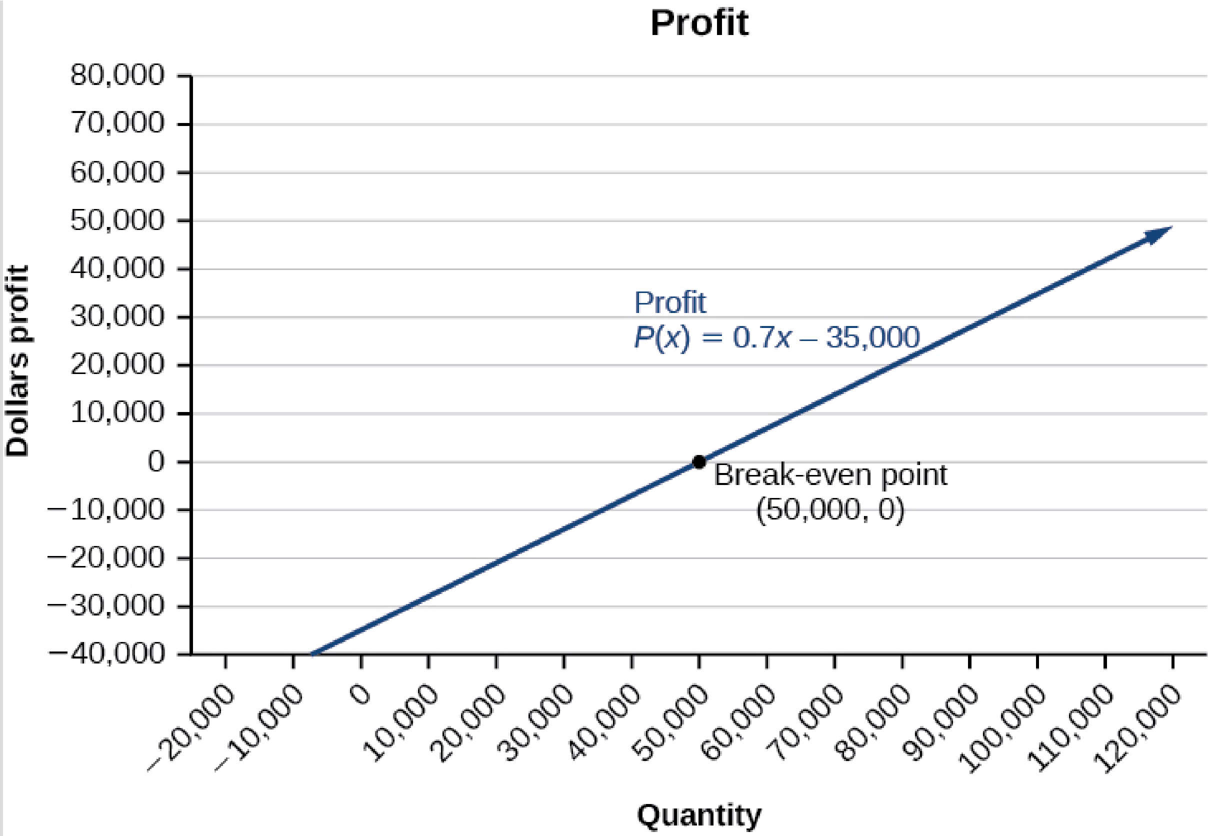
The profit function is $P(x) = 0.7x{-}35{,}000$.

Analysis

The cost to produce 50,000 units is $77,500, and the revenue from the sales of 50,000 units is also $77,500. To make a profit, the business must produce and sell more than 50,000 units. See [link].



We see from the graph in [link] that the profit function has a negative value until $x = 50{,}000$, when the graph crosses the *x*-axis. Then, the graph emerges into positive *y*-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.



Example:

Exercise:

Problem:

Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2,000$ and the total gate revenue is $\$70,000$. How many children and how many adults bought tickets?

Solution:

Let c = the number of children and a = the number of adults in attendance.

The total number of people is $2{,}000$. We can use this to write an equation for the number of people at the circus that day.
Equation:

$$c + a = 2{,}000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is$\$70{,}000$. We can use this to write an equation for the revenue.
Equation:

$$25c + 50a = 70{,}000$$

We now have a system of linear equations in two variables.
Equation:

$$\begin{aligned} c + a &= 2{,}000 \\ 25c + 50a &= 70{,}000 \end{aligned}$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.
Equation:

$$\begin{aligned} c + a &= 2{,}000 \\ a &= 2{,}000 - c \end{aligned}$$

Substitute the expression $2{,}000 - c$ in the second equation for a and solve for c.
Equation:

$$\begin{aligned} 25c + 50(2{,}000 - c) &= 70{,}000 \\ 25c + 100{,}000 - 50c &= 70{,}000 \\ -25c &= -30{,}000 \\ c &= 1{,}200 \end{aligned}$$

Substitute $c = 1{,}200$ into the first equation to solve for a.

Equation:

$$\begin{aligned} 1{,}200 + a &= 2{,}000 \\ a &= 800 \end{aligned}$$

We find that $1{,}200$ children and 800 adults bought tickets to the circus that day.

Note:

Exercise:

Problem:

Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If $1{,}650$ meal tickets were bought for a total of $\$14{,}200$, how many children and how many adults bought meal tickets?

Solution:

700 children, 950 adults

Note:

Access these online resources for additional instruction and practice with systems of linear equations.

Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See [link].
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See [link].
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See [link].
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See [link].
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See [link], [link], and [link].
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See [link].
- The solution to a system of dependent equations will always be true because both equations describe the same line. See [link].
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Can a system of linear equations have exactly two solutions? Explain why or why not.

Solution:

No, you can either have zero, one, or infinitely many. Examine graphs.

Exercise:

Problem:

If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company's profit margins.

Exercise:

Problem:

If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?

Solution:

This means there is no realistic break-even point. By the time the company produces one unit they are already making profit.

Exercise:

Problem:

If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?

Exercise:

Problem:

Given a system of equations, explain at least two different methods of solving that system.

Solution:

You can solve by substitution (isolating x or y), graphically, or by addition.

Algebraic

For the following exercises, determine whether the given ordered pair is a solution to the system of equations.

Exercise:

Problem: $\begin{aligned} 5x - y &= 4 \\ x + 6y &= 2 \end{aligned}$ and $(4, 0)$

Exercise:

Problem: $\begin{aligned} -3x - 5y &= 13 \\ -x + 4y &= 10 \end{aligned}$ and $(-6, 1)$

Solution:

Yes

Exercise:

Problem: $\begin{array}{r}3x+7y=1\\2x+4y=0\end{array}$ and $(2,3)$

Exercise:

Problem: $\begin{array}{r}-2x+5y=7\\2x+9y=7\end{array}$ and $(-1,1)$

Solution:

Yes

Exercise:

Problem: $\begin{array}{r}x+8y=43\\3x-2y=-1\end{array}$ and $(3,5)$

For the following exercises, solve each system by substitution.

Exercise:

Problem: $\begin{array}{r}x+3y=5\\2x+3y=4\end{array}$

Solution:

$(-1,2)$

Exercise:

Problem: $\begin{array}{r}3x-2y=18\\5x+10y=-10\end{array}$

Exercise:

Problem: $$\begin{aligned} 4x + 2y &= -10 \\ 3x + 9y &= 0 \end{aligned}$$

Solution:

$(-3, 1)$

Exercise:

Problem: $$\begin{aligned} 2x + 4y &= -3.8 \\ 9x - 5y &= 1.3 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} -2x + 3y &= 1.2 \\ -3x - 6y &= 1.8 \end{aligned}$$

Solution:

$\left(-\frac{3}{5}, 0\right)$

Exercise:

Problem: $$\begin{aligned} x - 0.2y &= 1 \\ -10x + 2y &= 5 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} 3x + 5y &= 9 \\ 30x + 50y &= -90 \end{aligned}$$

Solution:

No solutions exist.

Exercise:

Problem: $-3x + y = 2$
$12x - 4y = -8$

Exercise:

Problem: $\frac{1}{2}x + \frac{1}{3}y = 16$
$\frac{1}{6}x + \frac{1}{4}y = 9$

Solution:

$\left(\frac{72}{5}, \frac{132}{5}\right)$

Exercise:

Problem: $-\frac{1}{4}x + \frac{3}{2}y = 11$
$-\frac{1}{8}x + \frac{1}{3}y = 3$

For the following exercises, solve each system by addition.

Exercise:

Problem: $-2x + 5y = -42$
$7x + 2y = 30$

Solution:

$(6, -6)$

Exercise:

Problem: $$\begin{array}{l}6x-5y=-34\\ 2x+6y=4\end{array}$$

Exercise:

Problem: $$\begin{array}{r}5x-y=-2.6\\ -4x-6y=1.4\end{array}$$

Solution:

$\left(-\frac{1}{2},\frac{1}{10}\right)$

Exercise:

Problem: $$\begin{array}{l}7x-2y=3\\ 4x+5y=3.25\end{array}$$

Exercise:

Problem: $$\begin{array}{l}-\text{x}+2y=-1\\ 5x-10y=6\end{array}$$

Solution:

No solutions exist.

Exercise:

Problem: $$\begin{array}{r}7x+6y=2\\ -28x-24y=-8\end{array}$$

Exercise:

Problem: $$\begin{aligned}\frac{5}{6}x+\frac{1}{4}y&=0\\ \frac{1}{8}x-\frac{1}{2}y&=-\frac{43}{120}\end{aligned}$$

Solution:

$\left(-\frac{1}{5},\frac{2}{3}\right)$

Exercise:

Problem: $$\begin{aligned}\frac{1}{3}x+\frac{1}{9}y&=\frac{2}{9}\\ -\frac{1}{2}x+\frac{4}{5}y&=-\frac{1}{3}\end{aligned}$$

Exercise:

Problem: $$\begin{aligned}-0.2x+0.4y&=0.6\\ x-2y&=-3\end{aligned}$$

Solution:

$\left(x,\frac{x+3}{2}\right)$

Exercise:

Problem: $$\begin{aligned}-0.1x+0.2y&=0.6\\ 5x-10y&=1\end{aligned}$$

For the following exercises, solve each system by any method.

Exercise:

Problem: $$\begin{aligned}5x+9y&=16\\ x+2y&=4\end{aligned}$$

Solution:

$(-4, 4)$

Exercise:

Problem: $6x - 8y = -0.6$
$3x + 2y = 0.9$

Exercise:

Problem: $5x - 2y = 2.25$
$7x - 4y = 3$

Solution:

$\left(\frac{1}{2}, \frac{1}{8}\right)$

Exercise:

Problem: $x - \frac{5}{12}y = -\frac{55}{12}$
$-6x + \frac{5}{2}y = \frac{55}{2}$

Exercise:

Problem: $7x - 4y = \frac{7}{6}$
$2x + 4y = \frac{1}{3}$

Solution:

$\left(\frac{1}{6}, 0\right)$

Exercise:

Problem: $\begin{array}{l} 3x + 6y = 11 \\ 2x + 4y = 9 \end{array}$

Exercise:

Problem: $\begin{array}{r} \frac{7}{3}x - \frac{1}{6}y = 2 \\ -\frac{21}{6}x + \frac{3}{12}y = -3 \end{array}$

Solution:

$(x, 2(7x - 6))$

Exercise:

Problem: $\begin{array}{l} \frac{1}{2}x + \frac{1}{3}y = \frac{1}{3} \\ \frac{3}{2}x + \frac{1}{4}y = -\frac{1}{8} \end{array}$

Exercise:

Problem: $\begin{array}{l} 2.2x + 1.3y = -0.1 \\ 4.2x + 4.2y = 2.1 \end{array}$

Solution:

$\left(-\frac{5}{6}, \frac{4}{3}\right)$

Exercise:

Problem: $\begin{array}{r} 0.1x + 0.2y = 2 \\ 0.35x - 0.3y = 0 \end{array}$

Graphical

For the following exercises, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.

Exercise:

Problem:
$$\begin{aligned} 3x - y &= 0.6 \\ x - 2y &= 1.3 \end{aligned}$$

Solution:

Consistent with one solution

Exercise:

Problem:
$$\begin{aligned} -x + 2y &= 4 \\ 2x - 4y &= 1 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} x + 2y &= 7 \\ 2x + 6y &= 12 \end{aligned}$$

Solution:

Consistent with one solution

Exercise:

Problem:
$$\begin{aligned} 3x - 5y &= 7 \\ x - 2y &= 3 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 3x - 2y &= 5 \\ -9x + 6y &= -15 \end{aligned}$$

Solution:

Dependent with infinitely many solutions

Technology

For the following exercises, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.

Exercise:

Problem:
$$\begin{array}{r}0.1x+0.2y=0.3\\-0.3x+0.5y=1\end{array}$$

Exercise:

Problem:
$$\begin{array}{r}-0.01x+0.12y=0.62\\0.15x+0.20y=0.52\end{array}$$

Solution:

$(-3.08,4.91)$

Exercise:

Problem:
$$\begin{array}{r}0.5x+0.3y=4\\0.25x-0.9y=0.46\end{array}$$

Exercise:

Problem:
$$\begin{array}{r}0.15x+0.27y=0.39\\-0.34x+0.56y=1.8\end{array}$$

Solution:

$(-1.52, 2.29)$

Exercise:

Problem:
$$-0.71x + 0.92y = 0.13$$
$$0.83x + 0.05y = 2.1$$

Extensions

For the following exercises, solve each system in terms of $A, B, C, D, E,$ and F where $A – F$ are nonzero numbers. Note that $A \neq B$ and $AE \neq BD$.

Exercise:

Problem:
$$x + y = A$$
$$x - y = B$$

Solution:

$\left(\frac{A+B}{2}, \frac{A-B}{2}\right)$

Exercise:

Problem:
$$x + Ay = 1$$
$$x + By = 1$$

Exercise:

Problem:
$$Ax + y = 0$$
$$Bx + y = 1$$

Solution:

$\left(\frac{-1}{A-B}, \frac{A}{A-B}\right)$

Exercise:

Problem: $$\begin{aligned} Ax + By &= C \\ x + y &= 1 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} Ax + By &= C \\ Dx + Ey &= F \end{aligned}$$

Solution:

$\left(\frac{CE-BF}{BD-AE}, \frac{AF-CD}{BD-AE}\right)$

Real-World Applications

For the following exercises, solve for the desired quantity.

Exercise:

Problem:

A stuffed animal business has a total cost of production $C = 12x + 30$ and a revenue function $R = 20x.$ Find the break-even point.

Exercise:

Problem:

A fast-food restaurant has a cost of production $C(x) = 11x + 120$ and a revenue function $R(x) = 5x.$ When does the company start to turn a profit?

Solution:

They never turn a profit.

Exercise:

Problem:

A cell phone factory has a cost of production $C(x) = 150x + 10,000$ and a revenue function $R(x) = 200x.$ What is the break-even point?

Exercise:

Problem:

A musician charges $C(x) = 64x + 20{,}000,$ where x is the total number of attendees at the concert. The venue charges $80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

Solution:

$(1, 250, 100, 000)$

Exercise:

Problem:

A guitar factory has a cost of production $C(x) = 75x + 50{,}000.$ If the company needs to break even after 150 units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

For the following exercises, use a system of linear equations with two variables and two equations to solve.

Exercise:

Problem: Find two numbers whose sum is 28 and difference is 13.

Solution:

The numbers are 7.5 and 20.5.

Exercise:

Problem:

A number is 9 more than another number. Twice the sum of the two numbers is 10. Find the two numbers.

Exercise:

Problem:

The startup cost for a restaurant is $120,000, and each meal costs $10 for the restaurant to make. If each meal is then sold for $15, after how many meals does the restaurant break even?

Solution:

24,000

Exercise:

Problem:

A moving company charges a flat rate of $150, and an additional $5 for each box. If a taxi service would charge $20 for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?

Exercise:

Problem:

A total of 1,595 first- and second-year college students gathered at a pep rally. The number of freshmen exceeded the number of sophomores by 15. How many freshmen and sophomores were in attendance?

Solution:

790 sophomores, 805 freshman

Exercise:

Problem:

276 students enrolled in a freshman-level chemistry class. By the end of the semester, 5 times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.

Exercise:

Problem:

There were 130 faculty at a conference. If there were 18 more women than men attending, how many of each gender attended the conference?

Solution:

56 men, 74 women

Exercise:

Problem:

A jeep and BMW enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway 30 minutes before the BMW did, and traveled 7 mph slower than the BMW. After 2 hours from the time the BMW entered the highway, the cars were 306.5 miles apart. Find the speed of each car, assuming they were driven on cruise control.

Exercise:

Problem:

If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many gallons of 10% and 60% solutions were mixed?

Solution:

10 gallons of 10% solution, 15 gallons of 60% solution

Exercise:

Problem:

An investor earned triple the profits of what she earned last year. If she made $500,000.48 total for both years, how much did she earn in profits each year?

Exercise:

Problem:

An investor who dabbles in real estate invested 1.1 million dollars into two land investments. On the first investment, Swan Peak, her return was a 110% increase on the money she invested. On the second investment, Riverside Community, she earned 50% over what she invested. If she earned $1 million in profits, how much did she invest in each of the land deals?

Solution:

Swan Peak: $750,000, Riverside: $350,000

Exercise:

Problem:

If an investor invests a total of $25,000 into two bonds, one that pays 3% simple interest, and the other that pays $2\frac{7}{8}\%$ interest, and the investor earns $737.50 annual interest, how much was invested in each account?

Exercise:

Problem:

If an investor invests $23,000 into two bonds, one that pays 4% in simple interest, and the other paying 2% simple interest, and the investor earns $710.00 annual interest, how much was invested in each account?

Solution:

$12,500 in the first account, $10,500 in the second account.

Exercise:

Problem:

CDs cost $5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost if 5 CDs and 2 DVDs cost $127.73?

Exercise:

Problem:

A store clerk sold 60 pairs of sneakers. The high-tops sold for $98.99 and the low-tops sold for $129.99. If the receipts for the two types of sales totaled $6,404.40, how many of each type of sneaker were sold?

Solution:

High-tops: 45, Low-tops: 15

Exercise:

Problem:

A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was $12.50, and the price for an adult ticket was $16.00. The register confirms that $5,075 was taken in. How many student tickets and adult tickets were sold?

Exercise:

Problem:

Admission into an amusement park for 4 children and 2 adults is $116.90. For 6 children and 3 adults, the admission is $175.35. Assuming a different price for children and adults, what is the price of the child's ticket and the price of the adult ticket?

Solution:

Infinitely many solutions. We need more information.

Glossary

addition method
: an algebraic technique used to solve systems of linear equations in which the equations are added in a way that eliminates one variable, allowing the resulting equation to be solved for the remaining variable; substitution is then used to solve for the first variable

break-even point
: the point at which a cost function intersects a revenue function; where profit is zero

consistent system
: a system for which there is a single solution to all equations in the system and it is an independent system, or if there are an infinite number of solutions and it is a dependent system

cost function
: the function used to calculate the costs of doing business; it usually has two parts, fixed costs and variable costs

dependent system
: a system of linear equations in which the two equations represent the same line; there are an infinite number of solutions to a dependent system

inconsistent system
: a system of linear equations with no common solution because they represent parallel lines, which have no point or line in common

independent system
: a system of linear equations with exactly one solution pair (x, y)

profit function
: the profit function is written as $P(x) = R(x) - C(x)$,revenue minus cost

revenue function
: the function that is used to calculate revenue, simply written as $R = xp$,where $x =$ quantity and $p =$ price

substitution method
: an algebraic technique used to solve systems of linear equations in which one of the two equations is solved for one variable and then substituted into the second equation to solve for the second variable

system of linear equations
: a set of two or more equations in two or more variables that must be considered simultaneously.

Systems of Linear Equations: Three Variables
In this section, you will:

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

(credit: “Elembis,” Wikimedia Commons)

John received an inheritance of $12,000 that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $4,000 more in municipal funds than in municipal bonds. He earned $670 in interest the first year. How much did John invest in each type of fund?

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution (x, y, z), which we call an ordered triple. A system in upper triangular form looks like the following:
Equation:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

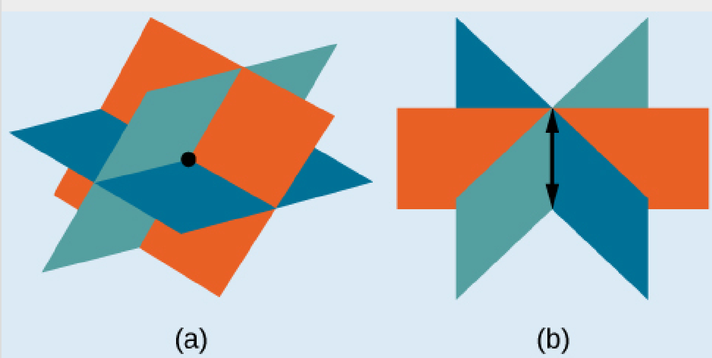
The third equation can be solved for z, and then we back-substitute to find y and x. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

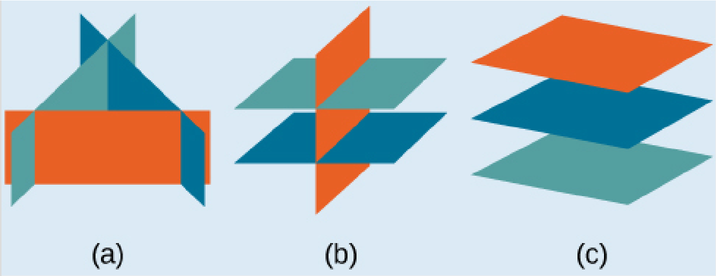
The **solution set** to a three-by-three system is an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

Note:
Number of Possible Solutions
[link] and [link] illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a **solution set** consisting of an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.



(a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.



All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

Example:
Exercise:

Problem:
Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.
Equation:

$$\begin{aligned} x + y + z &= 2 \\ 6x - 4y + 5z &= 31 \\ 5x + 2y + 2z &= 13 \end{aligned}$$

Solution:

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$$\begin{aligned} x + y + z &= 2 \\ (3) + (-2) + (1) &= 2 \\ \text{True} \end{aligned}$$

$$\begin{aligned} 6x - 4y + 5z &= 31 \\ 6(3) - 4(-2) + 5(1) &= 31 \\ 18 + 8 + 5 &= 31 \\ \text{True} \end{aligned}$$

$$\begin{aligned} 5x + 2y + 2z &= 13 \\ 5(3) + 2(-2) + 2(1) &= 13 \\ 15 - 4 + 2 &= 13 \\ \text{True} \end{aligned}$$

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

Note:
Given a linear system of three equations, solve for three unknowns.

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example:
Exercise:

Problem:
Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:
Equation:

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ -x + 3y - z &= -6 \quad (2) \\ 2x - 5y + 5z &= 17 \quad (3) \end{aligned}$$

Solution:

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).
Equation:

$$\begin{array}{rl} x - 2y + 3z = 9 & (1) \\ -x + 3y - z = -6 & (2) \\ \hline y + 2z = 3 & (3) \end{array}$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.
Equation:

$$\begin{array}{rl} -2x + 4y - 6z = -18 & (1)\ \text{multiplied by} \ -2 \\ 2x - 5y + 5z = 17 & (3) \\ \hline -y - z = -1 & (5) \end{array}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.
Equation:

$$\begin{array}{rl} y+2z=3 & (4) \\ -y-z=-1 & (5) \\ \hline z=2 & (6) \end{array}$$

Choosing one equation from each new system, we obtain the upper triangular form:
Equation:

$$\begin{aligned} x-2y+3z&=9 \quad (1) \\ y+2z&=3 \quad (4) \\ z&=2 \quad (6) \end{aligned}$$

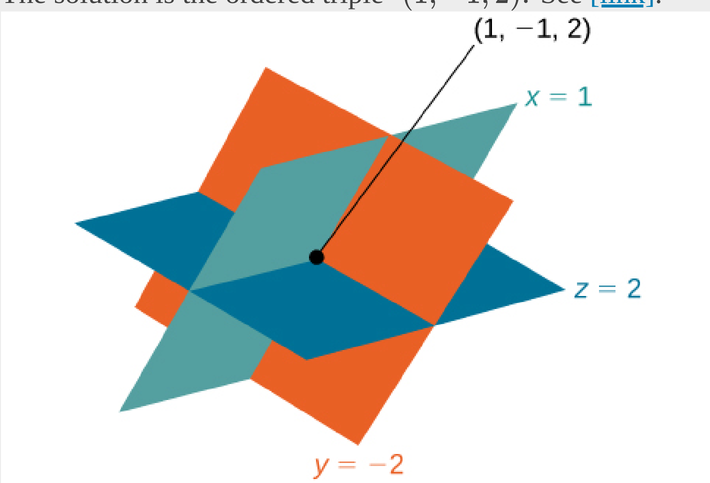
Next, we back-substitute $z=2$ into equation (4) and solve for y.
Equation:

$$\begin{aligned} y+2(2)&=3 \\ y+4&=3 \\ y&=-1 \end{aligned}$$

Finally, we can back-substitute $z=2$ and $y=-1$ into equation (1). This will yield the solution for x.
Equation:

$$\begin{aligned} x-2(-1)+3(2)&=9 \\ x+2+6&=9 \\ x&=1 \end{aligned}$$

The solution is the ordered triple $(1,-1,2)$. See [link].



Example:
Exercise:

Problem:
Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of $12,000 in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested $4,000 more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $670. How much did he invest in each type of fund?

Solution:

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:
Equation:

$$\begin{aligned} x &= \text{amount invested in money-market fund} \\ y &= \text{amount invested in municipal bonds} \\ z &= \text{amount invested in mutual funds} \end{aligned}$$

The first equation indicates that the sum of the three principal amounts is $12,000.
Equation:

$$x + y + z = 12{,}000$$

We form the second equation according to the information that John invested $4,000 more in mutual funds than he invested in municipal bonds.
Equation:

$$z = y + 4{,}000$$

The third equation shows that the total amount of interest earned from each fund equals $670.
Equation:

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.
Equation:

$$\begin{aligned} x + y + z &= 12{,}000 \\ -y + z &= 4{,}000 \\ 0.03x + 0.04y + 0.07z &= 670 \end{aligned}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,
Equation:

$$\begin{aligned} x + \quad y + z \ &= 12{,}000 \quad (1) \\ -y + z \ &= 4{,}000 \quad \ \ (2) \\ 3x + 4y + 7z &= 67{,}000 \quad (3) \end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.

Equation:

$$\begin{aligned} x + \ \ y + \ \ z &= 12{,}000 \\ 3x + 4y \ + 7z &= 67{,}000 \\ -y \ \ + \ \ z &= 4{,}000 \end{aligned}$$

Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.

Equation:

$$\begin{aligned} x + y + z \ &= 12{,}000 \\ y + 4z &= 31{,}000 \\ -y + z \ &= 4{,}000 \end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

Equation:

$$\begin{aligned} x + y + \ \ z &= 12{,}000 \\ y + 4z &= 31{,}000 \\ 5z \ &= 35{,}000 \end{aligned}$$

Step 4. Solve for z in equation (3). Back-substitute that value in equation (2) and solve for y. Then, back-substitute the values for z and y into equation (1) and solve for x.

Equation:

$$\begin{aligned} 5z &= 35{,}000 \\ z &= 7{,}000 \end{aligned}$$

$$\begin{aligned} y + 4(7{,}000) &= 31{,}000 \\ y &= 3{,}000 \end{aligned}$$

$$\begin{aligned} x + 3{,}000 + 7{,}000 &= 12{,}000 \\ x &= 2{,}000 \end{aligned}$$

John invested \$2,000 in a money-market fund, \$3,000 in municipal bonds, and \$7,000 in mutual funds.

Note:
Exercise:

Problem: Solve the system of equations in three variables.
Equation:

$$\begin{aligned}2x+y-2z&=-1\\3x-3y-z&=5\\x-2y+3z&=6\end{aligned}$$

Solution:

$(1,-1,1)$

Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3=7$ or some other contradiction.

Example:
Exercise:

Problem:
Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.
Equation:

$$\begin{aligned}x-3y+z&=4 \quad (1)\\-x+2y-5z&=3 \quad (2)\\5x-13y+13z&=8 \quad (3)\end{aligned}$$

Solution:

Looking at the coefficients of $x,$ we can see that we can eliminate x by adding equation (1) to equation (2).
Equation:

$$\begin{array}{r}x-3y+z=4 \quad (1)\\ \underline{-x+2y-5z=3 \quad (2)}\\ -y-4z=7 \quad (4)\end{array}$$

Next, we multiply equation (1) by -5 and add it to equation (3).
Equation:

$$\begin{array}{rl} -5x+15y-5z=-20 & (1)\text{ multiplied by }-5 \\ 5x-13y+13z=8 & (3) \\ \hline 2y+8z=-12 & (5) \end{array}$$

Then, we multiply equation (4) by 2 and add it to equation (5).
Equation:

$$\begin{array}{rl} -2y-8z=14 & (4)\text{ multiplied by }2 \\ 2y+8z=-12 & (5) \\ \hline 0=2 & \end{array}$$

The final equation $0=2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

Note:
Exercise:

Problem: Solve the system of three equations in three variables.
Equation:

$$\begin{aligned} x+y+z&=2 \\ y-3z&=1 \\ 2x+y+5z&=0 \end{aligned}$$

Solution:

No solution.

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be

the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

Example:
Exercise:

Problem:
Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.
Equation:

$$\begin{aligned} 2x + y - 3z &= 0 \quad (1) \\ 4x + 2y - 6z &= 0 \quad (2) \\ x - y + z &= 0 \quad (3) \end{aligned}$$

Solution:

First, we can multiply equation (1) by -2 and add it to equation (2).
Equation:

$$\begin{array}{rl} -4x - 2y + 6z = 0 & \text{equation } (1) \text{ multiplied by } -2 \\ 4x + 2y - 6z = 0 & \qquad (2) \\ \hline 0 = 0 & \end{array}$$

We do not need to proceed any further. The result we get is an identity, $0 = 0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2, and adding it to equation (1). We then perform the same steps as above and find the same result, $0 = 0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have
Equation:

$$\begin{array}{r} 2x + y - 3z = 0 \\ x - y + z = 0 \\ \hline 3x - 2z = 0 \end{array}$$

We then solve the resulting equation for z.
Equation:

$$\begin{aligned} 3x - 2z &= 0 \\ z &= \tfrac{3}{2}x \end{aligned}$$

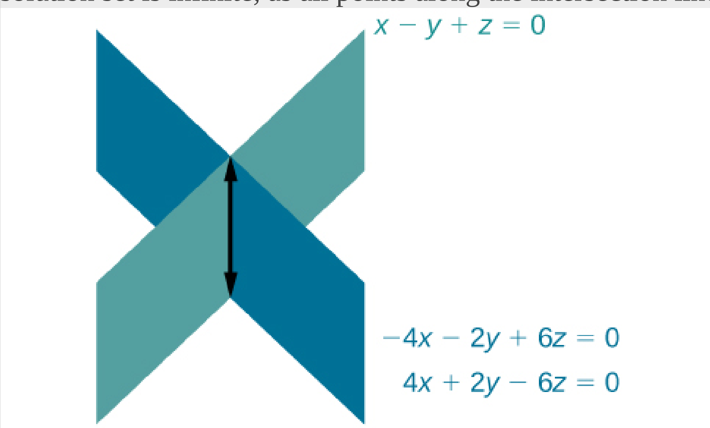
We back-substitute the expression for z into one of the equations and solve for y.
Equation:

$$\begin{aligned} 2x + y - 3\left(\frac{3}{2}x\right) &= 0 \\ 2x + y - \frac{9}{2}x &= 0 \\ y &= \frac{9}{2}x - 2x \\ y &= \frac{5}{2}x \end{aligned}$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

Analysis

As shown in [link], two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.



Note:
Does the generic solution to a dependent system always have to be written in terms of x?
No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

Note:
Exercise:

Problem: Solve the following system.
Equation:

$$\begin{aligned} x + y + z &= 7 \\ 3x - 2y - z &= 4 \\ x + 6y + 5z &= 24 \end{aligned}$$

Solution:

Infinite number of solutions of the form $(x, 4x-11, -5x+18)$.

Note:
Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination

Key Concepts

- A solution set is an ordered triple $\{(x, y, z)\}$ that represents the intersection of three planes in space. See [link].
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See [link].
- Systems of three equations in three variables are useful for solving many different types of real-world problems. See [link].
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See [link].
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See [link].
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

Section Exercises

Verbal

Exercise:

Problem:

Can a linear system of three equations have exactly two solutions? Explain why or why not

Solution:

No, there can be only one, zero, or infinitely many solutions.

Exercise:

Problem:

If a given ordered triple solves the system of equations, is that solution unique? If so, explain why. If not, give an example where it is not unique.

Exercise:

Problem:

If a given ordered triple does not solve the system of equations, is there no solution? If so, explain why. If not, give an example.

Solution:

Not necessarily. There could be zero, one, or infinitely many solutions. For example, $(0,0,0)$ is not a solution to the system below, but that does not mean that it has no solution.

$$\begin{aligned} 2x + 3y - 6z &= 1 \\ -4x - 6y + 12z &= -2 \\ x + 2y + 5z &= 10 \end{aligned}$$

Exercise:

Problem: Using the method of addition, is there only one way to solve the system?

Exercise:

Problem:

Can you explain whether there can be only one method to solve a linear system of equations? If yes, give an example of such a system of equations. If not, explain why not.

Solution:

Every system of equations can be solved graphically, by substitution, and by addition. However, systems of three equations become very complex to solve graphically so other methods are usually preferable.

Algebraic

For the following exercises, determine whether the ordered triple given is the solution to the system of equations.

Exercise:

Problem: $\begin{aligned} 2x - 6y + 6z &= -12 \\ x + 4y + 5z &= -1 \\ -x + 2y + 3z &= -1 \end{aligned}$ and $(0,1,-1)$

Exercise:

Problem: $\begin{array}{r} 6x - y + 3z = 6 \\ 3x + 5y + 2z = 0 \\ x + y = 0 \end{array}$ and $(3, -3, -5)$

Solution:

No

Exercise:

Problem: $\begin{array}{r} 6x - 7y + z = 2 \\ -x - y + 3z = 4 \\ 2x + y - z = 1 \end{array}$ and $(4, 2, -6)$

Exercise:

Problem: $\begin{array}{r} x - y = 0 \\ x - z = 5 \\ x - y + z = -1 \end{array}$ and $(4, 4, -1)$

Solution:

Yes

Exercise:

Problem: $\begin{array}{r} -x - y + 2z = 3 \\ 5x + 8y - 3z = 4 \\ -x + 3y - 5z = -5 \end{array}$ and $(4, 1, -7)$

For the following exercises, solve each system by substitution.

Exercise:

Problem: $\begin{array}{r} 3x - 4y + 2z = -15 \\ 2x + 4y + z = 16 \\ 2x + 3y + 5z = 20 \end{array}$

Solution:

$(-1, 4, 2)$

Exercise:

Problem: $\begin{array}{r} 5x - 2y + 3z = 20 \\ 2x - 4y - 3z = -9 \\ x + 6y - 8z = 21 \end{array}$

Exercise:

Problem: $$\begin{aligned} 5x+2y+4z&=9 \\ -3x+2y+z&=10 \\ 4x-3y+5z&=-3 \end{aligned}$$

Solution:

$\left(-\frac{85}{107},\frac{312}{107},\frac{191}{107}\right)$

Exercise:

Problem: $$\begin{aligned} 4x-3y+5z&=31 \\ -x+2y+4z&=20 \\ x+5y-2z&=-29 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} 5x-2y+3z&=4 \\ -4x+6y-7z&=-1 \\ 3x+2y-z&=4 \end{aligned}$$

Solution:

$\left(1,\frac{1}{2},0\right)$

Exercise:

Problem: $$\begin{aligned} 4x+6y+9z&=0 \\ -5x+2y-6z&=3 \\ 7x-4y+3z&=-3 \end{aligned}$$

For the following exercises, solve each system by Gaussian elimination.

Exercise:

Problem: $$\begin{aligned} 2x-y+3z&=17 \\ -5x+4y-2z&=-46 \\ 2y+5z&=-7 \end{aligned}$$

Solution:

$(4,-6,1)$

Exercise:

Problem: $$\begin{aligned} 5x-6y+3z&=50 \\ -x+4y&=10 \\ 2x-z&=10 \end{aligned}$$

Exercise:

Problem: $\begin{array}{c} 2x+3y-6z=1 \\ -4x-6y+12z=-2 \\ x+2y+5z=10 \end{array}$

Solution:

$\left(x, \frac{1}{27}(65-16x), \frac{x+28}{27}\right)$

Exercise:

Problem: $\begin{array}{c} 4x+6y-2z=8 \\ 6x+9y-3z=12 \\ -2x-3y+z=-4 \end{array}$

Exercise:

Problem: $\begin{array}{c} 2x+3y-4z=5 \\ -3x+2y+z=11 \\ -x+5y+3z=4 \end{array}$

Solution:

$\left(-\frac{45}{13}, \frac{17}{13}, -2\right)$

Exercise:

Problem: $\begin{array}{c} 10x+2y-14z=8 \\ -\text{x}-2y-4z=-1 \\ -12x-6y+6z=-12 \end{array}$

Exercise:

Problem: $\begin{array}{c} x+y+z=14 \\ 2y+3z=-14 \\ -16y-24z=-112 \end{array}$

Solution:

No solutions exist

Exercise:

Problem: $\begin{array}{c} 5x-3y+4z=-1 \\ -4x+2y-3z=0 \\ -\text{x}+5y+7z=-11 \end{array}$

Exercise:

Problem: $\begin{aligned} x + y + z &= 0 \\ 2x - y + 3z &= 0 \\ x - z &= 0 \end{aligned}$

Solution:

$(0, 0, 0)$

Exercise:

Problem: $\begin{aligned} 3x + 2y - 5z &= 6 \\ 5x - 4y + 3z &= -12 \\ 4x + 5y - 2z &= 15 \end{aligned}$

Exercise:

Problem: $\begin{aligned} x + y + z &= 0 \\ 2x - y + 3z &= 0 \\ x - z &= 1 \end{aligned}$

Solution:

$\left(\frac{4}{7}, -\frac{1}{7}, -\frac{3}{7}\right)$

Exercise:

Problem: $\begin{aligned} 3x - \frac{1}{2}y - z &= -\frac{1}{2} \\ 4x + z &= 3 \\ -x + \frac{3}{2}y &= \frac{5}{2} \end{aligned}$

Exercise:

Problem: $\begin{aligned} 6x - 5y + 6z &= 38 \\ \frac{1}{5}x - \frac{1}{2}y + \frac{3}{5}z &= 1 \\ -4x - \frac{3}{2}y - z &= -74 \end{aligned}$

Solution:

$(7, 20, 16)$

Exercise:

Problem: $\begin{aligned} \frac{1}{2}x - \frac{1}{5}y + \frac{2}{5}z &= -\frac{13}{10} \\ \frac{1}{4}x - \frac{2}{5}y - \frac{1}{5}z &= -\frac{7}{20} \\ -\frac{1}{2}x - \frac{3}{4}y - \frac{1}{2}z &= -\frac{5}{4} \end{aligned}$

Exercise:

Problem:
$$\begin{aligned}-\tfrac{1}{3}x-\tfrac{1}{2}y-\tfrac{1}{4}z&=\tfrac{3}{4}\\ -\tfrac{1}{2}x-\tfrac{1}{4}y-\tfrac{1}{2}z&=2\\ -\tfrac{1}{4}x-\tfrac{3}{4}y-\tfrac{1}{2}z&=-\tfrac{1}{2}\end{aligned}$$

Solution:

$(-6,2,1)$

Exercise:

Problem:
$$\begin{aligned}\tfrac{1}{2}x-\tfrac{1}{4}y+\tfrac{3}{4}z&=0\\ \tfrac{1}{4}x-\tfrac{1}{10}y+\tfrac{2}{5}z&=-2\\ \tfrac{1}{8}x+\tfrac{1}{5}y-\tfrac{1}{8}z&=2\end{aligned}$$

Exercise:

Problem:
$$\begin{aligned}\tfrac{4}{5}x-\tfrac{7}{8}y+\tfrac{1}{2}z&=1\\ -\tfrac{4}{5}x-\tfrac{3}{4}y+\tfrac{1}{3}z&=-8\\ -\tfrac{2}{5}x-\tfrac{7}{8}y+\tfrac{1}{2}z&=-5\end{aligned}$$

Solution:

$(5,12,15)$

Exercise:

Problem:
$$\begin{aligned}-\tfrac{1}{3}x-\tfrac{1}{8}y+\tfrac{1}{6}z&=-\tfrac{4}{3}\\ -\tfrac{2}{3}x-\tfrac{7}{8}y+\tfrac{1}{3}z&=-\tfrac{23}{3}\\ -\tfrac{1}{3}x-\tfrac{5}{8}y+\tfrac{5}{6}z&=0\end{aligned}$$

Exercise:

Problem:
$$\begin{aligned}-\tfrac{1}{4}x-\tfrac{5}{4}y+\tfrac{5}{2}z&=-5\\ -\tfrac{1}{2}x-\tfrac{5}{3}y+\tfrac{5}{4}z&=\tfrac{55}{12}\\ -\tfrac{1}{3}x-\tfrac{1}{3}y+\tfrac{1}{3}z&=\tfrac{5}{3}\end{aligned}$$

Solution:

$(-5,-5,-5)$

Exercise:

Problem:
$$\begin{aligned} \frac{1}{40}x + \frac{1}{60}y + \frac{1}{80}z &= \frac{1}{100} \\ -\frac{1}{2}x - \frac{1}{3}y - \frac{1}{4}z &= -\frac{1}{5} \\ \frac{3}{8}x + \frac{3}{12}y + \frac{3}{16}z &= \frac{3}{20} \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 0.1x - 0.2y + 0.3z &= 2 \\ 0.5x - 0.1y + 0.4z &= 8 \\ 0.7x - 0.2y + 0.3z &= 8 \end{aligned}$$

Solution:

$(10, 10, 10)$

Exercise:

Problem:
$$\begin{aligned} 0.2x + 0.1y - 0.3z &= 0.2 \\ 0.8x + 0.4y - 1.2z &= 0.1 \\ 1.6x + 0.8y - 2.4z &= 0.2 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 1.1x + 0.7y - 3.1z &= -1.79 \\ 2.1x + 0.5y - 1.6z &= -0.13 \\ 0.5x + 0.4y - 0.5z &= -0.07 \end{aligned}$$

Solution:

$\left(\frac{1}{2}, \frac{1}{5}, \frac{4}{5}\right)$

Exercise:

Problem:
$$\begin{aligned} 0.5x - 0.5y + 0.5z &= 10 \\ 0.2x - 0.2y + 0.2z &= 4 \\ 0.1x - 0.1y + 0.1z &= 2 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 0.1x + 0.2y + 0.3z &= 0.37 \\ 0.1x - 0.2y - 0.3z &= -0.27 \\ 0.5x - 0.1y - 0.3z &= -0.03 \end{aligned}$$

Solution:

$\left(\frac{1}{2}, \frac{2}{5}, \frac{4}{5}\right)$

Exercise:

Problem: $$\begin{aligned}0.5x-0.5y-0.3z&=0.13\\0.4x-0.1y-0.3z&=0.11\\0.2x-0.8y-0.9z&=-0.32\end{aligned}$$

Exercise:

Problem: $$\begin{aligned}0.5x+0.2y-0.3z&=1\\0.4x-0.6y+0.7z&=0.8\\0.3x-0.1y-0.9z&=0.6\end{aligned}$$

Solution:

$(2,0,0)$

Exercise:

Problem: $$\begin{aligned}0.3x+0.3y+0.5z&=0.6\\0.4x+0.4y+0.4z&=1.8\\0.4x+0.2y+0.1z&=1.6\end{aligned}$$

Exercise:

Problem: $$\begin{aligned}0.8x+0.8y+0.8z&=2.4\\0.3x-0.5y+0.2z&=0\\0.1x+0.2y+0.3z&=0.6\end{aligned}$$

Solution:

$(1,1,1)$

Extensions

For the following exercises, solve the system for x, y, and z.

Exercise:

Problem: $$\begin{aligned}x+y+z&=3\\\frac{x-1}{2}+\frac{y-3}{2}+\frac{z+1}{2}&=0\\\frac{x-2}{3}+\frac{y+4}{3}+\frac{z-3}{3}&=\frac{2}{3}\end{aligned}$$

Exercise:

Problem: $$\begin{aligned}5x-3y-\frac{z+1}{2}&=\frac{1}{2}\\6x+\frac{y-9}{2}+2z&=-3\\\frac{x+8}{2}-4y+z&=4\end{aligned}$$

Solution:

$\left(\frac{128}{557}, \frac{23}{557}, \frac{28}{557}\right)$

Exercise:

Problem: $$\begin{array}{c}\frac{x+4}{7}-\frac{y-1}{6}+\frac{z+2}{3}=1\\ \frac{x-2}{4}+\frac{y+1}{8}-\frac{z+8}{12}=0\\ \frac{x+6}{3}-\frac{y+2}{3}+\frac{z+4}{2}=3\end{array}$$

Exercise:

Problem: $$\begin{array}{c}\frac{x-3}{6}+\frac{y+2}{2}-\frac{z-3}{3}=2\\ \frac{x+2}{4}+\frac{y-5}{2}+\frac{z+4}{2}=1\\ \frac{x+6}{2}-\frac{y-3}{2}+z+1=9\end{array}$$

Solution:

$(6, -1, 0)$

Exercise:

Problem: $$\begin{array}{c}\frac{x-1}{3}+\frac{y+3}{4}+\frac{z+2}{6}=1\\ 4x+3y-2z=11\\ 0.02x+0.015y-0.01z=0.065\end{array}$$

Real-World Applications

Exercise:

Problem:

Three even numbers sum up to 108. The smaller is half the larger and the middle number is $\frac{3}{4}$ the larger. What are the three numbers?

Solution:

24, 36, 48

Exercise:

Problem:

Three numbers sum up to 147. The smallest number is half the middle number, which is half the largest number. What are the three numbers?

Exercise:

Problem:

At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?

Solution:

70 grandparents, 140 parents, 190 children

Exercise:

Problem:

An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?

Exercise:

Problem:

Your roommate, Sarah, offered to buy groceries for you and your other roommate. The total bill was $82. She forgot to save the individual receipts but remembered that your groceries were $0.05 cheaper than half of her groceries, and that your other roommate's groceries were $2.10 more than your groceries. How much was each of your share of the groceries?

Solution:

Your share was $19.95, Sarah's share was $40, and your other roommate's share was $22.05.

Exercise:

Problem:

Your roommate, John, offered to buy household supplies for you and your other roommate. You live near the border of three states, each of which has a different sales tax. The total amount of money spent was $100.75. Your supplies were bought with 5% tax, John's with 8% tax, and your third roommate's with 9% sales tax. The total amount of money spent without taxes is $93.50. If your supplies before tax were $1 more than half of what your third roommate's supplies were before tax, how much did each of you spend? Give your answer both with and without taxes.

Exercise:

Problem:

Three coworkers work for the same employer. Their jobs are warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $82,000. The office manager makes $4,000 more than the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $78,000. What is the annual salary of each of the co-workers?

Solution:

There are infinitely many solutions; we need more information

Exercise:

Problem:

At a carnival, $2,914.25 in receipts were taken at the end of the day. The cost of a child's ticket was $20.50, an adult ticket was $29.75, and a senior citizen ticket was $15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adult, and senior citizen tickets were sold?

Exercise:

Problem:

A local band sells out for their concert. They sell all 1,175 tickets for a total purse of $28,112.50. The tickets were priced at $20 for student tickets, $22.50 for children, and $29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type was sold?

Solution:

500 students, 225 children, and 450 adults

Exercise:

Problem:

In a bag, a child has 325 coins worth $19.50. There were three types of coins: pennies, nickels, and dimes. If the bag contained the same number of nickels as dimes, how many of each type of coin was in the bag?

Exercise:

Problem:

Last year, at Haven's Pond Car Dealership, for a particular model of BMW, Jeep, and Toyota, one could purchase all three cars for a total of $140,000. This year, due to inflation, the same cars would cost $151,830. The cost of the BMW increased by 8%, the Jeep by 5%, and the Toyota by 12%. If the price of last year's Jeep was $7,000 less than the price of last year's BMW, what was the price of each of the three cars last year?

Solution:

The BMW was $49,636, the Jeep was $42,636, and the Toyota was $47,727.

Exercise:

Problem:

A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested $80,500 into three accounts, one that paid 4% simple interest, one that paid $3\frac{1}{8}\%$ simple interest, and one that paid $2\frac{1}{2}\%$ simple interest. He earned $2,670 interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?

Exercise:

Problem:

You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $34,000 in interest. If you invest four times the money into the account that pays 3% compared to 2%, how much did you invest in each account?

Solution:

$400,000 in the account that pays 3% interest, $500,000 in the account that pays 4% interest, and $100,000 in the account that pays 2% interest.

Exercise:

Problem:

You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $3,650 in interest. If you invest five times the money in the account that pays 4% compared to 3%, how much did you invest in each account?

Exercise:

Problem:

The top three countries in oil consumption in a certain year are as follows: the United States, Japan, and China. In millions of barrels per day, the three top countries consumed 39.8% of the world's consumed oil. The United States consumed 0.7% more than four times China's consumption. The United States consumed 5% more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan, and China consume?[footnote]
"Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

Solution:

The United States consumed 26.3%, Japan 7.1%, and China 6.4% of the world's oil.

Exercise:

Problem:

The top three countries in oil production in the same year are Saudi Arabia, the United States, and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world's produced oil. Saudi Arabia and the United States combined for 22.1% of the world's production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?[footnote]
"Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

Exercise:

Problem:

The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. The three top countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from Canada. What percent of the United States oil imports were from these three countries?[footnote]
"Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

Solution:

Saudi Arabia imported 16.8%, Canada imported 15.1%, and Mexico 15.0%

Exercise:

Problem:

The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?[footnote]
"USA: The coming global oil crisis," accessed April 6, 2014, http://www.oilcrisis.com/us/.

Exercise:

Problem:

At one time, in the United States, 398 species of animals were on the endangered species list. The top groups were mammals, birds, and fish, which comprised 55% of the endangered species. Birds accounted for 0.7% more than fish, and fish accounted for 1.5% more than mammals. What percent of the endangered species came from mammals, birds, and fish?

Solution:

Birds were 19.3%, fish were 18.6%, and mammals were 17.1% of endangered species

Exercise:

Problem:

Meat consumption in the United States can be broken into three categories: red meat, poultry, and fish. If fish makes up 4% less than one-quarter of poultry consumption, and red meat consumption is 18.2% higher than poultry consumption, what are the percentages of meat consumption?[footnote]
"The United States Meat Industry at a Glance," accessed April 6, 2014, http://www.meatami.com/ht/d/sp/i/47465/pid/47465.

Glossary

solution set
: the set of all ordered pairs or triples that satisfy all equations in a system of equations

Matrices and Matrix Operations
In this section, you will:

- Find the sum and difference of two matrices.
- Find scalar multiples of a matrix.
- Find the product of two matrices.

(credit: “SD Dirk,” Flickr)

Two club soccer teams, the Wildcats and the Mud Cats, are hoping to obtain new equipment for an upcoming season. [link] shows the needs of both teams.

	Wildcats	**Mud Cats**
Goals	6	10
Balls	30	24
Jerseys	14	20

A goal costs $300; a ball costs $10; and a jersey costs $30. How can we find the total cost for the equipment needed for each team? In this section, we discover a method in which the data in the soccer equipment table can be displayed and used for calculating other information. Then, we will be able to calculate the cost of the equipment.

Finding the Sum and Difference of Two Matrices

To solve a problem like the one described for the soccer teams, we can use a matrix, which is a rectangular array of numbers. A row in a matrix is a set of numbers that are aligned horizontally. A column in a matrix is a set of numbers that are aligned vertically. Each number is an entry, sometimes called an element, of the matrix. Matrices (plural) are enclosed in [ ] or ( ), and are usually named with capital letters. For example, three matrices named A, B, and C are shown below.
Equation:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{matrix} 1 & 2 & 7 \\ 0 & -5 & 6 \\ 7 & 8 & 2 \end{matrix}, C = \begin{matrix} -1 & 3 \\ 0 & 2 \\ 3 & 1 \end{matrix}$$

Describing Matrices

A matrix is often referred to by its size or dimensions: $m \times n$ indicating m rows and n columns. Matrix entries are defined first by row and then by column. For example, to locate the entry in matrix A identified as a_{ij}, we look for the entry in row i, column j. In matrix A, shown below, the entry in row 2, column 3 is a_{23}.
Equation:

$$A = \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

A square matrix is a matrix with dimensions $n \times n$, meaning that it has the same number of rows as columns. The 3×3 matrix above is an example of a square matrix.

A row matrix is a matrix consisting of one row with dimensions $1 \times n$.
Equation:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

A column matrix is a matrix consisting of one column with dimensions $m \times 1$.
Equation:

$$\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix}$$

A matrix may be used to represent a system of equations. In these cases, the numbers represent the coefficients of the variables in the system. Matrices often make solving systems of equations easier because they are not encumbered with variables. We will investigate this idea further in the next section, but first we will look at basic matrix operations.

Note:
Matrices

A **matrix** is a rectangular array of numbers that is usually named by a capital letter: A, B, C, and so on. Each entry in a matrix is referred to as a_{ij},such that i represents the row and j represents the column. Matrices are often referred to by their dimensions: $m \times n$ indicating m rows and n columns.

Example:
Exercise:

Problem:
Finding the Dimensions of the Given Matrix and Locating Entries

Given matrix A :

a. What are the dimensions of matrix A?
b. What are the entries at a_{31} and a_{22}?
Equation:

$$A = \begin{matrix} 2 & 1 & 0 \\ 2 & 4 & 7 \\ 3 & 1 & -2 \end{matrix}$$

Solution:

a. The dimensions are 3×3 because there are three rows and three columns.
b. Entry a_{31} is the number at row 3, column 1, which is 3. The entry a_{22} is the number at row 2, column 2, which is 4. Remember, the row comes first, then the column.

Adding and Subtracting Matrices

We use matrices to list data or to represent systems. Because the entries are numbers, we can perform operations on matrices. We add or subtract matrices by adding or subtracting corresponding entries.

In order to do this, the entries must correspond. Therefore, *addition and subtraction of matrices is only possible when the matrices have the same dimensions*. We can add or subtract a 3×3 matrix and another 3×3 matrix, but we cannot add or subtract a 2×3 matrix and a 3×3 matrix because some entries in one matrix will not have a corresponding entry in the other matrix.

Note:
Adding and Subtracting Matrices
Given matrices A and B of like dimensions, addition and subtraction of A and B will produce matrix C or matrix D of the same dimension.
Equation:

$$A + B = C \text{ such that } a_{ij} + b_{ij} = c_{ij}$$

Equation:

$$A - B = D \text{ such that } a_{ij} - b_{ij} = d_{ij}$$

Matrix addition is commutative.
Equation:

$$A + B = B + A$$

It is also associative.
Equation:

$$(A + B) + C = A + (B + C)$$

Example:
Exercise:

Problem:
Finding the Sum of Matrices

Find the sum of A and B, given
Equation:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Solution:

Add corresponding entries.
Equation:

$$\begin{aligned} A + B &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &= \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix} \end{aligned}$$

Example:
Exercise:

Problem:
Adding Matrix *A* and Matrix *B*

Find the sum of A and B.
Equation:

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix}$$

Solution:

Add corresponding entries. Add the entry in row 1, column 1, a_{11}, of matrix A to the entry in row 1, column 1, b_{11},of B. Continue the pattern until all entries have been added.

Equation:

$$\begin{aligned} A+B &= \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 4+5 & 1+9 \\ 3+0 & 2+7 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 10 \\ 3 & 9 \end{bmatrix} \end{aligned}$$

Example:
Exercise:

Problem:
Finding the Difference of Two Matrices

Find the difference of A and B.
Equation:

$$A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

Solution:

We subtract the corresponding entries of each matrix.
Equation:

$$\begin{aligned} A-B &= \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -2-8 & 3-1 \\ 0-5 & 1-4 \end{bmatrix} \\ &= \begin{bmatrix} -10 & 2 \\ -5 & -3 \end{bmatrix} \end{aligned}$$

Example:
Exercise:

Problem:
Finding the Sum and Difference of Two 3 x 3 Matrices

Given A and B :

a. Find the sum.
b. Find the difference.

Equation:

$$A = \begin{matrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{matrix} \text{ and } B = \begin{matrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{matrix}$$

Solution:

a. Add the corresponding entries.
Equation:

$$\begin{aligned} A + B &= \begin{matrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{matrix} + \begin{matrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{matrix} \\ &= \begin{matrix} 2+6 & -10+10 & -2-2 \\ 14+0 & 12-12 & 10-4 \\ 4-5 & -2+2 & 2-2 \end{matrix} \\ &= \begin{matrix} 8 & 0 & -4 \\ 14 & 0 & 6 \\ -1 & 0 & 0 \end{matrix} \end{aligned}$$

b. Subtract the corresponding entries.
Equation:

$$\begin{aligned} A - B &= \begin{matrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{matrix} - \begin{matrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{matrix} \\ &= \begin{matrix} 2-6 & -10-10 & -2+2 \\ 14-0 & 12+12 & 10+4 \\ 4+5 & -2-2 & 2+2 \end{matrix} \\ &= \begin{matrix} -4 & -20 & 0 \\ 14 & 24 & 14 \\ 9 & -4 & 4 \end{matrix} \end{aligned}$$

Note:
Exercise:

Problem: Add matrix A and matrix B.
Equation:

$$A = \begin{matrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{matrix} \text{ and } B = \begin{matrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{matrix}$$

Solution:
Equation:

$$A+B=\begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix}+\begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix}=\begin{bmatrix} 2+3 & 6+(-2) \\ 1+1 & 0+5 \\ 1+(-4) & -3+3 \end{bmatrix}=\begin{bmatrix} 5 & 4 \\ 2 & 5 \\ -3 & 0 \end{bmatrix}$$

Finding Scalar Multiples of a Matrix

Besides adding and subtracting whole matrices, there are many situations in which we need to multiply a matrix by a constant called a scalar. Recall that a scalar is a real number quantity that has magnitude, but not direction. For example, time, temperature, and distance are scalar quantities. The process of scalar multiplication involves multiplying each entry in a matrix by a scalar. A **scalar multiple** is any entry of a matrix that results from scalar multiplication.

Consider a real-world scenario in which a university needs to add to its inventory of computers, computer tables, and chairs in two of the campus labs due to increased enrollment. They estimate that 15% more equipment is needed in both labs. The school's current inventory is displayed in [link].

	Lab A	**Lab B**
Computers	15	27
Computer Tables	16	34
Chairs	16	34

Converting the data to a matrix, we have
Equation:

$$C_{2013}=\begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix}$$

To calculate how much computer equipment will be needed, we multiply all entries in matrix C by 0.15.
Equation:

$$(0.15)C_{2013}=\begin{bmatrix} (0.15)15 & (0.15)27 \\ (0.15)16 & (0.15)34 \\ (0.15)16 & (0.15)34 \end{bmatrix}=\begin{bmatrix} 2.25 & 4.05 \\ 2.4 & 5.1 \\ 2.4 & 5.1 \end{bmatrix}$$

We must round up to the next integer, so the amount of new equipment needed is
Equation:

$$\begin{matrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{matrix}$$

Adding the two matrices as shown below, we see the new inventory amounts.
Equation:

$$\begin{matrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{matrix} + \begin{matrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{matrix} = \begin{matrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{matrix}$$

This means
Equation:

$$C_{2014} = \begin{matrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{matrix}$$

Thus, Lab A will have 18 computers, 19 computer tables, and 19 chairs; Lab B will have 32 computers, 40 computer tables, and 40 chairs.

Note:
Scalar Multiplication
Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given
Equation:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the scalar multiple cA is
Equation:

$$\begin{aligned} cA &= c \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix} \end{aligned}$$

Scalar multiplication is distributive. For the matrices A, B,and C with scalars a and b,
Equation:

$$\begin{aligned} a(A+B) &= aA + aB \\ (a+b)A &= aA + bA \end{aligned}$$

Example:
Exercise:

Problem:
Multiplying the Matrix by a Scalar

Multiply matrix A by the scalar 3.
Equation:

$$A = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

Solution:

Multiply each entry in A by the scalar 3.
Equation:

$$\begin{aligned} 3A &= 3\begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 8 & 3 \cdot 1 \\ 3 \cdot 5 & 3 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 24 & 3 \\ 15 & 12 \end{bmatrix} \end{aligned}$$

Note:
Exercise:

Problem: Given matrix B, find $-2B$ where
Equation:

$$B = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

Solution:

$-2B = \begin{bmatrix} -8 & -2 \\ -6 & -4 \end{bmatrix}$

Example:
Exercise:

Problem:
Finding the Sum of Scalar Multiples

Find the sum $3A + 2B$.
Equation:

$$A = \begin{matrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6 \end{matrix} \text{ and } B = \begin{matrix} -1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 1 & -4 \end{matrix}$$

Solution:

First, find $3A$, then $2B$.
Equation:

$$\begin{aligned} 3A &= \begin{matrix} 3 \cdot 1 & 3(-2) & 3 \cdot 0 \\ 3 \cdot 0 & 3(-1) & 3 \cdot 2 \\ 3 \cdot 4 & 3 \cdot 3 & 3(-6) \end{matrix} \\ &= \begin{matrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{matrix} \end{aligned}$$

Equation:

$$\begin{aligned} 2B &= \begin{matrix} 2(-1) & 2 \cdot 2 & 2 \cdot 1 \\ 2 \cdot 0 & 2(-3) & 2 \cdot 2 \\ 2 \cdot 0 & 2 \cdot 1 & 2(-4) \end{matrix} \\ &= \begin{matrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{matrix} \end{aligned}$$

Now, add $3A + 2B$.
Equation:

$$\begin{aligned} 3A + 2B &= \begin{matrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{matrix} + \begin{matrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{matrix} \\ &= \begin{matrix} 3-2 & -6+4 & 0+2 \\ 0+0 & -3-6 & 6+4 \\ 12+0 & 9+2 & -18-8 \end{matrix} \\ &= \begin{matrix} 1 & -2 & 2 \\ 0 & -9 & 10 \\ 12 & 11 & -26 \end{matrix} \end{aligned}$$

Finding the Product of Two Matrices

In addition to multiplying a matrix by a scalar, we can multiply two matrices. Finding the product of two matrices is only possible when the inner dimensions are the same, meaning that the number of columns of the first matrix is equal to the number of rows of the second matrix. If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product matrix AB is an $m \times n$ matrix. For example, the product AB is possible because the number of columns in A is the same as the number of rows in B. If the inner dimensions do not match, the product is not defined.

$$\begin{array}{ccc} A & \cdot & B \\ 2\times 3 & & 3\times 3 \end{array}$$

$$\text{same}$$

We multiply entries of A with entries of B according to a specific pattern as outlined below. The process of matrix multiplication becomes clearer when working a problem with real numbers.

To obtain the entries in row i of AB, we multiply the entries in row i of A by column j in B and add. For example, given matrices A and B, where the dimensions of A are 2×3 and the dimensions of B are 3×3, the product of AB will be a 2×3 matrix.
Equation:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{matrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{matrix}$$

Multiply and add as follows to obtain the first entry of the product matrix AB.

1. To obtain the entry in row 1, column 1 of AB, multiply the first row in A by the first column in B,and add.
   Equation:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{matrix} b_{11} \\ b_{21} \\ b_{31} \end{matrix} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}$$

2. To obtain the entry in row 1, column 2 of AB, multiply the first row of A by the second column in B,and add.
   Equation:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{matrix} b_{12} \\ b_{22} \\ b_{32} \end{matrix} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}$$

3. To obtain the entry in row 1, column 3 of AB, multiply the first row of A by the third column in B,and add.
   Equation:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{matrix} b_{13} \\ b_{23} \\ b_{33} \end{matrix} = a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33}$$

We proceed the same way to obtain the second row of AB. In other words, row 2 of A times column 1 of B; row 2 of A times column 2 of B; row 2 of A times column 3 of B. When complete, the product matrix will be

Equation:

$$AB = \begin{matrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \end{matrix}$$

Note:
Properties of Matrix Multiplication
For the matrices A, B, and C the following properties hold.

- Matrix multiplication is associative: $(AB)C = A\,(BC)$.
- Matrix multiplication is distributive: $C(A+B) = CA + CB,$
  $(A+B)C = AC + BC.$

Note that matrix multiplication is not commutative.

Example:
Exercise:

Problem:
Multiplying Two Matrices

Multiply matrix A and matrix B.
Equation:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution:

First, we check the dimensions of the matrices. Matrix A has dimensions 2×2 and matrix B has dimensions 2×2. The inner dimensions are the same so we can perform the multiplication. The product will have the dimensions 2×2.

We perform the operations outlined previously.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{bmatrix} \\ &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \end{aligned}$$

Example:
Exercise:

Problem:
Multiplying Two Matrices

Given A and B :

a. Find AB.
b. Find BA.

Equation:

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \text{ and } B = \begin{matrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{matrix}$$

Solution:

a. As the dimensions of A are 2×3 and the dimensions of B are 3×2, these matrices can be multiplied together because the number of columns in A matches the number of rows in B. The resulting product will be a 2×2 matrix, the number of rows in A by the number of columns in B.
Equation:

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{matrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{matrix} \\ &= \begin{bmatrix} -1(5) + 2(-4) + 3(2) & -1(-1) + 2(0) + 3(3) \\ 4(5) + 0(-4) + 5(2) & 4(-1) + 0(0) + 5(3) \end{bmatrix} \\ &= \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \end{aligned}$$

b. The dimensions of B are 3×2 and the dimensions of A are 2×3. The inner dimensions match so the product is defined and will be a 3×3 matrix.
Equation:

$$\begin{aligned} BA &= \begin{matrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{matrix} \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \\ &= \begin{matrix} 5(-1) + -1(4) & 5(2) + -1(0) & 5(3) + -1(5) \\ -4(-1) + 0(4) & -4(2) + 0(0) & -4(3) + 0(5) \\ 2(-1) + 3(4) & 2(2) + 3(0) & 2(3) + 3(5) \end{matrix} \\ &= \begin{matrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{matrix} \end{aligned}$$

Analysis

Notice that the products AB and BA are not equal.
Equation:

$$AB = \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \neq \begin{matrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{matrix} = BA$$

This illustrates the fact that matrix multiplication is not commutative.

Note:
Is it possible for *AB* to be defined but not *BA*?
Yes, consider a matrix A with dimension 3×4 *and matrix B with dimension* 4×2. *For the product AB the inner dimensions are 4 and the product is defined, but for the product BA the inner dimensions are 2 and 3 so the product is undefined.*

Example:
Exercise:

Problem:
Using Matrices in Real-World Problems

Let's return to the problem presented at the opening of this section. We have [link], representing the equipment needs of two soccer teams.

	Wildcats	**Mud Cats**
Goals	6	10
Balls	30	24
Jerseys	14	20

We are also given the prices of the equipment, as shown in [link].

Goal	\$300
Ball	\$10
Jersey	\$30

We will convert the data to matrices. Thus, the equipment need matrix is written as
Equation:

$$E = \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix}$$

The cost matrix is written as
Equation:

$$C = \begin{bmatrix} 300 & 10 & 30 \end{bmatrix}$$

We perform matrix multiplication to obtain costs for the equipment.

Equation:

$$\begin{aligned} CE &= \begin{bmatrix} 300 & 10 & 30 \end{bmatrix} \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix} \\ &= \begin{bmatrix} 300(6)+10(30)+30(14) & 300(10)+10(24)+30(20) \end{bmatrix} \\ &= \begin{bmatrix} 2{,}520 & 3{,}840 \end{bmatrix} \end{aligned}$$

The total cost for equipment for the Wildcats is $2,520, and the total cost for equipment for the Mud Cats is $3,840.

Note:
Given a matrix operation, evaluate using a calculator.

1. Save each matrix as a matrix variable $[A], [B], [C], \ldots$
2. Enter the operation into the calculator, calling up each matrix variable as needed.
3. If the operation is defined, the calculator will present the solution matrix; if the operation is undefined, it will display an error message.

Example:
Exercise:

Problem:
Using a Calculator to Perform Matrix Operations

Find $AB - C$ given
Equation:

$$A = \begin{bmatrix} -15 & 25 & 32 \\ 41 & -7 & -28 \\ 10 & 34 & -2 \end{bmatrix}, B = \begin{bmatrix} 45 & 21 & -37 \\ -24 & 52 & 19 \\ 6 & -48 & -31 \end{bmatrix}, \text{and } C = \begin{bmatrix} -100 & -89 & -98 \\ 25 & -56 & 74 \\ -67 & 42 & -75 \end{bmatrix}.$$

Solution:

On the matrix page of the calculator, we enter matrix A above as the matrix variable $[A]$,matrix B above as the matrix variable $[B]$,and matrix C above as the matrix variable $[C]$.

On the home screen of the calculator, we type in the problem and call up each matrix variable as needed.
Equation:

$$[A] \times [B] - [C]$$

The calculator gives us the following matrix.
Equation:

$$\begin{bmatrix} -983 & -462 & 136 \\ 1,820 & 1,897 & -856 \\ -311 & 2,032 & 413 \end{bmatrix}$$

Note:
Access these online resources for additional instruction and practice with matrices and matrix operations.

- Dimensions of a Matrix
- Matrix Addition and Subtraction
- Matrix Operations
- Matrix Multiplication

Key Concepts

- A matrix is a rectangular array of numbers. Entries are arranged in rows and columns.
- The dimensions of a matrix refer to the number of rows and the number of columns. A 3×2 matrix has three rows and two columns. See [link].
- We add and subtract matrices of equal dimensions by adding and subtracting corresponding entries of each matrix. See [link], [link], [link], and [link].
- Scalar multiplication involves multiplying each entry in a matrix by a constant. See [link].
- Scalar multiplication is often required before addition or subtraction can occur. See [link].
- Multiplying matrices is possible when inner dimensions are the same—the number of columns in the first matrix must match the number of rows in the second.
- The product of two matrices, A and B,is obtained by multiplying each entry in row 1 of A by each entry in column 1 of B; then multiply each entry of row 1 of A by each entry in columns 2 of B, and so on. See [link] and [link].
- Many real-world problems can often be solved using matrices. See [link].
- We can use a calculator to perform matrix operations after saving each matrix as a matrix variable. See [link].

Section Exercises

Verbal

Exercise:

Problem:

Can we add any two matrices together? If so, explain why; if not, explain why not and give an example of two matrices that cannot be added together.

Solution:

No, they must have the same dimensions. An example would include two matrices of different dimensions. One cannot add the following two matrices because the first is a 2×2 matrix and the second is a 2×3 matrix. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ has no sum.

Exercise:

Problem: Can we multiply any column matrix by any row matrix? Explain why or why not.

Exercise:

Problem: Can both the products AB and BA be defined? If so, explain how; if not, explain why.

Solution:

Yes, if the dimensions of A are $m \times n$ and the dimensions of B are $n \times m$, both products will be defined.

Exercise:

Problem:

Can any two matrices of the same size be multiplied? If so, explain why, and if not, explain why not and give an example of two matrices of the same size that cannot be multiplied together.

Exercise:

Problem:

Does matrix multiplication commute? That is, does $AB = BA$? If so, prove why it does. If not, explain why it does not.

Solution:

Not necessarily. To find AB, we multiply the first row of A by the first column of B to get the first entry of AB. To find BA, we multiply the first row of B by the first column of A to get the first entry of BA. Thus, if those are unequal, then the matrix multiplication does not commute.

Algebraic

For the following exercises, use the matrices below and perform the matrix addition or subtraction. Indicate if the operation is undefined.

Equation:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 7 \end{bmatrix}, B = \begin{bmatrix} 2 & 14 \\ 22 & 6 \end{bmatrix}, C = \begin{matrix} 1 & 5 \\ 8 & 92 \\ 12 & 6 \end{matrix}, D = \begin{matrix} 10 & 14 \\ 7 & 2 \\ 5 & 61 \end{matrix}, E = \begin{bmatrix} 6 & 12 \\ 14 & 5 \end{bmatrix}, F = \begin{matrix} 0 & 9 \\ 78 & 17 \\ 15 & 4 \end{matrix}$$

Exercise:

Problem: $A + B$

Exercise:

Problem: $C + D$

Solution:

$$\begin{matrix} 11 & 19 \\ 15 & 94 \\ 17 & 67 \end{matrix}$$

Exercise:

Problem: $A + C$

Exercise:

Problem: $B - E$

Solution:

$$\begin{bmatrix} -4 & 2 \\ 8 & 1 \end{bmatrix}$$

Exercise:

Problem: $C + F$

Exercise:

Problem: $D - B$

Solution:

Undidentified; dimensions do not match

For the following exercises, use the matrices below to perform scalar multiplication.
Equation:

$$A = \begin{bmatrix} 4 & 6 \\ 13 & 12 \end{bmatrix}, B = \begin{matrix} 3 & 9 \\ 21 & 12 \\ 0 & 64 \end{matrix}, C = \begin{bmatrix} 16 & 3 & 7 & 18 \\ 90 & 5 & 3 & 29 \end{bmatrix}, D = \begin{matrix} 18 & 12 & 13 \\ 8 & 14 & 6 \\ 7 & 4 & 21 \end{matrix}$$

Exercise:

Problem: $5A$

Exercise:

Problem: $3B$

Solution:

$$\begin{matrix} 9 & 27 \\ 63 & 36 \\ 0 & 192 \end{matrix}$$

Exercise:

Problem: $-2B$

Exercise:

Problem: $-4C$

Solution:

$$\begin{bmatrix} -64 & -12 & -28 & -72 \\ -360 & -20 & -12 & -116 \end{bmatrix}$$

Exercise:

Problem: $\frac{1}{2}C$

Exercise:

Problem: $100D$

Solution:

$$\begin{matrix} 1,800 & 1,200 & 1,300 \\ 800 & 1,400 & 600 \\ 700 & 400 & 2,100 \end{matrix}$$

For the following exercises, use the matrices below to perform matrix multiplication.

Equation:

$$A = \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 6 & 4 \\ -8 & 0 & 12 \end{bmatrix}, C = \begin{matrix} 4 & 10 \\ -2 & 6 \\ 5 & 9 \end{matrix}, D = \begin{matrix} 2 & -3 & 12 \\ 9 & 3 & 1 \\ 0 & 8 & -10 \end{matrix}$$

Exercise:

Problem: AB

Exercise:

Problem: BC

Solution:

$$\begin{bmatrix} 20 & 102 \\ 28 & 28 \end{bmatrix}$$

Exercise:

Problem: CA

Exercise:

Problem: BD

Solution:

$$\begin{bmatrix} 60 & 41 & 2 \\ -16 & 120 & -216 \end{bmatrix}$$

Exercise:

Problem: DC

Exercise:

Problem: CB

Solution:

$$\begin{matrix} -68 & 24 & 136 \\ -54 & -12 & 64 \\ -57 & 30 & 128 \end{matrix}$$

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed.

Equation:

$$A = \begin{bmatrix} 2 & -5 \\ 6 & 7 \end{bmatrix}, B = \begin{bmatrix} -9 & 6 \\ -4 & 2 \end{bmatrix}, C = \begin{bmatrix} 0 & 9 \\ 7 & 1 \end{bmatrix}, D = \begin{matrix} -8 & 7 & -5 \\ 4 & 3 & 2 \\ 0 & 9 & 2 \end{matrix}, E = \begin{matrix} 4 & 5 & 3 \\ 7 & -6 & -5 \\ 1 & 0 & 9 \end{matrix}$$

Exercise:

Problem: $A + B - C$

Exercise:

Problem: $4A + 5D$

Solution:

Undefined; dimensions do not match.

Exercise:

Problem: $2C + B$

Exercise:

Problem: $3D + 4E$

Solution:

$$\begin{matrix} -8 & 41 & -3 \\ 40 & -15 & -14 \\ 4 & 27 & 42 \end{matrix}$$

Exercise:

Problem: $C - 0.5D$

Exercise:

Problem: $100D - 10E$

Solution:

$$\begin{matrix} -840 & 650 & -530 \\ 330 & 360 & 250 \\ -10 & 900 & 110 \end{matrix}$$

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2 = A \cdot A$)

Equation:

$$A = \begin{bmatrix} -10 & 20 \\ 5 & 25 \end{bmatrix}, B = \begin{bmatrix} 40 & 10 \\ -20 & 30 \end{bmatrix}, C = \begin{matrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \end{matrix}$$

Exercise:

Problem: AB

Exercise:

Problem: BA

Solution:

$$\begin{bmatrix} -350 & 1,050 \\ 350 & 350 \end{bmatrix}$$

Exercise:

Problem: CA

Exercise:

Problem: BC

Solution:

Undefined; inner dimensions do not match.

Exercise:

Problem: A^2

Exercise:

Problem: B^2

Solution:

$$\begin{bmatrix} 1,400 & 700 \\ -1,400 & 700 \end{bmatrix}$$

Exercise:

Problem: C^2

Exercise:

Problem: B^2A^2

Solution:

$$\begin{bmatrix} 332,500 & 927,500 \\ -227,500 & 87,500 \end{bmatrix}$$

Exercise:

Problem: A^2B^2

Exercise:

Problem: $(AB)^2$

Solution:

$$\begin{bmatrix} 490,000 & 0 \\ 0 & 490,000 \end{bmatrix}$$

Exercise:

Problem: $(BA)^2$

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2 = A \cdot A$)

Equation:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} -2 & 3 & 4 \\ -1 & 1 & -5 \end{bmatrix}, C = \begin{matrix} 0.5 & 0.1 \\ 1 & 0.2 \\ -0.5 & 0.3 \end{matrix}, D = \begin{matrix} 1 & 0 & -1 \\ -6 & 7 & 5 \\ 4 & 2 & 1 \end{matrix}$$

Exercise:

Problem: AB

Solution:

$$\begin{bmatrix} -2 & 3 & 4 \\ -7 & 9 & -7 \end{bmatrix}$$

Exercise:

Problem: BA

Exercise:

Problem: BD

Solution:

$$\begin{bmatrix} -4 & 29 & 21 \\ -27 & -3 & 1 \end{bmatrix}$$

Exercise:

Problem: DC

Exercise:

Problem: D^2

Solution:

$$\begin{matrix} -3 & -2 & -2 \\ -28 & 59 & 46 \\ -4 & 16 & 7 \end{matrix}$$

Exercise:

Problem: A^2

Exercise:

Problem: D^3

Solution:

$$\begin{matrix} 1 & -18 & -9 \\ -198 & 505 & 369 \\ -72 & 126 & 91 \end{matrix}$$

Exercise:

Problem: $(AB)C$

Exercise:

Problem: $A(BC)$

Solution:

$$\begin{bmatrix} 0 & 1.6 \\ 9 & -1 \end{bmatrix}$$

Technology

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. Use a calculator to verify your solution.

Equation:

$$A = \begin{matrix} -2 & 0 & 9 \\ 1 & 8 & -3 \\ 0.5 & 4 & 5 \end{matrix}, B = \begin{matrix} 0.5 & 3 & 0 \\ -4 & 1 & 6 \\ 8 & 7 & 2 \end{matrix}, C = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$$

Exercise:

Problem: AB

Exercise:

Problem: BA

Solution:

$$\begin{matrix} 2 & 24 & -4.5 \\ 12 & 32 & -9 \\ -8 & 64 & 61 \end{matrix}$$

Exercise:

Problem: CA

Exercise:

Problem: BC

Solution:

$$\begin{matrix} 0.5 & 3 & 0.5 \\ 2 & 1 & 2 \\ 10 & 7 & 10 \end{matrix}$$

Exercise:

Problem: ABC

Extensions

For the following exercises, use the matrix below to perform the indicated operation on the given matrix.

Equation:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Exercise:

Problem: B^2

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise:

Problem: B^3

Exercise:

Problem: B^4

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise:

Problem: B^5

Exercise:

Problem:

Using the above questions, find a formula for $B^n.$ Test the formula for B^{201} and B^{202}, using a calculator.

Solution:

$$B^n = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & n\text{ even,} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, & n\text{ odd.} \end{cases}$$

Glossary

column
: a set of numbers aligned vertically in a matrix

entry
: an element, coefficient, or constant in a matrix

matrix
: a rectangular array of numbers

row
: a set of numbers aligned horizontally in a matrix

scalar multiple
: an entry of a matrix that has been multiplied by a scalar

Solving Systems with Inverses
In this section, you will:

- Find the inverse of a matrix.
- Solve a system of linear equations using an inverse matrix.

Nancy plans to invest $10,500 into two different bonds to spread out her risk. The first bond has an annual return of 10%, and the second bond has an annual return of 6%. In order to receive an 8.5% return from the two bonds, how much should Nancy invest in each bond? What is the best method to solve this problem?

There are several ways we can solve this problem. As we have seen in previous sections, systems of equations and matrices are useful in solving real-world problems involving finance. After studying this section, we will have the tools to solve the bond problem using the inverse of a matrix.

Finding the Inverse of a Matrix

We know that the multiplicative inverse of a real number a is a^{-1}, and $aa^{-1} = a^{-1}a = \left(\frac{1}{a}\right)a = 1$. For example, $2^{-1} = \frac{1}{2}$ and $\left(\frac{1}{2}\right)2 = 1$. The multiplicative inverse of a matrix is similar in concept, except that the product of matrix A and its inverse A^{-1} equals the identity matrix. The identity matrix is a square matrix containing ones down the main diagonal and zeros everywhere else. We identify identity matrices by I_n where n represents the dimension of the matrix. [link] and the following equations.

Equation:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equation:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The identity matrix acts as a 1 in matrix algebra. For example, $AI = IA = A$.

A matrix that has a multiplicative inverse has the properties

Equation:

$$\begin{aligned} AA^{-1} &= I \\ A^{-1}A &= I \end{aligned}$$

A matrix that has a multiplicative inverse is called an invertible matrix. Only a square matrix may have a multiplicative inverse, as the reversibility, $AA^{-1} = A^{-1}A = I$, is a requirement.

Not all square matrices have an inverse, but if A is invertible, then A^{-1} is unique. We will look at two methods for finding the inverse of a 2×2 matrix and a third method that can be used on both 2×2 and 3×3 matrices.

Note:
The Identity Matrix and Multiplicative Inverse
The **identity matrix**, I_n, is a square matrix containing ones down the main diagonal and zeros everywhere else.
Equation:

$$\underset{2\,\times\,2}{I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \qquad \underset{3\,\times\,3}{I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

If A is an $n \times n$ matrix and B is an $n \times n$ matrix such that $AB = BA = I_n$, then $B = A^{-1}$, the **multiplicative inverse of a matrix** A.

Example:
Exercise:

Problem:
Showing That the Identity Matrix Acts as a 1

Given matrix A, show that $AI = IA = A$.
Equation:

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Solution:

Use matrix multiplication to show that the product of A and the identity is equal to the product of the identity and A.
Equation:

$$AI = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot 1 \\ -2 \cdot 1 + 5 \cdot 0 & -2 \cdot 0 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Equation:

$$AI = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 0 \cdot (-2) & 1 \cdot 4 + 0 \cdot 5 \\ 0 \cdot 3 + 1 \cdot (-2) & 0 \cdot 4 + 1 \cdot 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Note:
Given two matrices, show that one is the multiplicative inverse of the other.

1. Given matrix A of order $n \times n$ and matrix B of order $n \times n$ multiply AB.
2. If $AB = I$, then find the product BA. If $BA = I$, then $B = A^{-1}$ and $A = B^{-1}$.

Example:
Exercise:

Problem:
Showing That Matrix *A* Is the Multiplicative Inverse of Matrix *B*

Show that the given matrices are multiplicative inverses of each other.
Equation:

$$A = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix}, B = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix}$$

Solution:

Multiply AB and BA. If both products equal the identity, then the two matrices are inverses of each other.
Equation:

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \cdot \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-9) + 5(2) & 1(-5) + 5(1) \\ -2(-9) - 9(2) & -2(-5) - 9(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Equation:

$$\begin{aligned} BA &= \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \\ &= \begin{bmatrix} -9(1)-5(-2) & -9(5)-5(-9) \\ 2(1)+1(-2) & 2(-5)+1(-9) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

A and$B$are inverses of each other.

Note:
Exercise:

Problem: Show that the following two matrices are inverses of each other.
Equation:

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}, B = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

Solution:
Equation:

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1(-3)+4(1) & 1(-4)+4(1) \\ -1(-3)+-3(1) & -1(-4)+-3(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ BA &= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -3(1)+-4(-1) & -3(4)+-4(-3) \\ 1(1)+1(-1) & 1(4)+1(-3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Finding the Multiplicative Inverse Using Matrix Multiplication

We can now determine whether two matrices are inverses, but how would we find the inverse of a given matrix? Since we know that the product of a matrix and its inverse is the identity matrix, we can find the inverse of a matrix by setting up an equation using matrix multiplication.

Example:
Exercise:

Problem:
Finding the Multiplicative Inverse Using Matrix Multiplication

Use matrix multiplication to find the inverse of the given matrix.
Equation:

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

Solution:

For this method, we multiply A by a matrix containing unknown constants and set it equal to the identity.
Equation:

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find the product of the two matrices on the left side of the equal sign.
Equation:

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a-2c & 1b-2d \\ 2a-3c & 2b-3d \end{bmatrix}$$

Next, set up a system of equations with the entry in row 1, column 1 of the new matrix equal to the first entry of the identity, 1. Set the entry in row 2, column 1 of the new matrix equal to the corresponding entry of the identity, which is 0.
Equation:

$$\begin{array}{ll} 1a-2c=1 & R_1 \\ 2a-3c=0 & R_2 \end{array}$$

Using row operations, multiply and add as follows: $(-2)R_1 + R_2 \rightarrow R_2$. Add the equations, and solve for c.
Equation:

$$\begin{aligned} 1a - 2c &= 1 \\ 0 + 1c &= -2 \\ c &= -2 \end{aligned}$$

Back-substitute to solve for a.
Equation:

$$a - 2(-2) = 1$$
$$a + 4 = 1$$
$$a = -3$$

Write another system of equations setting the entry in row 1, column 2 of the new matrix equal to the corresponding entry of the identity, 0. Set the entry in row 2, column 2 equal to the corresponding entry of the identity.

Equation:

$$\begin{aligned} 1b - 2d &= 0 \quad R_1 \\ 2b - 3d &= 1 \quad R_2 \end{aligned}$$

Using row operations, multiply and add as follows: $(-2)R_1 + R_2 = R_2$. Add the two equations and solve for d.

Equation:

$$1b - 2d = 0$$
$$\frac{0 + 1d = 1}{d = 1}$$

Once more, back-substitute and solve for b.

Equation:

$$b - 2(1) = 0$$
$$b - 2 = 0$$
$$b = 2$$

Equation:

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

Finding the Multiplicative Inverse by Augmenting with the Identity

Another way to find the multiplicative inverse is by augmenting with the identity. When matrix A is transformed into I, the augmented matrix I transforms into A^{-1}.

For example, given

Equation:

$$A=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}$$

augment A with the identity
Equation:

$$\left[\begin{array}{cc|cc}2 & 1 & 1 & 0\\ 5 & 3 & 0 & 1\end{array}\right]$$

Perform row operations with the goal of turning A into the identity.

1. Switch row 1 and row 2.
   Equation:

$$\left[\begin{array}{cc|cc}5 & 3 & 0 & 1\\ 2 & 1 & 1 & 0\end{array}\right]$$

2. Multiply row 2 by -2 and add to row 1.
   Equation:

$$\left[\begin{array}{cc|cc}1 & 1 & -2 & 1\\ 2 & 1 & 1 & 0\end{array}\right]$$

3. Multiply row 1 by -2 and add to row 2.
   Equation:

$$\left[\begin{array}{cc|cc}1 & 1 & -2 & 1\\ 0 & -1 & 5 & -2\end{array}\right]$$

4. Add row 2 to row 1.
   Equation:

$$\left[\begin{array}{cc|cc}1 & 0 & 3 & -1\\ 0 & -1 & 5 & -2\end{array}\right]$$

5. Multiply row 2 by -1.
   Equation:

$$\left[\begin{array}{cc|cc}1 & 0 & 3 & -1\\ 0 & 1 & -5 & 2\end{array}\right]$$

The matrix we have found is A^{-1}.
Equation:

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Finding the Multiplicative Inverse of 2×2 Matrices Using a Formula

When we need to find the multiplicative inverse of a 2×2 matrix, we can use a special formula instead of using matrix multiplication or augmenting with the identity.

If A is a 2×2 matrix, such as
Equation:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the multiplicative inverse of A is given by the formula
Equation:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $ad - bc \neq 0$. If $ad - bc = 0,$ then A has no inverse.

Example:
Exercise:

Problem:
Using the Formula to Find the Multiplicative Inverse of Matrix *A*

Use the formula to find the multiplicative inverse of
Equation:

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

Solution:

Using the formula, we have
Equation:

$$\begin{aligned} A^{-1} &= \frac{1}{(1)(-3)-(-2)(2)}\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{-3+4}\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \end{aligned}$$

Analysis

We can check that our formula works by using one of the other methods to calculate the inverse. Let's augment A with the identity.
Equation:

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$$

Perform row operations with the goal of turning A into the identity.

1. Multiply row 1 by -2 and add to row 2.
   Equation:

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

2. Multiply row 1 by 2 and add to row 1.
   Equation:

$$\left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

So, we have verified our original solution.
Equation:

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

Note:
Exercise:

Problem:

Use the formula to find the inverse of matrix A. Verify your answer by augmenting with the identity matrix.
Equation:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

Solution:

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Example:
Exercise:

Problem:
Finding the Inverse of the Matrix, If It Exists

Find the inverse, if it exists, of the given matrix.
Equation:

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

Solution:

We will use the method of augmenting with the identity.
Equation:

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

1. Switch row 1 and row 2.
   Equation:

$$\left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 3 & 6 & 1 & 0 \end{array}\right]$$

2. Multiply row 1 by −3 and add it to row 2.
   Equation:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right]$$

3. There is nothing further we can do. The zeros in row 2 indicate that this matrix has no inverse.

Finding the Multiplicative Inverse of 3×3 Matrices

Unfortunately, we do not have a formula similar to the one for a 2×2 matrix to find the inverse of a 3×3 matrix. Instead, we will augment the original matrix with the identity matrix and use row operations to obtain the inverse.

Given a 3×3 matrix
Equation:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

augment A with the identity matrix
Equation:

$$A \mid I = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

To begin, we write the augmented matrix with the identity on the right and A on the left. Performing elementary row operations so that the identity matrix appears on the left, we will obtain the inverse matrix on the right. We will find the inverse of this matrix in the next example.

Note:
Given a 3×3 matrix, find the inverse

1. Write the original matrix augmented with the identity matrix on the right.
2. Use elementary row operations so that the identity appears on the left.
3. What is obtained on the right is the inverse of the original matrix.
4. Use matrix multiplication to show that $AA^{-1} = I$ and $A^{-1}A = I$.

Example:
Exercise:

Problem:
Finding the Inverse of a 3 × 3 Matrix

Given the 3×3 matrix $A,$ find the inverse.
Equation:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

Solution:

Augment A with the identity matrix, and then begin row operations until the identity matrix replaces $A.$ The matrix on the right will be the inverse of $A.$
Equation:

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Interchange } R_2 \text{ and } R_1} \left[\begin{array}{ccc|ccc} 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

Equation:

$$-R_2 + R_1 = R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

Equation:

$$-R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

Equation:

$$R_3 \leftrightarrow R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right]$$

Equation:

$$-2R_1 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0 \end{array}\right]$$

Equation:

$$-3R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

Thus,
Equation:

$$A^{-1} = B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

Analysis

To prove that $B = A^{-1}$, let's multiply the two matrices together to see if the product equals the identity, if$AA^{-1} = I$ and $A^{-1}A = I$.
Equation:

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 2(-1) + 3(-1) + 1(6) & 2(1) + 3(0) + 1(-2) & 2(0) + 3(1) + 1(-3) \\ 3(-1) + 3(-1) + 1(6) & 3(1) + 3(0) + 1(-2) & 3(0) + 3(1) + 1(-3) \\ 2(-1) + 4(-1) + 1(6) & 2(1) + 4(0) + 1(-2) & 2(0) + 4(1) + 1(-3) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Equation:

$$A^{-1}A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1(2)+1(3)+0(2) & -1(3)+1(3)+0(4) & -1(1)+1(1)+0(1) \\ -1(2)+0(3)+1(2) & -1(3)+0(3)+1(4) & -1(1)+0(1)+1(1) \\ 6(2)+-2(3)+-3(2) & 6(3)+-2(3)+-3(4) & 6(1)+-2(1)+-3(1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note:
Exercise:

Problem: Find the inverse of the 3×3 matrix.
Equation:

$$A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$$

Solution:

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

Solving a System of Linear Equations Using the Inverse of a Matrix

Solving a system of linear equations using the inverse of a matrix requires the definition of two new matrices: X is the matrix representing the variables of the system, and B is the matrix representing the constants. Using matrix multiplication, we may define a system of equations with the same number of equations as variables as
Equation:

$$AX = B$$

To solve a system of linear equations using an inverse matrix, let A be the coefficient matrix, let X be the variable matrix, and let B be the constant matrix. Thus, we want to solve a system $AX = B.$ For example, look at the following system of equations.
Equation:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

From this system, the coefficient matrix is
Equation:

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The variable matrix is
Equation:

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

And the constant matrix is
Equation:

$$B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then $AX = B$ looks like
Equation:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Recall the discussion earlier in this section regarding multiplying a real number by its inverse, $\left(2^{-1}\right) 2 = \left(\frac{1}{2}\right) 2 = 1.$ To solve a single linear equation $ax = b$ for $x,$ we would simply multiply both sides of the equation by the multiplicative inverse (reciprocal) of $a.$ Thus,
Equation:

$$
\begin{aligned}
ax &= b \\
\left(\tfrac{1}{a}\right)ax &= \left(\tfrac{1}{a}\right)b \\
(a^{-1})ax &= (a^{-1})b \\
[(a^{-1})a]x &= (a^{-1})b \\
1x &= (a^{-1})b \\
x &= (a^{-1})b
\end{aligned}
$$

The only difference between a solving a linear equation and a system of equations written in matrix form is that finding the inverse of a matrix is more complicated, and matrix multiplication is a longer process. However, the goal is the same—to isolate the variable.

We will investigate this idea in detail, but it is helpful to begin with a 2×2 system and then move on to a 3×3 system.

Note:
Solving a System of Equations Using the Inverse of a Matrix
Given a system of equations, write the coefficient matrix A, the variable matrix X, and the constant matrix B. Then
Equation:

$$AX = B$$

Multiply both sides by the inverse of A to obtain the solution.
Equation:

$$
\begin{aligned}
\left(A^{-1}\right)AX &= \left(A^{-1}\right)B \\
\left[\left(A^{-1}\right)A\right]X &= \left(A^{-1}\right)B \\
IX &= \left(A^{-1}\right)B \\
X &= \left(A^{-1}\right)B
\end{aligned}
$$

Note:
If the coefficient matrix does not have an inverse, does that mean the system has no solution?
No, if the coefficient matrix is not invertible, the system could be inconsistent and have no solution, or be dependent and have infinitely many solutions.

Example:
Exercise:

Problem:
Solving a 2 × 2 System Using the Inverse of a Matrix

Solve the given system of equations using the inverse of a matrix.
Equation:

$$\begin{aligned} 3x + 8y &= 5 \\ 4x + 11y &= 7 \end{aligned}$$

Solution:

Write the system in terms of a coefficient matrix, a variable matrix, and a constant matrix.
Equation:

$$A = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Then
Equation:

$$\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

First, we need to calculate A^{-1}. Using the formula to calculate the inverse of a 2 by 2 matrix, we have:
Equation:

$$\begin{aligned} A^{-1} &= \tfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \tfrac{1}{3(11)-8(4)} \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \\ &= \tfrac{1}{1} \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \end{aligned}$$

So,
Equation:

$$A^{-1} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

Now we are ready to solve. Multiply both sides of the equation by A^{-1}.
Equation:

$$\left(A^{-1}\right)AX = \left(A^{-1}\right)B$$
$$\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11(5)+(-8)7 \\ -4(5)+3(7) \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The solution is $(-1, 1)$.

Note:
Can we solve for X by finding the product BA^{-1}?
No, recall that matrix multiplication is not commutative, so $A^{-1}B \neq BA^{-1}$. Consider our steps for solving the matrix equation.
Equation:

$$\begin{aligned} \left(A^{-1}\right)AX &= \left(A^{-1}\right)B \\ \left[\left(A^{-1}\right)A\right]X &= \left(A^{-1}\right)B \\ IX &= \left(A^{-1}\right)B \\ X &= \left(A^{-1}\right)B \end{aligned}$$

Notice in the first step we multiplied both sides of the equation by A^{-1}, but the A^{-1} was to the left of A on the left side and to the left of B on the right side. Because matrix multiplication is not commutative, order matters.

Example:
Exercise:

Problem:
Solving a 3 × 3 System Using the Inverse of a Matrix

Solve the following system using the inverse of a matrix.
Equation:

$$\begin{aligned} 5x + 15y + 56z &= 35 \\ -4x - 11y - 41z &= -26 \\ -x - 3y - 11z &= -7 \end{aligned}$$

Solution:

Write the equation $AX = B$.
Equation:

$$\begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

First, we will find the inverse of A by augmenting with the identity.
Equation:

$$\left[\begin{array}{rrr|rrr} 5 & 15 & 56 & 1 & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by $\frac{1}{5}$.
Equation:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by 4 and add to row 2.
Equation:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Add row 1 to row 3.
Equation:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array}\right]$$

Multiply row 2 by −3 and add to row 1.
Equation:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array}\right]$$

Multiply row 3 by 5.
Equation:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

Multiply row 3 by $\frac{1}{5}$ and add to row 1.
Equation:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

Multiply row 3 by $-\frac{19}{5}$ and add to row 2.
Equation:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & 1 & -19 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

So,
Equation:

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix}$$

Multiply both sides of the equation by A^{-1}. We want $A^{-1}AX = A^{-1}B$:
Equation:

$$\begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

Thus,
Equation:

$$A^{-1}B = \begin{bmatrix} -70 + 78 - 7 \\ -105 - 26 + 133 \\ 35 + 0 - 35 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The solution is $(1, 2, 0)$.

Note:
Exercise:

Problem: Solve the system using the inverse of the coefficient matrix.
Equation:

$$\begin{aligned} 2x - 17y + 11z &= 0 \\ -x + 11y - 7z &= 8 \\ 3y - 2z &= -2 \end{aligned}$$

Solution:

$$X = \begin{bmatrix} 4 \\ 38 \\ 58 \end{bmatrix}$$

Note:
Given a system of equations, solve with matrix inverses using a calculator.

1. Save the coefficient matrix and the constant matrix as matrix variables $[A]$ and $[B]$.
2. Enter the multiplication into the calculator, calling up each matrix variable as needed.
3. If the coefficient matrix is invertible, the calculator will present the solution matrix; if the coefficient matrix is not invertible, the calculator will present an error message.

Example:
Exercise:

Problem:

Using a Calculator to Solve a System of Equations with Matrix Inverses

Solve the system of equations with matrix inverses using a calculator

Equation:

$$\begin{aligned} 2x + 3y + z &= 32 \\ 3x + 3y + z &= -27 \\ 2x + 4y + z &= -2 \end{aligned}$$

Solution:

On the matrix page of the calculator, enter the coefficient matrix as the matrix variable $[A]$, and enter the constant matrix as the matrix variable $[B]$.

Equation:

$$[A] = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}, \quad [B] = \begin{bmatrix} 32 \\ -27 \\ -2 \end{bmatrix}$$

On the home screen of the calculator, type in the multiplication to solve for X, calling up each matrix variable as needed.

Equation:

$$[A]^{-1} \times [B]$$

Evaluate the expression.

Equation:

$$\begin{bmatrix} -59 \\ -34 \\ 252 \end{bmatrix}$$

Note:
Access these online resources for additional instruction and practice with solving systems with inverses.

- The Identity Matrix
- Determining Inverse Matrices
- Using a Matrix Equation to Solve a System of Equations

Key Equations

Identity matrix for a2×2matrix	$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Identity matrix for a3×3matrix	$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Multiplicative inverse of a2×2matrix	$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$

Key Concepts

- An identity matrix has the property $AI = IA = A.$ See [link].
- An invertible matrix has the property $AA^{-1} = A^{-1}A = I.$ See [link].
- Use matrix multiplication and the identity to find the inverse of a 2×2 matrix. See [link].
- The multiplicative inverse can be found using a formula. See [link].
- Another method of finding the inverse is by augmenting with the identity. See [link].
- We can augment a 3×3 matrix with the identity on the right and use row operations to turn the original matrix into the identity, and the matrix on the right becomes the inverse. See [link].
- Write the system of equations as $AX = B$, and multiply both sides by the inverse of $A : A^{-1}AX = A^{-1}B.$ See [link] and [link].
- We can also use a calculator to solve a system of equations with matrix inverses. See [link].

Section Exercises

Verbal

Exercise:

Problem:

In a previous section, we showed that matrix multiplication is not commutative, that is, $AB \neq BA$ in most cases. Can you explain why matrix multiplication is commutative for matrix inverses, that is, $A^{-1}A = AA^{-1}$?

Solution:

If A^{-1} is the inverse of $A,$ then $AA^{-1} = I,$ the identity matrix. Since A is also the inverse of A^{-1}, $A^{-1}A = I.$ You can also check by proving this for a 2×2 matrix.

Exercise:

Problem:

Does every 2×2 matrix have an inverse? Explain why or why not. Explain what condition is necessary for an inverse to exist.

Exercise:

Problem:

Can you explain whether a 2×2 matrix with an entire row of zeros can have an inverse?

Solution:

No, because ad and bc are both 0, so $ad - bc = 0,$ which requires us to divide by 0 in the formula.

Exercise:

Problem:

Can a matrix with an entire column of zeros have an inverse? Explain why or why not.

Exercise:

Problem:

Can a matrix with zeros on the diagonal have an inverse? If so, find an example. If not, prove why not. For simplicity, assume a 2×2 matrix.

Solution:

Yes. Consider the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The inverse is found with the following calculation:

$A^{-1} = \frac{1}{0(0)-1(1)} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

Algebraic

In the following exercises, show that matrix A is the inverse of matrix $B.$

Exercise:

Problem: $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Exercise:

Problem: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Solution:

$$AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Exercise:

Problem: $A = \begin{bmatrix} 4 & 5 \\ 7 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & \frac{1}{7} \\ \frac{1}{5} & -\frac{4}{35} \end{bmatrix}$

Exercise:

Problem: $A = \begin{bmatrix} -2 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix}$

Solution:

$$AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Exercise:

Problem: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \frac{1}{2}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

Exercise:

Problem: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \\ 1 & 6 & 9 \end{bmatrix}$, $B = \frac{1}{4}\begin{bmatrix} 6 & 0 & -2 \\ 17 & -3 & -5 \\ -12 & 2 & 4 \end{bmatrix}$

Solution:

$$AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Exercise:

Problem: $A = \begin{bmatrix} 3 & 8 & 2 \\ 1 & 1 & 1 \\ 5 & 6 & 12 \end{bmatrix}, B = \frac{1}{36}\begin{bmatrix} -6 & 84 & -6 \\ 7 & -26 & 1 \\ -1 & -22 & 5 \end{bmatrix}$

For the following exercises, find the multiplicative inverse of each matrix, if it exists.

Exercise:

Problem: $\begin{bmatrix} 3 & -2 \\ 1 & 9 \end{bmatrix}$

Solution:

$\frac{1}{29}\begin{bmatrix} 9 & 2 \\ -1 & 3 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} -3 & 7 \\ 9 & 2 \end{bmatrix}$

Solution:

$\frac{1}{69}\begin{bmatrix} -2 & 7 \\ 9 & 3 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} -4 & -3 \\ -5 & 8 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Solution:

There is no inverse

Exercise:

Problem: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} 0.5 & 1.5 \\ 1 & -0.5 \end{bmatrix}$

Solution:

$\frac{4}{7}\begin{bmatrix} 0.5 & 1.5 \\ 1 & -0.5 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} 1 & 0 & 6 \\ -2 & 1 & 7 \\ 3 & 0 & 2 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} 0 & 1 & -3 \\ 4 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

Solution:

$\frac{1}{17}\begin{bmatrix} -5 & 5 & -3 \\ 20 & -3 & 12 \\ 1 & -1 & 4 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 1 \\ -2 & -4 & -5 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} 1 & 9 & -3 \\ 2 & 5 & 6 \\ 4 & -2 & 7 \end{bmatrix}$

Solution:

$$\frac{1}{209}\begin{bmatrix} 47 & -57 & 69 \\ 10 & 19 & -12 \\ -24 & 38 & -13 \end{bmatrix}$$

Exercise:

Problem: $\begin{bmatrix} 1 & -2 & 3 \\ -4 & 8 & -12 \\ 1 & 4 & 2 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \end{bmatrix}$

Solution:

$$\begin{bmatrix} 18 & 60 & -168 \\ -56 & -140 & 448 \\ 40 & 80 & -280 \end{bmatrix}$$

Exercise:

Problem: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

For the following exercises, solve the system using the inverse of a 2×2 matrix.

Exercise:

Problem: $\begin{array}{r} 5x - 6y = -61 \\ 4x + 3y = -2 \end{array}$

Solution:

$(-5, 6)$

Exercise:

Problem: $\begin{array}{l} 8x + 4y = -100 \\ 3x - 4y = 1 \end{array}$

Exercise:

Problem: $$\begin{aligned} 3x - 2y &= 6 \\ -x + 5y &= -2 \end{aligned}$$

Solution:

$(2,0)$

Exercise:

Problem: $$\begin{aligned} 5x - 4y &= -5 \\ 4x + y &= 2.3 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} -3x - 4y &= 9 \\ 12x + 4y &= -6 \end{aligned}$$

Solution:

$\left(\frac{1}{3}, -\frac{5}{2}\right)$

Exercise:

Problem: $$\begin{aligned} -2x + 3y &= \frac{3}{10} \\ -x + 5y &= \frac{1}{2} \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} \frac{8}{5}x - \frac{4}{5}y &= \frac{2}{5} \\ -\frac{8}{5}x + \frac{1}{5}y &= \frac{7}{10} \end{aligned}$$

Solution:

$\left(-\frac{2}{3}, -\frac{11}{6}\right)$

Exercise:

Problem: $$\begin{aligned} \frac{1}{2}x + \frac{1}{5}y &= -\frac{1}{4} \\ \frac{1}{2}x - \frac{3}{5}y &= -\frac{9}{4} \end{aligned}$$

For the following exercises, solve a system using the inverse of a 3×3 matrix.

Exercise:

Problem:
$$\begin{aligned} 3x - 2y + 5z &= 21 \\ 5x + 4y &= 37 \\ x - 2y - 5z &= 5 \end{aligned}$$

Solution:

$\left(7, \frac{1}{2}, \frac{1}{5}\right)$

Exercise:

Problem:
$$\begin{aligned} 4x + 4y + 4z &= 40 \\ 2x - 3y + 4z &= -12 \\ -x + 3y + 4z &= 9 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 6x - 5y - z &= 31 \\ -x + 2y + z &= -6 \\ 3x + 3y + 2z &= 13 \end{aligned}$$

Solution:

$(5, 0, -1)$

Exercise:

Problem:
$$\begin{aligned} 6x - 5y + 2z &= -4 \\ 2x + 5y - z &= 12 \\ 2x + 5y + z &= 12 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 4x - 2y + 3z &= -12 \\ 2x + 2y - 9z &= 33 \\ 6y - 4z &= 1 \end{aligned}$$

Solution:

$\frac{1}{34}(-35, -97, -154)$

Exercise:

Problem:
$$\begin{aligned} \frac{1}{10}x - \frac{1}{5}y + 4z &= \frac{-41}{2} \\ \frac{1}{5}x - 20y + \frac{2}{5}z &= -101 \\ \frac{3}{10}x + 4y - \frac{3}{10}z &= 23 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} \frac{1}{2}x - \frac{1}{5}y + \frac{1}{5}z &= \frac{31}{100} \\ -\frac{3}{4}x - \frac{1}{4}y + \frac{1}{2}z &= \frac{7}{40} \\ -\frac{4}{5}x - \frac{1}{2}y + \frac{3}{2}z &= \frac{1}{4} \end{aligned}$$

Solution:

$\frac{1}{690}(65, -1136, -229)$

Exercise:

Problem:
$$\begin{aligned} 0.1x + 0.2y + 0.3z &= -1.4 \\ 0.1x - 0.2y + 0.3z &= 0.6 \\ 0.4y + 0.9z &= -2 \end{aligned}$$

Technology

For the following exercises, use a calculator to solve the system of equations with matrix inverses.

Exercise:

Problem:
$$\begin{aligned} 2x - y &= -3 \\ -x + 2y &= 2.3 \end{aligned}$$

Solution:

$\left(-\frac{37}{30}, \frac{8}{15}\right)$

Exercise:

Problem:
$$\begin{aligned} -\frac{1}{2}x - \frac{3}{2}y &= -\frac{43}{20} \\ \frac{5}{2}x + \frac{11}{5}y &= \frac{31}{4} \end{aligned}$$

Exercise:

Problem:
$$\begin{array}{r} 12.3x-2y-2.5z=2 \\ 36.9x+7y-7.5z=-7 \\ 8y-5z=-10 \end{array}$$

Solution:

$\left(\frac{10}{123}, -1, \frac{2}{5}\right)$

Exercise:

Problem:
$$\begin{array}{r} 0.5x-3y+6z=-0.8 \\ 0.7x-2y=-0.06 \\ 0.5x+4y+5z=0 \end{array}$$

Extensions

For the following exercises, find the inverse of the given matrix.

Exercise:

Problem: $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Solution:

$\frac{1}{2}\begin{bmatrix} 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & -1 & 0 \\ 1 & -3 & 0 & 1 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 4 & -2 & 3 \\ -5 & 0 & 1 & 1 \end{bmatrix}$

Solution:

$$\frac{1}{39}\begin{bmatrix} 3 & 2 & 1 & -7 \\ 18 & -53 & 32 & 10 \\ 24 & -36 & 21 & 9 \\ -9 & 46 & -16 & -5 \end{bmatrix}$$

Exercise:

Problem: $\begin{bmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$

Exercise:

Problem: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

Real-World Applications

For the following exercises, write a system of equations that represents the situation. Then, solve the system using the inverse of a matrix.

Exercise:

Problem:

2,400 tickets were sold for a basketball game. If the prices for floor 1 and floor 2 were different, and the total amount of money brought in is $64,000, how much was the price of each ticket?

Exercise:

Problem:

In the previous exercise, if you were told there were 400 more tickets sold for floor 2 than floor 1, how much was the price of each ticket?

Solution:

Infinite solutions.

Exercise:

Problem:

A food drive collected two different types of canned goods, green beans and kidney beans. The total number of collected cans was 350 and the total weight of all donated food was 348 lb, 12 oz. If the green bean cans weigh 2 oz less than the kidney bean cans, how many of each can was donated?

Exercise:

Problem:

Students were asked to bring their favorite fruit to class. 95% of the fruits consisted of banana, apple, and oranges. If oranges were twice as popular as bananas, and apples were 5% less popular than bananas, what are the percentages of each individual fruit?

Solution:

50% oranges, 25% bananas, 20% apples

Exercise:

Problem:

A sorority held a bake sale to raise money and sold brownies and chocolate chip cookies. They priced the brownies at $1 and the chocolate chip cookies at $0.75. They raised $700 and sold 850 items. How many brownies and how many cookies were sold?

Exercise:

Problem:

A clothing store needs to order new inventory. It has three different types of hats for sale: straw hats, beanies, and cowboy hats. The straw hat is priced at \$13.99, the beanie at \$7.99, and the cowboy hat at \$14.49. If 100 hats were sold this past quarter, \$1,119 was taken in by sales, and the amount of beanies sold was 10 more than cowboy hats, how many of each should the clothing store order to replace those already sold?

Solution:

10 straw hats, 50 beanies, 40 cowboy hats

Exercise:

Problem:

Anna, Ashley, and Andrea weigh a combined 370 lb. If Andrea weighs 20 lb more than Ashley, and Anna weighs 1.5 times as much as Ashley, how much does each girl weigh?

Exercise:

Problem:

Three roommates shared a package of 12 ice cream bars, but no one remembers who ate how many. If Tom ate twice as many ice cream bars as Joe, and Albert ate three less than Tom, how many ice cream bars did each roommate eat?

Solution:

Tom ate 6, Joe ate 3, and Albert ate 3.

Exercise:

Problem:

A farmer constructed a chicken coop out of chicken wire, wood, and plywood. The chicken wire cost \$2 per square foot, the wood \$10 per square foot, and the plywood \$5 per square foot. The farmer spent a total of \$51, and the total amount of materials used was $14\ \text{ft}^2.$ He used $3\ \text{ft}^2$ more chicken wire than plywood. How much of each material in did the farmer use?

Exercise:

Problem:

Jay has lemon, orange, and pomegranate trees in his backyard. An orange weighs 8 oz, a lemon 5 oz, and a pomegranate 11 oz. Jay picked 142 pieces of fruit weighing a total of 70 lb, 10 oz. He picked 15.5 times more oranges than pomegranates. How many of each fruit did Jay pick?

Solution:

124 oranges, 10 lemons, 8 pomegranates

Glossary

identity matrix
: a square matrix containing ones down the main diagonal and zeros everywhere else; it acts as a 1 in matrix algebra

multiplicative inverse of a matrix
: a matrix that, when multiplied by the original, equals the identity matrix